Boundary Calculations in Relative E-Theory

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1. Introduction

In this paper we develop some applications of the relative E-theory groups [Gue2]. The applications we consider are closely modeled on the applications of the relative K-homology groups [BD2] considered by Baum, Douglas, and Taylor [BDT].

After a preliminary section in which we recall the basic definitions and results of E-theory and relative E-theory, in Section 3 we study in detail how a self-adjoint extension of a first-order, elliptic differential operator on an open manifold determines an element of an E-theory group. In the cases of manifolds with boundary and complete manifolds, and under additional assumptions, an operator determines an element of a relative E-theory group. We describe some invariance properties of the E-theory classes associated to elliptic operators which propagate, via the excision isomorphism, to invariance properties of relative E-theory classes.

In Section 4 we discuss the boundary map in relative E-theory in greater detail. We begin by recalling the relevant constructions and then give the abstract boundary calculation for relative E-theory classes represented by compact asymptotic morphisms.

The abstract boundary calculation is specialized in Section 5 to calculate the image under the boundary map of the class of the Dolbeault operator on a strongly pseudoconvex domain in $\mathbb{C}^n$. Our approach is based on a vanishing theorem for a twisted Dolbeault operator on a strongly pseudoconvex domain equipped with a "Bergman-type" metric obtained independently by Donnelly [Don] and myself and Higson [GH].

In the final section we make a few remarks on the case of operators on manifolds with boundary considered by Baum, Douglas, and Taylor. We shall see how our results in the context of relative E-theory reproduce those of relative K-homology. In particular, we recover a number of results from [BDT].

The material in this paper formed part of my Ph.D. thesis at the Pennsylvania State University, although the construction of the boundary map has been greatly simplified and the material in Section 6 is presented here in a completely different manner. I am greatly indebted to my advisor N. Higson and would like to thank him for his invaluable help and encouragement.

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2. *E*-Theory

In this section we present a brief review of the definitions of asymptotic morphisms, E-theory, and relative E-theory groups. It is not our intention to give a thorough survey of the subject, but rather to collect for the reader's convenience results we will use in the sequel. For more extensive information as well as detailed proofs, the reader is encouraged to consult one of the many references on the subject. Our treatment follows most closely that of Dadarlat [Dad], which is reviewed briefly in [Gue2], the primary source for the material on relative E-theory. Other sources include [CH1; CH2; Con].

In this paper all C^*-algebras are assumed separable unless specifically stated otherwise. Let A and B be separable C^*-algebras. An *asymptotic morphism* is a family of functions $\{\varphi_t\}\colon A \to B$, indexed by $t \in [1,\infty)$, satisfying the continuity condition for all $a \in A$:

$$t \mapsto \varphi_t(a)\colon [1,\infty) \to B \text{ is continuous,}$$

as well as the following asymptotic conditions for all $a, a' \in A$ and $\lambda \in \mathbb{C}$:

$$\begin{aligned} \varphi_t(aa') - \varphi_t(a)\varphi_t(a') &\longrightarrow 0, \\ \varphi_t(a+\lambda a') - \varphi_t(a) - \lambda\varphi_t(a') &\longrightarrow 0, \\ \varphi_t(a)^* - \varphi_t(a^*) &\longrightarrow 0 \quad \text{as } t \to \infty. \end{aligned}$$

A continuous family of $*$-homomorphisms from A to B is an asymptotic morphism. In particular, a single $*$-homomorphism from A to B is the "constant" asymptotic morphism.

We introduce the following notational conventions. Denote by BX the C^*-algebra of continuous B-valued functions vanishing at infinity on the locally compact space X. Thus, $BX \cong C_0(X) \otimes B$. Most often we use this notation when X is an interval.

Two asymptotic morphisms $\{\varphi_t^0\}$ and $\{\varphi_t^1\}\colon A \to B$ are *asymptotically equivalent* or simply *equivalent* if, for all $a \in A$,

$$\varphi_t^0(a) - \varphi_t^1(a) \to 0 \quad \text{as } t \to \infty.$$

They are *homotopic* if there is an asymptotic morphism $\{\varphi_t\}\colon A \to B[0,1]$ such that

$$\mathrm{ev}_0 \circ \varphi_t = \varphi_t^0 \quad \text{and} \quad \mathrm{ev}_1 \circ \varphi_t = \varphi_t^1,$$

where ev_0 and ev_1 are evaluation at 0 and 1, respectively. Asymptotic equivalence and homotopy are equivalence relations on the set of asymptotic morphisms from A to B. The set of homotopy classes is denoted $[\![A, B]\!]$.

The *suspension* of an asymptotic morphism $\{\varphi_t\}\colon A \to B$ is the asymptotic morphism $\{1 \otimes \varphi_t\}\colon A(0,1) \to B(0,1)$ determined (up to equivalence) by the assignment

$$\{1 \otimes \varphi_t\} : (f \mapsto \varphi_t \circ f) : A(0,1) \to B(0,1).$$

If $\{\varphi_t\}$ is an equicontinuous family of functions, this is easily seen to determine an asymptotic morphism as required. If not, make use of the fact that every asymptotic morphism is equivalent to one given by an equicontinuous family of functions. The operation of suspension is well-defined on asymptotic equivalence and homotopy classes. Up to homotopy we denote $A(0,1)$ by $\mathcal{S}A$ and $\{1 \otimes \varphi_t\}$ by $\{\mathcal{S}\varphi_t\}$, so that suspension defines a map $\mathcal{S}\colon [\![A, B]\!] \to [\![\mathcal{S}A, \mathcal{S}B]\!]$.

The set $[\![A, \mathcal{S}^k B]\!]$ is a group under loop composition for $k \geq 1$. For $k \geq 2$ this group is abelian. The suspension maps $\mathcal{S}$ are group homomorphisms.

Definition 2.1. The E-theory groups of the C^*-algebra A are defined by

$$E^n(A) = [\![A, \mathcal{K}]\!]_n = \varinjlim_{k\in\mathbb{N}} [\![\mathcal{S}^{k+n}A, \mathcal{S}^k\mathcal{K}]\!],$$

where $\mathcal{K}$ is the algebra of compact operators on a separable Hilbert space and the direct limit is taken with respect to the suspension maps.

Remark. This definition is equivalent to (although not the same as) the one given by Connes and Higson [CH1, Con]. For a discussion of the equivalence of these definitions, see Section 2 of [Gue2].

A fundamental result of the theory is the existence of long exact sequences for arbitrary ideals of C^*-algebras.

Theorem 2.2. *Let I be an ideal of the C^*-algebra A. There is a boundary map $\delta\colon E^n(A) \to E^{n+1}(A/I)$ and a long exact sequence*

$$\cdots \longrightarrow E^n(A) \longrightarrow E^n(I) \xrightarrow{\delta} E^{n+1}(A/I) \longrightarrow E^{n+1}(A) \longrightarrow \cdots.$$

The maps in the sequence other than δ are induced by appropriate inclusions or projections.

Proof. Corollary 2.13 of [Gue2] and Theorem 14 of [Dad]. □

A *pair of C^*-algebras* consists of a C^*-algebra A and a closed two-sided ideal I. We introduce the notation $A \rhd I$ for a pair of C^*-algebras. A *relative asymptotic morphism* is an asymptotic morphism $\{\varphi_t\}\colon A \to B$ such that $\varphi_t(I) \subset J$ for all $t \geq 1$. We use the notation $\{\varphi_t\}\colon A \rhd I \to B \rhd J$.

Just as for ordinary asymptotic morphisms, there are notions of (asymptotic) equivalence, homotopy, and suspension of relative asymptotic morphisms. We denote the set of homotopy classes of relative asymptotic morphisms by $[\![A \rhd I, B \rhd J]\!]$. Introducing the notation $\mathcal{S}(A \rhd I)$ for the pair $\mathcal{S}A \rhd \mathcal{S}I$ suspension gives a map $\mathcal{S}\colon [\![A \rhd I, B \rhd J]\!] \to [\![\mathcal{S}(A \rhd I), \mathcal{S}(B \rhd I)]\!]$.

Definition 2.3. The relative E-theory groups of the pair $A \rhd I$ are defined by

$$E^n_{\mathrm{rel}}(A; I) = [\![A \rhd I, \mathcal{B} \rhd \mathcal{K}]\!]_n = \varinjlim_{k\in\mathbb{N}} [\![\mathcal{S}^{k+n}(A \rhd I), \mathcal{S}^k(\mathcal{B} \rhd \mathcal{K})]\!],$$

where $\mathcal{K}$ and $\mathcal{B}$ are the algebras of compact and bounded operators on a separable Hilbert space and the direct limit is taken with respect to the suspension maps.

The fundamental results of relative E-theory are the existence of a boundary map that fits into a long exact sequence and the excision isomorphism. We shall return to a discussion of the construction of the boundary map $\mathrm{Bdy}^n : E^n_{\mathrm{rel}}(A; I) \to E^{n+1}(A/I)$ in Section 4. At this stage we are content to state the following theorem.

THEOREM 2.4. *Let $A \rhd I$ be a pair of C^*-algebras. There is a commutative diagram with exact rows*:

$$\begin{tikzcd}
E^n(A/I) \arrow[r] \arrow[d, equal] & E^n(A) \arrow[r] \arrow[d, equal] & E^n_{\mathrm{rel}}(A; I) \arrow[r, "\mathrm{Bdy}^n"] \arrow[d, "\mathrm{Ex}^n"] & E^{n+1}(A/I) \arrow[r] \arrow[d, equal] & E^{n+1}(A) \arrow[d, equal] \\
E^n(A/I) \arrow[r] & E^n(A) \arrow[r] & E^n(I) \arrow[r, "\delta"] & E^{n+1}(A/I) \arrow[r] & E^{n+1}(A)
\end{tikzcd}$$

Furthermore, the excision map Ex^n *is an isomorphism*.

Proof. Theorem 6.15 and Corollary 6.16 of [Gue2]. □

In applications we consider only commutative C^*-algebras. Consequently we introduce the following notation. For a locally compact, metrizable topological space Z we use $E_{-n}(Z)$ to denote $E^n(C_0(Z))$. For a compact, metrizable topological space X and closed subspace Y we use $E_{-n}(X; Y)$ to denote $E^n_{\mathrm{rel}}(C(X); C_0(X \setminus Y))$. With this notation the boundary map and excision isomorphism take the form $E_{-n}(X; Y) \to E_{-n-1}(Y)$ and $E_{-n}(X, Y) \cong E_{-n}(X \setminus Y)$.

3. Elliptic Operators

In this section we discuss how a self-adjoint extension of a first-order, elliptic differential operator D on an open manifold M determines an element of the E-theory group $E_n(M)$, where $n = 0, 1$ according as the dimension of M is even or odd (in the even-dimensional case it is of course necessary to assume that the operator D is odd with respect to a grading operator). Although the formula used in defining the asymptotic morphism associated to D is known, it has not appeared in the literature and we go into some detail. We proceed to show that the element is independent of the choice of self-adjoint extension, so that we may unambiguously write

$$[D] \in E_n(M).$$

We will describe the following stability results for the class $[D] \in E_n(M)$:

(a) $[D]$ depends only on the principal symbol of D; and
(b) if D is a "geometric operator" then $[D]$ is independent of the choice of Riemannian structure on M.

In our applications the manifold M will typically be the interior of a manifold with boundary, or a complete Riemannian manifold. In these cases, and under certain additional conditions, the operator D determines an element of a relative E-theory group

$$[D] \in E_0(\bar{M}, \partial M),$$

where $\bar{M}$ is a suitable compactification of M and $\partial M = \bar{M} \setminus M$ is the boundary of $\bar{M}$.

3.1. The Class of an Operator

Let M be an open, Riemannian manifold. Let D be a first-order differential operator acting on smooth sections of a Hermitian vector bundle S on M. Assume that D is formally self-adjoint. We consider D as an unbounded symmetric operator on the Hilbert space $L^2(S)$, with domain $C_c^\infty(S)$, the smooth compactly supported sections of S.

To each $\varphi \in C^\infty(M)$ we associate the operator of multiplication by φ on $L^2(S)$. To simplify notation we denote this operator also by φ, although occasionally we write M_φ for emphasis. Each of the operators M_φ maps the domain of D into itself. Thus, the commutator $[D, \varphi]$ is defined on domain(D) and satisfies the *symbol identity,*

$$[D, \varphi]s(x) = \mathrm{sym}(D)(x, d\varphi_x)s(x), \tag{1}$$

where $s \in C_c^\infty(S)$ and $\mathrm{sym}(D)$ is the principal symbol of D. It follows that the commutator $[D, \varphi]$ is a pointwise skew-adjoint multiplication operator with domain $C_c^\infty(S)$. It may or may not extend by continuity to a bounded operator on $L^2(S)$, depending on the particular choice of operator D and smooth function φ.

The *local propagation speed* of the operator D at $x \in M$ is

$$\mathrm{Prop}_x(D) = \sup\{\, \|\mathrm{sym}(D)(x, \xi)\| : (x, \xi) \in T_x^*M,\ \|\xi\| = 1\},$$

where the norm on the right is the operator norm on $\mathrm{End}(S_x)$.

LEMMA 3.1. *Let φ be a smooth compactly supported function on M. Then the commutator $[D, \varphi]$ extends to a bounded operator $\overline{[D, \varphi]}$ on $L^2(S)$ and*

$$\|\overline{[D, \varphi]}\| \le \sup\{\, \|d\varphi_x\|\, \mathrm{Prop}_x(D) : x \in M\,\}.$$

Proof. Immediate from the definitions and the symbol identity (1). □

To obtain an element of the E-theory group $E_0(M)$ we must associate an asymptotic morphism to D. Since our formula will involve the functional calculus for unbounded self-adjoint operators, we are led to consider extensions of D to unbounded self-adjoint operators on $L^2(S)$.

Prior to discussing these extensions, we fix some notation. The *minimal extension* of D is denoted by $D^{\min}$ and the *maximal extension* of D is denoted by $D^{\max}$. We assume that D is formally self-adjoint, so we have $D^{\max} = D^*$, the *adjoint* of D [BDT].

LEMMA 3.2. *Multiplication by $\varphi \in C_c^\infty(M)$ maps the domains of $D^{\min}$ and $D^{\max}$ into themselves, and the commutators $[D^{\min}, \varphi]$ and $[D^{\max}, \varphi]$ extend to bounded operators on $L^2(S)$. There is an identity of bounded operators on $L^2(S)$:*

$$[D^{\min}, \varphi] = \overline{[D, \varphi]} = [D^{\max}, \varphi].$$

Proof. Straightforward calculation from the definitions. □

From now on we assume that the operator D is *elliptic*.

PROPOSITION 3.3. *Let $s \in L^2(S)$ be compactly supported. Then*

$$s \in H^1_{\mathrm{comp}}(S) \iff s \in \mathrm{domain}(D^{\min}) \iff s \in \mathrm{domain}(D^{\max}).$$

Sketch of Proof. Using the previous lemma and a finite partition of unity for a compact neighborhood of the support of s, the proof may be reduced to the case of a compactly supported operator on $\mathbb{R}^n$. The proof for this case is a standard Friedrich's mollifier argument (see e.g. [Tay; Fol]). □

It follows immediately from these propositions that multiplication by smooth compactly supported functions maps the domain of $D^{\max}$ into the domain of $D^{\min}$. We consider an extension $\tilde{D}$ of D to an unbounded self-adjoint operator on $L^2(S)$. Such an extension necessarily satisfies $D^{\min} \subset \tilde{D} \subset D^{\max}$. For $\varphi \in C_c^\infty(M)$ we therefore conclude that:

(a) multiplication by φ maps the domain of $\tilde{D}$ into itself; and
(b) the commutator $[\tilde{D}, \varphi]$, an operator on $\mathrm{domain}(\tilde{D})$, extends to a bounded operator on $L^2(S)$ equal to $\overline{[D, \varphi]}$ (since $[\tilde{D}, \varphi]$ and $[D^{\max}, \varphi]$ agree on $\mathrm{domain}(\tilde{D}) \subset \mathrm{domain}(D^{\max})$).

We need to add a little extra structure, typical of operators on even-dimensional manifolds, before continuing. A *grading operator* on the vector bundle S is a self-adjoint endomorphism ε of S satisfying $\varepsilon^2 = 1$. The vector bundle decomposes as $S = S_+ \oplus S_-$, where $S_\pm$ is the ± 1-eigenbundle of ε. There are similar decompositions on spaces of smooth compactly supported and square integrable sections of S. An operator D is *odd with respect to the grading* if $\varepsilon D = -D\varepsilon$. In this case D is represented by the off-diagonal matrix

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$$

with respect to the decomposition $C_c^\infty(S) = C_c^\infty(S_+) \oplus C_c^\infty(S_-)$.

Assume that the Hermitian vector bundle S is graded, with grading operator ε, and that the operator D is odd with respect to the grading.

REMARK. Under the assumptions just outlined,

$$\tilde{D} = \begin{pmatrix} 0 & D_-^{\min} \\ D_+^{\max} & 0 \end{pmatrix}$$

defines a self-adjoint extension of D [BDT]. It will be used frequently in the sequel.

We now explain how to associate an asymptotic morphism, and E-theory class, to the operator D. As mentioned in Section 1, this result is part of the folklore and originally appeared without proof in the unpublished manuscript [CH1].

THEOREM 3.4. *An extension $\tilde{D}$ of D to an unbounded self-adjoint operator determines an element $[\tilde{D}]$ of the E-theory group $E_0(M)$ by the assignment*

$$\{\mathcal{A}_t^{\tilde{D}}\}\colon\ f \otimes \varphi \mapsto M_\varphi f(t^{-1}\tilde{D} + x\varepsilon), \quad f \in C_0(\mathbb{R}),\ \ \varphi \in C_0(M).$$

We write M_φ for the bounded operator on $L^2(S)$ of multiplication by φ. The bounded operator $f(t^{-1}\tilde{D} + x\varepsilon)$ on $L^2(S)$ is defined by the functional calculus.

In the proof of this theorem we need the following result. For the sake of completeness we shall record a proof (but see also [Hig2]).

LEMMA 3.5. *For $f \in C_0(\mathbb{R})$ and $\varphi \in C_0(M)$, the operator $\varphi f(\tilde{D})$ is a compact operator on $L^2(S)$.*

Proof. It suffices to consider the case where $\varphi \in C_c^\infty(M)$ and f is one of the resolvent functions $r_\pm(x) = (x \pm \sqrt{-1})^{-1}$. Note that $r_\pm(\tilde{D})$ maps $L^2(S)$ into the domain of $\tilde{D}$ and hence that $\varphi r_\pm(\tilde{D})$ maps $L^2(S)$ into $H^1(\text{support}(\varphi), S)$.

It follows from the Rellich lemma that $H^1(\text{support}(\varphi), S) \to L^2(S)$ is a compact inclusion. From the boundedness of this inclusion we see that $\varphi r_\pm(\tilde{D})$, considered as an operator into $H^1(\text{support}(\varphi), S)$, has closed graph. By the closed graph theorem, $\varphi r_\pm(\tilde{D})$ is bounded as an operator into $H^1(\text{support}(\varphi), S)$. By the compactness of the inclusion we conclude that $\varphi r_\pm(\tilde{D})$ is compact as an operator on $L^2(S)$. $\square$

For future reference we record the *resolvent identity* for a self-adjoint unbounded operator T (where $r_\pm$ are as before):

$$[r_\pm(T), \varphi] = r_\pm(T)[\varphi, T]r_\pm(T). \tag{2}$$

Proof of Theorem 3.4. We must show that $\{\mathcal{A}_t^{\tilde{D}}\}$ determines an asymptotic morphism (up to equivalence) from $C_0(\mathbb{R}) \otimes C_0(M)$ to $C_0(\mathbb{R}) \otimes \mathcal{K}$, where $\mathcal{K}$ is the algebra of compact operators on $L^2(S)$. Denote by $\mathcal{B}$ the algebra of bounded operators on $L^2(S)$. The proof comprises three steps:

(i) the assignment $\varphi \mapsto M_\varphi$ is a $*$-homomorphism $C_0(M) \to \mathcal{B}$;
(ii) the assignment $f \mapsto f(t^{-1}\tilde{D} + x\varepsilon)$ is a continuous family of $*$-homomorphisms $C_0(\mathbb{R}) \to C_0(\mathbb{R}) \otimes \mathcal{B}$; and
(iii) $[\varphi, f(t^{-1}D + x\varepsilon)] \to 0$ as $t \to \infty$ for $\varphi \in C_0(M)$ and $f \in C_0(\mathbb{R})$.

The theorem then follows from the previous lemma and Lemma 7.1 of Section 7. (In applying this lemma, take $A = C_0(\mathbb{R})$, $B = J = C_0(M)$, $C = C_b(\mathbb{R}, B)$ and $K = C_0(\mathbb{R})$.)

Of these three points, (i) is obvious. For (ii), we begin by checking for $f \in C_0(\mathbb{R})$ that $f(t^{-1}\tilde{D} + x\varepsilon)$ is a continuous operator-valued function of x vanishing at infinity; by the spectral theorem and an approximation argument, it suffices to check this for the resolvent functions $r_\pm$. The difference

$$r_\pm(t^{-1}\tilde{D} + x\varepsilon) - r_\pm(t^{-1}\tilde{D} + y\varepsilon) = r_\pm(t^{-1}\tilde{D} + x\varepsilon)((y - x)\varepsilon)r_\pm(t^{-1}\tilde{D} + y\varepsilon)$$

has norm bounded by $|x - y|$. It follows that $r_\pm(t^{-1}\tilde{D} + x\varepsilon)$ is continuous in x. Notice now that

$$(t^{-1}\tilde{D}+x\varepsilon)^2 = t^{-2}\tilde{D}^2 + x^2 \geq x^2$$

independently of $t \geq 1$. Hence, for each $t \geq 1$, the spectrum of $t^{-1}\tilde{D}+x\varepsilon$ is contained in the complement of $(-x, x)$. We conclude that $r_{\pm}(t^{-1}\tilde{D}+x\varepsilon)$ vanishes at infinity. In fact,

$$\|r_{\pm}(t^{-1}\tilde{D}+x\varepsilon)\| \leq \sup\{\,|r_{\pm}(y)| : |y| \geq |x|\,\} = (x^2+1)^{-1/2}.$$

Next we must check that for $f \in C_0(\mathbb{R})$ the family of operator-valued functions $f(t^{-1}\tilde{D}+x\varepsilon)$ is continuous in $t \in [1,\infty)$. Again we may reduce to the case $f = r_{\pm}$. It follows from the factorization

$$\begin{aligned} &r_{\pm}(t^{-1}\tilde{D}+x\varepsilon) - r_{\pm}(s^{-1}\tilde{D}+x\varepsilon)\\ &\qquad = r_{\pm}(t^{-1}\tilde{D}+x\varepsilon)(s^{-1}-t^{-1})\,\tilde{D}r_{\pm}(s^{-1}\tilde{D}+x\varepsilon)\\ &\qquad = r_{\pm}(t^{-1}\tilde{D}+x\varepsilon)(1-st^{-1})(s^{-1}\tilde{D}+x\varepsilon)r_{\pm}(s^{-1}\tilde{D}+x\varepsilon)\\ &\qquad\quad + r_{\pm}(t^{-1}\tilde{D}+x\varepsilon)(1-st^{-1})(-x\varepsilon)r_{\pm}(s^{-1}\tilde{D}+x\varepsilon) \end{aligned}$$

that, for each $x \in \mathbb{R}$,

$$\|r_{\pm}(t^{-1}\tilde{D}+x\varepsilon) - r_{\pm}(s^{-1}\tilde{D}+x\varepsilon)\| \leq (1-st^{-1})\left(\frac{|x|}{x^2+1} + \frac{1}{(x^2+1)^{1/2}}\right).$$

Hence, for $s \to t$, the operator-valued functions $r_{\pm}(s^{-1}\tilde{D}+x\varepsilon)$ of $x \in \mathbb{R}$ converge uniformly to $r_{\pm}(t^{-1}\tilde{D}+x\varepsilon)$.

Turning now to (iii), it suffices to consider $\varphi \in C_c^{\infty}(M)$ and $f = r_{\pm}$. It follows from the resolvent identity (2) that

$$\|[r_{\pm}(t^{-1}\tilde{D}+x\varepsilon), \varphi]\| \leq t^{-1}\|[\varphi, \tilde{D}]\| \to 0 \quad \text{as } t \to \infty. \qquad \square$$

REMARK. Of the three points discussed in the proof the first two hold in great generality; the first for the C^*-algebra of continuous bounded functions, and the second for any self-adjoint unbounded operator T on $L^2(S)$. Generalizations of the third will concern us later.

REMARK. It is often useful to observe that the asymptotic morphism $\{\mathcal{A}_t^{\tilde{D}}\}$ is also defined (up to equivalence) by the assignment

$$f \otimes \varphi \mapsto f(t^{-1}D + x\varepsilon)M_{\varphi}, \qquad f \in C_0(\mathbb{R}),\ \ \varphi \in C_0(M).$$

This follows from an approximation argument and the resolvent identity (2), as in the previous proof.

PROPOSITION 3.6. *Different self-adjoint extensions of the operator D determine the same element of the E-theory group $E_0(M)$. In fact, the asymptotic morphisms associated to different self-adjoint extensions of D are asymptotically equivalent.*

Proof. By an approximation argument and the previous remark, it suffices to show that for $\varphi \in C_c^{\infty}(M)$,

$$\mathcal{A}_t^{\tilde{D}}(r_{\pm} \otimes \varphi) - r_{\pm}(t^{-1}\hat{D}+x\varepsilon)\varphi \to 0 \quad \text{as } t \to \infty. \tag{3}$$

But for $\varphi \in C_c^\infty(M)$ we have $\varphi\,\mathrm{domain}(\tilde{D}) \subset \mathrm{domain}(D^{\min}) \subset \mathrm{domain}(\hat{D})$, so we may factor

$$\begin{aligned}
&\mathcal{A}_t^{\tilde{D}}(r_\pm \otimes \varphi) - r_\pm(t^{-1}\hat{D} + x\varepsilon)\varphi \\
&\qquad = r_\pm(t^{-1}\hat{D} + x\varepsilon)t^{-1}(\hat{D}\varphi - \varphi\tilde{D})r_\pm(t^{-1}\tilde{D} + x\varepsilon) \\
&\qquad = r_\pm(t^{-1}\hat{D} + x\varepsilon)t^{-1}\big((\hat{D} - \tilde{D})\varphi + [\tilde{D}, \varphi]\big)r_\pm(t^{-1}\tilde{D} + x\varepsilon).
\end{aligned}$$

Because $\hat{D}$ and $\tilde{D}$ agree on $\mathrm{domain}(D^{\min})$, we see that

$$(\hat{D} - \tilde{D})\varphi r_\pm(t^{-1}\tilde{D} + x\varepsilon) = 0.$$

Hence, the norm of (3) is bounded by $t^{-1}\|[\tilde{D}, \varphi]\| \to 0$ as $t \to \infty$. □

Armed with this proposition, we write $[D] \in E_0(M)$ for the E-theory class unambiguously assigned to the operator D by Theorem 3.4.

We now describe the stability results for the class $[D]$ mentioned at the beginning of this section. We limit ourselves to a description of the relevant results, which are not surprising in their content to the experienced reader. For a detailed discussion and proofs we refer to [Gue1].

We begin by observing that, just as the index of a Fredholm operator is stable under compact perturbations, the E-theory class of $[D]$ depends only on the principal symbol of D. Let V be a "zeroth-order potential" on M, that is, a pointwise self-adjoint, smooth endomorphism of the Hermitian bundle S. Pointwise multiplication by V determines an operator on $L^2(S)$. We make no assumptions about the boundedness of the potential V, and this operator is not necessarily bounded. It is, however, formally self-adjoint on the domain $C_c^\infty(S)$. We do assume that each $V(x)$, which is an endomorphism of the finite-dimensional vector space S_x, is odd with respect to the grading. We are interested in comparing the E-theory classes associated to the operators D and $D_V = D + V$.

PROPOSITION 3.7. *The E-theory classes associated to the operators D and D_V are equal*:

$$[D] = [D_V] \in E_0(M).$$

In fact, the asymptotic morphisms associated to the operators D and D_V are asymptotically equivalent.

Sketch of Proof. We use the self-adjoint extensions $\tilde{D}$ and $\tilde{D}_V$ described in the remark immediately preceding the statement of Theorem 3.4. As a representative of $[D] \in E_0(M)$ we take the asymptotic morphism associated to the extension $\tilde{D}$:

$$\{\mathcal{A}_t^{\tilde{D}}\}\colon\ f \otimes \varphi \mapsto M_\varphi f(t^{-1}\tilde{D} + x\varepsilon). \tag{4}$$

As a representative of $[D_V] \in E_0(M)$ we take the asymptotic morphism associated to $\tilde{D}_V$, using the remark following the proof of Theorem 3.4:

$$f \otimes \varphi \mapsto f(t^{-1}\tilde{D}_V + x\varepsilon)M_\varphi. \tag{5}$$

Factorizations similar to those in proof of the previous proposition show that, for $\varphi \in C_c^\infty(M)$ and $f = r_\pm$, the norm of the difference of (4) and (5) is bounded by $t^{-1}(\|V\varphi\| + \|[D, M_\varphi]\|) \to 0$ as $t \to \infty$. $\square$

We now turn to our second stability result. We show that the E-theory class associated to an elliptic operator depends only on its homotopy class and not, for example, on the Hermitian structure of the vector bundle S or the Riemannian structure of the manifold M.

An *operator homotopy* $\{D_s\}$ is a first-order differential operator on the cross-product $M \times [0, 1]$ acting on smooth compactly supported sections of the pulled-back vector bundle S such that, with respect to local coordinates $x = (x_1, \dots, x_n)$ on M and a local trivialization of S over M,

$$D_s = \sum_{j=1}^{n} a_j(x, s) \frac{\partial}{\partial x_j} + b(x, s),$$

where $a_j(x, s)$ and $b(x, s)$ are smooth matrix-valued functions of $(x, s) \in M \times [0, 1]$ and where $\sum a_j(x, s)\xi_j$ is invertible for all $0 \neq \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ and $(x, s) \in M \times [0, 1]$.

REMARK. There is no derivative in the s-direction, so an operator homotopy does restrict, for each $s \in [0, 1]$, to an operator on M. The definition has been phrased so that there is some control over the zeroth-order part of the individual operators D_s.

REMARK. An operator homotopy is nothing other than a homotopy within first-order elliptic symbols from the principal symbol of D_0 to that of D_1. More precisely, the principal symbol of an operator homotopy can be used to construct a homotopy of the principal symbols of the restricted operators. Conversely, a first-order differential operator associated to a homotopy of principal symbols, considered as a first-order symbol on $M \times [0, 1]$, is an operator homotopy as in the definition.

PROPOSITION 3.8. *For the fixed Riemannian structure on M and Hermitian structure on S, the class of the operator D depends only on its homotopy class.*

Sketch of Proof. Let $\{D_s\}$ be an operator homotopy. We must show that $[D_0] = [D_1] \in E_0(M)$. The idea of the proof is to show that the family of asymptotic morphisms for $s \in [0, 1]$,

$$\{\mathcal{A}_t^{D_s}\} \colon f \otimes \varphi \mapsto M_\varphi f(t^{-1}D_s + x\varepsilon),$$

fit together to form a homotopy of asymptotic morphisms. The asymptotic properties of the $\mathcal{A}_t^{D_s}$ hold uniformly in s. Hence, the family $\{\mathcal{A}_t^{D_s}\}$ would define an asymptotic morphism as required if it were continuous in the homotopy variable s. Observe that it is continuous in s as $t \to \infty$ in the sense that, for all $f \in C_0(\mathbb{R})$, $\varphi \in C_c^\infty(M)$, and $\varepsilon > 0$, there exist $T > 0$ and $\delta > 0$ such that

$$\|\mathcal{A}_t^{D_s}(f \otimes \varphi) - \mathcal{A}_t^{D_{s'}}(f \otimes \varphi)\| < \varepsilon \quad \forall t > T \text{ and } \forall |s - s'| < \delta,$$

the norm being taken in $C_0(\mathbb{R}) \otimes \mathcal{K}$. From this it follows that the family $\{\mathcal{A}_t^{D_s}\}$ may be adjusted up to equivalence to obtain an asymptotic morphism as required. $\square$

The *positive part* of the operator $D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$ is D_+. To study the dependence of the E-theory class of D on the Riemannian structure of M and the Hermitian structure of S, we change focus slightly and consider D_+ instead of D. To emphasize this change of focus we refer to the E-theory class of D_+.

Our assumption that D be formally self-adjoint on the domain $C_c^\infty(S)$ forces this change of focus; we are free to change neither the Riemannian structure of M nor the Hermitian structure of S without altering D in some way. On the other hand, D and D_+ determine each other. Clearly, D determines D_+. Conversely, a Riemannian structure on M and a Hermitian structure on S determine an inner product on $C_c^\infty(S)$. We assume that D is formally self-adjoint on this domain, so D is determined by D_+.

PROPOSITION 3.9. *The class of the operator D_+ is independent of the choice of Riemannian structure on the underlying manifold.*

Sketch of Proof. Let M be equipped with a Riemannian structure and S with a Hermitian structure, and let D be an operator as before. Let M' denote M but with a different Riemannian structure. The volume forms of M and M' are related by multiplication by a positive real-valued smooth function u^2. Multiplication by u defines a unitary isomorphism $U\colon L^2(M, S) \to L^2(M', S)$.

Since U commutes with multiplication by smooth bounded functions on M, we see that there exists an operator D' such that UDU^{-1} and D' have the same positive part and differ by a zeroth-order potential. The result now follows from Proposition 3.7. $\square$

PROPOSITION 3.10. *The class of the operator D_+ is independent of the Hermitian structure on the vector bundle S.*

Sketch of Proof. Let M be equipped with a Riemannian structure and S with a Hermitian structure, and let D be an operator as before. Let S' denote S but with a different Hermitian structure, and let D' be an operator on S' such that $\mathrm{sym}(D'_+) = \mathrm{sym}(D_+)$. Note that any two such D' differ by a zeroth-order term and so, by Proposition 3.7, $[D'] \in E_0(M)$ is well-defined. We must show $[D'] = [D]$.

Let $\pi\colon M \times [0,1] \to M$ be the projection. Denote by π^*S the pullback of S with the constant Hermitian structure of S. Denote by S'' the pullback of S equipped with a Hermitian structure smoothly varying from that of S at $s = 0$ to that of S' at $s = 1$.

Extend the principal symbol of D_+, pulled back over S'', to a skew-adjoint endomorphism $\sigma(x, \xi)$ of S'', and let $D'' = \{D''_s\}$ be any first-order differential operator with principal symbol σ.

There is a unitary bundle isomorphism from π^*S to S'' that is the identity over $s = 0$ and conjugates the operator D'' to an operator homotopy $\{D_s\}$ over π^*S.

This homotopy is from $D_0 = D$ to D_1, and D_1 is unitarily equivalent to D'. Now apply Proposition 3.8. $\square$

3.2. Complete Manifolds

Let M be a complete, open Riemannian manifold. Let D be an operator of the type considered in Section 3.1. That is, D is a first-order, elliptic differential operator, formally self-adjoint on the domain of smooth compactly supported sections of a Hermitian vector bundle S on M. We are interested in associating to D an element of the relative E-theory groups $E_0(\bar{M}, \partial M)$ for suitable compactifications $\bar{M}$ of M. In order to ensure that the formula of Theorem 3.4 defines a relative asymptotic morphism, we need to place restrictions on the operator D as well as on the compactification $\bar{M}$.

The operator D has *finite propagation speed* if its symbol is bounded on the cosphere bundle of M. Equivalently, the local propagation $\operatorname{Prop}_x(D)$ defined earlier is bounded independently of $x \in M$. The *propagation bound* of D is then defined by

$$\operatorname{Prop}(D) = \sup_{x \in M} \operatorname{Prop}_x(D).$$

Let $C_b(M)$ be the C^*-algebra of continuous bounded functions on M, and let $C_h(M)$ be the commutative unital C^*-subalgebra of $C_b(M)$ generated by the smooth bounded functions on M whose gradients vanish at infinity. (The notation $C_h(M)$ is taken from [Roe3].) A *metric compactification* of M is the maximal ideal space $\bar{M}$ of a separable, unital C^*-subalgebra of $C_h(M)$. The boundary $\partial\bar{M} = \bar{M} \setminus M$ is a *metric corona* of M. We have $C_0(M) \lhd C(\bar{M}) \subset C_h(M)$ and $C(\partial\bar{M}) \cong C(\bar{M})/C_0(M)$. Further, $\bar{M}$ is a compactification of M in the usual sense, meaning that $\bar{M}$ is a compact topological space containing M as an open dense subset.

As suggested by Higson [Hig1; Hig2] and Roe [Roe3], the notions of metric compactification and finite propagation speed are dual for the purposes of K-homology.

LEMMA 3.11. *Let φ be a smooth function on M with bounded gradient. Then the commutator $[D, \varphi]$ extends to a bounded operator $\overline{[D, \varphi]}$ on $L^2(S)$ and*

$$\|\overline{[D, \varphi]}\| \le \|d\varphi_x\|_\infty \operatorname{Prop}(D).$$

Furthermore, multiplication by φ maps the domain $D^{\min}$ into itself. The commutator $[D^{\min}, \varphi]$ extends continuously to the bounded operator $\overline{[D, \varphi]}$ on $L^2(S)$.

Proof. The first assertion follows from the symbol identity (1). The second is a direct calculation from the definitions (cf. Lemmas 3.1 and 3.2). $\square$

Let now $\bar{M}$ be a metric compactification of M, and assume that $\dim M$ is even. Let D be an operator as before. Assume that the bundle S is graded with grading operator ε and that D has odd grading degree. The assumption of finite propogation speed ensures that D is essentially self-adjoint [Che; Wol; GL] and allows us to dispense with the various extensions of D considered previously. With an abuse of notation we denote the unique self-adjoint extension of D also by D.

THEOREM 3.12. *The operator D determines an element of the relative E-theory group $[D] \in E_0(\bar{M}, \partial M)$. The element $[D]$ is determined by the assignment*

$$\{\mathcal{A}_t^D\}\colon f \otimes \varphi \mapsto M_\varphi f(t^{-1}D + x\varepsilon), \quad f \in C_0(\mathbb{R}), \;\; \varphi \in C(\bar{M}).$$

Proof. We must show that $\{\mathcal{A}_t^D\}$ determines a relative asymptotic morphism (up to equivalence) from $C_0(\mathbb{R}) \otimes C(\bar{M}) \rhd C_0(\mathbb{R}) \otimes C_0(M)$ to $C_0(\mathbb{R}) \otimes \mathcal{B} \rhd C_0(\mathbb{R}) \otimes \mathcal{K}$, where $\mathcal{K}$ and $\mathcal{B}$ are the algebras of compact and bounded operators on the Hilbert space $L^2(S)$, respectively.

The proof of Theorem 3.4 generalizes immediately to this situation. We need simply observe that, for $\varphi \in C^\infty(M)$ bounded with bounded gradient, the resolvent identity (2) shows that

$$\|[r_\pm(t^{-1}D + x\varepsilon), \varphi]\| \le t^{-1}\|[\varphi, D]\| \le t^{-1}\|d\varphi\|_\infty \operatorname{Prop}(D) \to 0 \quad \text{as } t \to \infty.$$

Now invoke Lemma 7.1. □

A close inspection of the proofs reveals that we have not exploited in full the definition of the metric compactification. In particular, these results remain true if we consider bounded functions φ with bounded gradient. The interplay between the functions of vanishing gradient and operators of finite propagation will be exploited in calculating the image of the class $[D]$ under the boundary map $E_0(\bar{M}; \partial M) \to E_{-1}(\partial \bar{M})$. In this regard, the following proposition will prove useful in a subsequent section.

PROPOSITION 3.13. *Let φ be a smooth function on M with gradient vanishing at infinity. Then, for D an operator with finite propagation speed and $f \in C_0(\mathbb{R})$, the commutator $[f(D), \varphi]$ is compact.*

Proof. It suffices to consider the case $f = r_\pm$. Recall the resolvent identity (2):

$$[r_\pm(D), \varphi] = r_\pm(D)[\varphi, D]r_\pm(D).$$

Since the gradient of φ vanishes at infinity, it follows from the symbol identity (1) that the commutator $[\varphi, D]$ is multiplication by a smooth section of the endomorphism bundle of S vanishing at infinity. Arguing as in the proof of Proposition 3.5, we see that the product $[\varphi, D]r_\pm(D)$ is a compact operator. □

3.3. Manifolds with Boundary

Let $\bar{M}$ be a Riemannian manifold with boundary and let M denote the interior of $\bar{M}$. Let D be an operator of the type considered previously. That is, D is a first-order, elliptic differential operator on M, formally self-adjoint on the domain of smooth compactly supported sections of a Hermitian vector bundle S. Additionally, assume that the operator D extends to a closed Riemannian manifold M' containing M as a submanifold with smooth boundary.

REMARK. This added assumption is not a serious restriction. If D is a Dirac-type operator (i.e., associated to a Clifford module on $\bar{M}$), then D extends to an operator D' on the double $M' = M \cup_{\partial M} M$ of M. All operators we typically consider in applications (e.g., the deRham ($\sqrt{2}$ times), Dolbeault, and Dirac operators) are of Dirac type.

Since we have sacrificed the assumption of completeness of M, the operator D need not be essentially self-adjoint. Extensions of D to unbounded self-adjoint operators on $L^2(S)$ are customarily given by imposing boundary conditions on D. To compensate for the fact that we are considering a class of functions broader than $C_0(M)$, we restrict the class of extensions. We consider self-adjoint extensions $\tilde{D}$ of D satisfying

$$\varphi\,\mathrm{domain}(\tilde{D}) \subset \mathrm{domain}(\tilde{D}), \quad \varphi \in C^\infty(\bar{M}).$$

Such extensions were considered in [BDT], where they were said to be given by "(generalized) local boundary conditions".

THEOREM 3.14. *An extension $\tilde{D}$ of D given by generalized local boundary conditions determines an element of the relative E-theory group $[\tilde{D}] \in E_0(\bar{M}, \partial M)$ by the assignment*

$$\{\mathcal{A}_t^{\tilde{D}}\}\colon f \otimes \varphi \mapsto \varphi f(t^{-1}\tilde{D} + x\varepsilon), \quad f \in C_0(\mathbb{R}), \ \ \varphi \in C(\bar{M}).$$

Further, this class is independent of the choice of extension, and we obtain a uniquely defined class denoted $[D] \in E_0(\bar{M}, \partial M)$.

Proof. Observe that (i) $\varphi \in C^\infty(\bar{M})$ is bounded with bounded gradient and (ii) D has finite propagation speed—the first since φ is the restriction to $\bar{M}$ of a smooth function on M', and the second since D is the restriction to $C_c^\infty(S)$ of D' defined on all of the closed manifold M'.

The proof that we obtain a relative E-theory class is now completely analogous to the proofs of Theorems 3.4 and 3.12. That this class is independent of the choice of extension follows immediately from Proposition 3.6 and the excision isomorphism $E_0(\bar{M}, \partial M) \cong E_0(M)$. □

REMARK. Under the excision isomorphism $E_0(\bar{M}, \partial M) \cong E_0(M)$, the relative E-theory class defined by D corresponds to its E-theory class. Accordingly, the stability properties described in Section 3.1 extend to the relative E-theory class of D. This remark holds for operators on complete Riemannian manifolds considered in Section 3.2, as well as for operators on manifolds with boundary considered in this section.

In order to analyze the image of the class $[D]$ under the boundary morphism $E_0(\bar{M}, \partial M) \to E_{-1}(\partial M)$, we would like to see (as in Section 3.2) that $[f(\tilde{D}), \varphi]$ is compact for $\varphi \in C(\bar{M})$ and $f \in C_0(\mathbb{R})$. Despite the fact that the gradient of φ is no longer assumed to vanish at infinity, we can recover this property for suitable choices of extension $\tilde{D}$ (cf. [BDT]).

The case that is of particular interest for our applications is the self-adjoint extension considered in the remark immediately preceeding the statement of Theorem 3.4,

$$\tilde{D} = \begin{pmatrix} 0 & D_-^{\min} \\ D_+^{\max} & 0 \end{pmatrix}.$$

We have previously observed that multiplication by a smooth bounded function with bounded gradient maps the domain of $\tilde{D}$ into itself (compare to Lemma 3.2). That is, this extension is defined by generalized local boundary conditions.

A self-adjoint operator A has a *punctured gap* in its spectrum if there exists a constant $\gamma > 0$ such that $\mathrm{spec}(A) \cap (-\gamma, \gamma) \subset \{0\}$. The following lemma is the "principle of convergence transfer" [Roe2].

CONVERGENCE TRANSFER. *Let T be a closed unbounded operator. If T^*T is bounded below then* 0 *is isolated in the spectrum of the operator*

$$A = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}.$$

Proof. If T^*T is bounded below then, since it is self-adjoint, there exists a $\gamma > 0$ such that $\mathrm{spec}(T^*T) \subset [\gamma^2, \infty)$. By the polar decomposition for closed unbounded operators, the spectra of TT^* and T^*T agree (except possibly for 0). Hence

$$A^2 = \begin{pmatrix} T^*T & 0 \\ 0 & TT^* \end{pmatrix}$$

has a punctured gap of width γ^2 in its spectrum. By the spectral theorem, A thus has a punctured gap of width γ. □

REMARK. It is similarly proved that if T^*T has compact resolvent then TT^* has compact resolvent on the orthogonal complement of the kernel of T^*. Thus, if one of T^*T or TT^* has compact resolvent then A has compact resolvent on the orthogonal complement of its kernel.

PROPOSITION 3.15. *Let $\tilde{D}$ be the foregoing extension of D to an unbounded self-adjoint operator on $L^2(S)$. Then $\tilde{D}$ has compact resolvent on the orthogonal complement of* $\mathrm{kernel}(\tilde{D}) \cap L^2(S_+) = \mathrm{kernel}(D_+^{\max})$.

Proof. From the basic elliptic estimate for D' [Roe1],

$$\|s\|_{H^1} \leq C(\|s\| + \|D's\|),$$

it follows that the domain of $D_-^{\min}$ is the closure of $C_c^\infty(M, S_-)$ in the Sobolev space $H^1(M', S'_-)$. By the Rellich lemma the inclusion $H^1(M', S'_-) \to L^2(S'_-)$ is compact. Since $L^2(S_-) \subset L^2(S'_-)$ is a closed subspace, we surmise that the inclusion

$$\mathrm{domain}(D_-^{\min}) \to L^2(S_-)$$

is compact.

By a theorem of von Neumann, $D_+^{\max} D_-^{\min}$ is a self-adjoint unbounded operator whose domain is a core for $D_-^{\min}$ [RS; Kat]. The resolvent $r_\pm(D_+^{\max} D_-^{\min})$ is therefore defined and maps $L^2(S_-)$ into $\mathrm{domain}(D_+^{\max} D_-^{\min}) \subset \mathrm{domain}(D_-^{\min})$. Arguing now as in the proof of Lemma 3.5, we conclude that $r_\pm(D_+^{\max} D_-^{\min})$ is compact. That is, $D_+^{\max} D_-^{\min}$ has compact resolvent. In particular, $\mathrm{kernel}(D_-^{\min}) = \mathrm{kernel}(D_+^{\max} D_-^{\min})$ is finite-dimensional.

We calculate

$$\tilde{D}^2 = \begin{pmatrix} D_-^{\min} D_+^{\max} & 0 \\ 0 & D_+^{\max} D_-^{\min} \end{pmatrix}.$$

By the previous remark, $\tilde{D}$ has compact resolvent on the orthogonal complement of its kernel in $L^2(S)$. Since

$$\mathrm{kernel}(\tilde{D}) = \mathrm{kernel}(D_+^{\max}) \oplus \mathrm{kernel}(D_-^{\min})$$

and $\mathrm{kernel}(D_-^{\min})$ is finite-dimensional, the proposition is proved. □

PROPOSITION 3.16. *Let φ be a smooth function on $\bar{M}$, and let $\tilde{D}$ be the above extension of D to an unbounded self-adjoint operator on $L^2(S)$. Then, for $f \in C_0(\mathbb{R})$, the commutator $[f(\tilde{D}), \varphi]$ is compact.*

Proof. By an approximation argument it suffices to consider the case $f = r_\pm$. We analyze the resolvent identity

$$[r_\pm(\tilde{D}), \varphi] = r_\pm(\tilde{D})[\varphi, \tilde{D}] r_\pm(\tilde{D})$$

on the kernel of $D_+^{\max}$ and its orthogonal complement in $L^2(S)$.

On the orthogonal complement of the kernel of $D_+^{\max}$, we see that the commutator $[r_\pm(\tilde{D}), \varphi]$ is the product of

(a) the compact operator $r_\pm(\tilde{D})\colon \mathrm{kernel}(D_+^{\max})^\perp \to L^2(S)$, and
(b) the bounded operator $r_\pm(\tilde{D})[\varphi, \tilde{D}]$ on $L^2(S)$.

Hence, $[r_\pm(\tilde{D}), \varphi]$ is compact on $\mathrm{kernel}(D_+^{\max})^\perp$.

On the kernel of $D_+^{\max}$ in $L^2(S)$, the operator $r_\pm(\tilde{D})$ is multiplication by $\mp\sqrt{-1}$. The commutator $[\varphi, \tilde{D}]$ is odd and hence maps $\mathrm{kernel}(D_+^{\max})$ into $L^2(S_-)$. Thus, the commutator $[r_\pm(\tilde{D}), \varphi]$ is the product of

(a) the bounded operator $[\varphi, \tilde{D}] r_\pm(\tilde{D})\colon \mathrm{kernel}(D_+^{\max}) \to L^2(S_-)$, and
(b) the compact operator $r_\pm(\tilde{D})\colon L^2(S_-) \to L^2(S)$.

Hence, $[r_\pm(\tilde{D}), \varphi]$ is compact on $\mathrm{kernel}(D_+^{\max})$. □

4. The Abstract Boundary Calculation

In this section we outline a general procedure used to calculate the image of a relative E-theory class associated to an elliptic operator under the boundary map. An analogous treatment of a special case is contained in the discussion surrounding Proposition 4.12 of [Gue2].

We begin by recalling, for the convenience of the reader, the definition of the boundary map and related facts we will be using. Complete details may be found in [Gue2].

Let $A \rhd I$ and $B \rhd J$ be pairs of C^*-algebras. A *quotient* of a relative asymptotic morphism $\{\varphi_t\}\colon A \rhd I \to B \rhd J$ is an asymptotic morphism $\{\bar{\varphi}_t\}\colon A/I \to B/J$, making the diagram

$$\begin{array}{ccc} A & \xrightarrow{\{\varphi_t\}} & B \\ \big\downarrow & & \big\downarrow \\ A/I & \xrightarrow{\{\bar{\varphi}_t\}} & B/J \end{array}$$

asymptotically commute. Quotients are defined only up to equivalence (hence the indefinite article). Let $p\colon B \to B/J$ be the projection and let s be any set theoretic section of $A \to A/I$. The simple formula $\bar{\varphi}_t(\bar{a}) = p \circ \varphi_t(s(\bar{a}))$ defines a quotient of $\{\varphi_t\}$. It is easily verified that, as defined by this formula, $\{\bar{\varphi}_t\}$ is (up to equivalence) independent of the choice of s.

A relative asymptotic morphism $\{\varphi_t\}\colon A \rhd I \to B \rhd J$ is *compact* if its quotient $\{\bar{\varphi}_t\}$ is a continuous familiy of $*$-homomorphisms. (This definition is not the same as the one given in [Gue2]; the precise relationship between these definitions is given in Section 7.) Compact relative asymptotic morphisms are called *compact asymptotic morphisms* for short. We will prove later that the relative asymptotic morphism defining the relative E-theory class of an operator as in the previous section is compact.

Let $\mathcal{Q} = \mathcal{B}/\mathcal{K}$ be the Calkin algebra. The boundary map in relative E-theory is the composition of two maps:

$$E^n_{\mathrm{rel}}(A, I) \xrightarrow{\ \alpha\ } [\![A/I, \mathcal{Q}]\!]_n \xrightarrow{\ \beta\ } E^{n+1}(A/I),$$

where α is induced by the quotient construction and β is the connecting map in the second variable for the stable homotopy theory of asymptotic morphisms [Dad; Gue2].

Both α and β are homomorphisms of abelian groups, where the group operation is loop composition. In each case the group operation is also given by "diagonal sum". For the ordinary E-theory group on the right, this is a familiar result. For the relative E-theory group $E^n_{\mathrm{rel}}(A, I)$ and the stable homotopy group $[\![A/I, \mathcal{Q}]\!]_n$, this is proven in [Gue2].

With these preliminaries out of the way, we return to our open Riemannian manifold M and compactification $\bar{M}$. Our methods apply equally to both situations discussed in the previous section. Thus, $\bar{M}$ is either a manifold with boundary ∂M and interior M, or a metric compactification of the complete Riemannian manifold M. Let D be a first-order, elliptic differential operator acting on smooth compactly supported sections of a Hermitian vector bundle S on M. Assume that S is graded, with grading operator ε, and that D is odd with respect to the grading. We consider, as usual, a fixed extension of D to an unbounded self-adjoint operator. With an abuse of notation we denote this extension also by D.

We need to introduce two additional concepts to unify the cases of manifolds with boundary and complete manifolds. The operator D is *commutator compact* if, for $f \in C_0(\mathbb{R})$ and $\varphi \in C(\bar{M})$, the commutator $[f(D), \varphi]$ is compact. We will shortly see that this ensures the compactness of the associated asymptotic morphism. The operator D is *spectrally isolated* if it has a punctured gap in its spectrum, that is, if there exists a $\gamma > 0$ such that $\operatorname{spec}(D) \cap (-\gamma, \gamma) \subset \{0\}$.

We make the additional assumption that D is commutator compact and spectrally isolated.

REMARK. If $\bar{M}$ is a manifold with boundary and if D is a first-order, elliptic differential operator with initial domain the space of smooth sections, compactly supported on the interior M of $\bar{M}$, then the self-adjoint extension

$$\tilde{D} = \begin{pmatrix} 0 & D_-^{\min} \\ D_+^{\max} & 0 \end{pmatrix}$$

is a commutator compact and spectrally isolated by Propositions 3.16 and 3.15, respectively.

REMARK. If D is an operator with finite propogation on the complete Riemannian manifold M, and if $\bar{M}$ is a metric compactification of M, then by Proposition 3.13 the closure of D is commutator compact. In this case we must verify by hand that D is spectrally isolated (after perhaps making additional assumptions).

If D is commutator compact and spectrally isolated then we may define Toeplitz operators on the kernel of D. For $\varphi \in C(\bar{M})$, the Toeplitz operator T_φ with symbol φ is defined as the composition

$$\operatorname{kernel}(D) \xrightarrow{\text{multiply by } \varphi} L^2(S) \xrightarrow{\text{project}} \operatorname{kernel}(D).$$

In a similar manner we define Toeplitz operators $T_\varphi^\pm$ on the kernels of $D_\pm$. These are related by

$$T_\varphi = \begin{pmatrix} T_\varphi^+ & 0 \\ 0 & T_\varphi^- \end{pmatrix}.$$

PROPOSITION 4.1. *The assignments $\varphi \mapsto T_\varphi^\pm$ define $*$-homomorphisms from $C(\bar{M})$ to the Calkin algebras $\mathcal{Q}(\operatorname{kernel}(D_\pm))$. These pass to $*$-homomorphisms*

$$\mathfrak{T}^\pm \colon C(\partial M) \to \mathcal{Q}(\operatorname{kernel}(D_\pm)).$$

Proof. It suffices to prove the analogous statements for T_φ. Since

$$T_\varphi T_\psi = T_{\varphi\psi} - P[P, \varphi]\psi, \quad \varphi, \psi \in C(\bar{M}),$$

it suffices to show that the commutator $[P, \varphi]$ is compact for $\varphi \in C(\bar{M})$. But if $f \in C_0(\mathbb{R})$ is such that $f(0) = 1$ and f is identically zero on $\operatorname{spec}(D) \setminus \{0\}$ then we have $P = f(D)$ and, by our assumptions on D, $[f(D), \varphi]$ is compact.

The remainder of the proposition follows because, for $\varphi \in C_0(M)$, the operator $\varphi f(D)$ is compact and $C(\partial M) \cong C(\bar{M})/C_0(M)$. □

LEMMA 4.2. *For $\varphi \in C(\bar{M})$, the operators $(1-P)\varphi P$ and $P\varphi(1-P)$ are compact.*

Proof. As observed in the proof of the previous proposition, the commutator $[P, \varphi]$ is a compact operator on $L^2(S)$. The result now follows from

$$(1-P)\varphi P = (1-P)P\varphi - (1-P)[P,\varphi] = (P-1)[P,\varphi]. \qquad \square$$

The Toeplitz extensions $\mathfrak{T}^{\pm}\colon C(\partial M) \to \mathcal{Q}(\mathrm{kernel}(D_{\pm}))$ determine E-theory classes

$$[\mathfrak{T}^{\pm}] \in E_{-1}(\partial M)$$

that admit two distinct descriptions. To describe these we recall that Connes and Higson associate to a short exact sequence $0 \to I \to A \to A/I \to 0$ of separable C^*-algebras a unique homotopy class of asymptotic morphisms $\mathcal{S}(A/I) \to I$ [CH2].

The first description is as the homotopy class of the asymptotic morphism associated to the pulled-back extension of C^*-algebras (we suppress the kernel$(D_{\pm})$ for notational convenience):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{E}_{\pm} & \longrightarrow & C(\partial M) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \mathfrak{T}^{\pm}} & & \\ 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{B} & \longrightarrow & \mathcal{Q} & \longrightarrow & 0. \end{array}$$

Note that the separability of $C(\partial M)$ implies that of $\mathcal{E}_{\pm}$. The second description is as the compositions

$$C_0(0,1) \otimes C(\partial M) \xrightarrow{1 \otimes \mathfrak{T}^{\pm}} C_0(0,1) \otimes \mathcal{E}'_{\pm}/\mathcal{K} \longrightarrow \mathcal{K},$$

where $\mathcal{E}'_{\pm}$ is the image in $\mathcal{B}(\mathrm{kernel}(D_{\pm}))$ of $\mathcal{E}_{\pm}$ and the second map is an asymptotic morphism associated as before to a short exact sequence. Notice that $\mathcal{E}'_{\pm}$ is the separable subalgebra of $\mathcal{B}(\mathrm{kernel}(D_{\pm}))$ generated by $\mathcal{K}(\mathrm{kernel}(D_{\pm}))$ and the Toeplitz operators $T^{\pm}_{\tilde{\varphi}}$, $\varphi \in C(\partial M)$.

The equivalence of these descriptions is not hard to establish and embodies the naturality of the construction of Connes and Higson with respect to pullbacks. For details compare Theorem 10 of [Dad] or the discussion surrounding Proposition 4.12 of [Gue2]. We shall use the second description.

PROPOSITION 4.3. *Let D be commutator compact and spectrally isolated. Under the boundary map $E_0(\bar{M}, \partial M) \to E_{-1}(\partial M)$, the class of D maps to the difference $[\mathfrak{T}^+] - [\mathfrak{T}^-]$.*

We prepare for the proof of this proposition with a few results.

LEMMA 4.4. *Let T be an unbounded self-adjoint operator and let A be a bounded self-adjoint operator such that $\|A\| < 1$. If B is a bounded operator such that the commutators $[(T+\sqrt{-1})^{-1}, B]$ and $[A, B]$ are compact, then the commutator $[(T+A+\sqrt{-1})^{-1}, B]$ is compact.*

Proof. Begin by noting that the operator $T + A$ is self-adjoint on domain(T) so that $T + A + \sqrt{-1}$ is surely invertible. However, by our assumptions on A, there is an equality of bounded operators

$$(T + A + \sqrt{-1})^{-1} = (T + \sqrt{-1})^{-1} \sum_{n=0}^{\infty} \big(A(T + \sqrt{-1})^{-1}\big)^n$$

(the series converges in the operator norm topology). It follows from the identity

$$[A(T + \sqrt{-1})^{-1}, B] = A[(T + \sqrt{-1})^{-1}, B] + [A, B](T + \sqrt{-1})^{-1}$$

that the commutator of B with $A(T + \sqrt{-1})^{-1}$ is compact, and by induction on n that the commutator of B with each term of the series is compact. Hence the commutator of B with $(T + A + \sqrt{-1})^{-1}$ is compact. □

PROPOSITION 4.5. *Let D be commutator compact. Then, for $f \in C_0(\mathbb{R})$ and $\varphi \in C(\bar{M})$, the commutator $[f(t^{-1}D+x\varepsilon), \varphi]$ is compact. Thus, the relative E-theory class of D is represented by a compact asymptotic morphism.*

Proof. By an approximation argument it suffices to consider the case where $f = r_\pm$ and φ is a smooth function.

We begin by considering the case $x = 0$. The commutator $[r_+(t^{-1}D), \varphi]$ is compact for $t = 1$. We calculate

$$r_+(t^{-1}D) = ts^{-1} r_+\big(s^{-1}D + (ts^{-1} - 1)\sqrt{-1}\big)^{-1}.$$

By the previous lemma, if $[r_+(s^{-1}D), \varphi]$ is compact and if $0 < t < 2s$, so that $|ts^{-1} - 1| < 1$, then $[r_+(t^{-1}D), \varphi]$ is compact. It follows easily that $[r_+(t^{-1}D), \varphi]$ is compact for all $t \geq 1$.

For general x note that it follows, again from the previous lemma, that if $|x - y| < 1$ and $[r_+(t^{-1}D + y\varepsilon), \varphi]$ is compact then so is $[r_+(t^{-1}D + x\varepsilon), \varphi]$. The final assertion follows from the definitions and Lemma 7.2. □

Proof of Proposition 4.3. The boundary map is the composition

$$E_0(\bar{M}, \partial M) \xrightarrow{\alpha} [\![C(\partial M), \mathcal{Q}]\!]_0 \xrightarrow{\beta} E_{-1}(\partial M).$$

The image of $[D]$ under the first of these maps is represented by the asymptotic morphism $\mathcal{S}C(\partial M) \to \mathcal{S}\mathcal{Q}$;

$$f \otimes \varphi \mapsto p(f(t^{-1}D + x\varepsilon)\tilde{\varphi}), \quad \varphi \in C(\partial M),\ f \in C_0(\mathbb{R}),$$

where $\tilde{\varphi} \in C(\bar{M})$ is any continuous extension of φ to all of $\bar{M}$ and p is the quotient map $\mathcal{B} \to \mathcal{Q}$. By the previous proposition, this asymptotic morphism is in fact a continuous family of $*$-homomorphisms and hence homotopic to the constant asymptotic morphism obtained by setting $t = 1$:

$$\alpha[D] = [\![f \otimes \varphi \mapsto p(f(D + x\varepsilon)\tilde{\varphi})]\!] \in [\![C(\partial M), \mathcal{Q}]\!]_0. \tag{6}$$

By definition, $[\mathfrak{T}^\pm] = \beta[\![\mathfrak{T}^\pm]\!]$, where $[\![\mathfrak{T}^\pm]\!]$ is the class of the $*$-homomorphism $\mathfrak{T}^\pm$ in $[\![C(\partial M), \mathcal{Q}]\!]_0$. (The map β is induced by composition with asymptotic morphisms $\mathcal{S}(\mathcal{E}/\mathcal{K}) \to \mathcal{K}$ associated to separable subalgebras $\mathcal{K} \subset E \subset \mathcal{B}$.) Using the various descriptions of addition in this group, we see that

$$\begin{aligned}
[\![\mathfrak{T}^+]\!] - [\![\mathfrak{T}^-]\!] &= \left[\!\!\left[\varphi \mapsto \begin{pmatrix} T^+_{\tilde\varphi} & 0 \\ 0 & 0 \end{pmatrix}\right]\!\!\right] - \left[\!\!\left[\varphi \mapsto \begin{pmatrix} T^-_{\tilde\varphi} & 0 \\ 0 & 0 \end{pmatrix}\right]\!\!\right] \\
&= \left[\!\!\left[f \otimes \varphi \mapsto \begin{pmatrix} f(x)T^+_{\tilde\varphi} & 0 \\ 0 & 0 \end{pmatrix}\right]\!\!\right] \\
&\quad + \left[\!\!\left[f \otimes \varphi \mapsto \begin{pmatrix} f(-x)T^-_{\tilde\varphi} & 0 \\ 0 & 0 \end{pmatrix}\right]\!\!\right] \\
&= \left[\!\!\left[f \otimes \varphi \mapsto \begin{pmatrix} f(x\varepsilon)T_{\tilde\varphi} & 0 \\ 0 & 0 \end{pmatrix}\right]\!\!\right]. \qquad (7)
\end{aligned}$$

The matrices in this equation are written with respect to the orthogonal decomposition $L^2(S) = \mathrm{kernel}(D) \oplus \mathrm{kernel}(D)^\perp$ or to a similar decomposition for $L^2(S_\pm)$ as appropriate. Note that since D is odd with respect to the grading, the operator ε maps the kernel of D into itself.

We conclude the proof by constructing an explicit homotopy of asymptotic morphisms from (6) to (7). A homotopy of asymptotic morphisms will be determined by a homotopy of $*$-homomorphisms. An explicit homotopy of $*$-homomorphisms is given by

$$f \otimes \varphi \mapsto \begin{cases} p(f(s^{-1}D + x\varepsilon)\tilde\varphi), & s > 0, \\ p\begin{pmatrix} f(x\varepsilon)T_{\tilde\varphi} & 0 \\ 0 & 0 \end{pmatrix}, & s = 0. \end{cases}$$

It follows (as in Proposition 4.5) that for each $s > 0$ we obtain a $*$-homomorphism and (from Proposition 4.1) that for $s = 0$ we obtain a $*$-homomorphism. Further, as was shown in the proof of Theorem 3.4, the family $\{f(s^{-1}D + x\varepsilon)\}$ is continuous in s for $s > 0$. It remains only to check continuity at $s = 0$. That is, it remains to show, for $f \in C_0(\mathbb{R})$ and $\tilde\varphi \in C(\bar M)$, that

$$p(f(s^{-1}D + x\varepsilon)\tilde\varphi) \to p\begin{pmatrix} f(x\varepsilon)T_{\tilde\varphi} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{as } s \to 0. \qquad (8)$$

To do this write $s^{-1}D + x\varepsilon$ and $\tilde\varphi$ as 2×2 matrices with respect to the decomposition $L^2(S) = \mathrm{kernel}(D) \oplus \mathrm{kernel}(D)^\perp$. For the first we have

$$s^{-1}D + x\varepsilon = \begin{pmatrix} x\varepsilon & 0 \\ 0 & T(s,x) \end{pmatrix},$$

where $T(s,x)$ is an unbounded self-adjoint operator. By virtue of the identity

$$(s^{-1}D + x\varepsilon)^2 = s^{-2}D^2 + x^2 \geq s^{-2}D^2,$$

$T(s,x)$ is bounded below by $s^{-1}\gamma$, where γ is the width of the gap in the spectrum of D. It follows that

$$\|f(T(s,x))\| \leq \sup\{\,|f(y)| : |y| \geq s^{-1}\gamma\,\} \to 0 \quad \text{as } s \to 0$$

and hence that

$$f(s^{-1}D + x\varepsilon) = \begin{pmatrix} f(x\varepsilon) & 0 \\ 0 & f(T(s,x)) \end{pmatrix} \to \begin{pmatrix} f(x\varepsilon) & 0 \\ 0 & 0 \end{pmatrix} \quad \text{as } s \to 0. \qquad (9)$$

For the second we have (as operators) using Lemma 4.2

$$\tilde{\varphi} = \begin{pmatrix} T_{\tilde{\varphi}} & P\tilde{\varphi}(1-P) \\ (1-P)\tilde{\varphi}P & (1-P)\tilde{\varphi}(1-P) \end{pmatrix} \underset{\operatorname{mod}\mathcal{K}}{\sim} \begin{pmatrix} T_{\tilde{\varphi}} & 0 \\ 0 & (1-P)\tilde{\varphi}(1-P) \end{pmatrix}. \quad (10)$$

The proof is concluded by noting that (8) follows immediately from (9) and (10). □

5. Strongly Pseudoconvex Domains

In this section we present a new approach to one of the results of Baum, Douglas, and Taylor. Let Ω be a strongly pseudoconvex domain with smooth boundary. Our goal is to prove the identity

$$\partial[D] = [\mathfrak{T}] \in E_{-1}(\partial\Omega).$$

Here, $[\mathfrak{T}]$ is the E-theory class associated to the Toeplitz extension on the Bergman space of square integrable holomorphic functions on Ω, and $[D]$ is the E-theory class associated to the Dolbeault operator of Ω.

This result was obtained by Baum, Douglas, and Taylor in the setting of relative K-homology theory. The contrast between their methods and those employed in this section is interesting. In their calculations it is necessary to find the appropriate local boundary conditions for an operator on a manifold with boundary, here the $\bar{\partial}$-Neumann conditions. We, however, trade these analytic aspects for geometric ones—namely, finding an appropriate vanishing theorem for an operator on a complete manifold. We note that the approach adopted here is easily modified to fit in their framework.

Our method is rather straightforward; we check that the Dolbeault operator D, constructed with respect to an appropriate complete Kähler metric on the strongly pseudoconvex domain Ω, is spectrally isolated. This completes the assumptions of Section 4 and enables us to apply the abstract boundary calculation of that section to obtain the desired identity.

Let Ω be a bounded domain in $\mathbb{C}^n$ with smooth boundary. A *defining function* r for Ω is a smooth, real-valued function on $\mathbb{C}^n$ such that $\Omega = \{ r(p) > 0 \}$ and such that $\operatorname{grad}(r)$ is nowhere vanishing on $\partial\Omega$. (Our conventions for defining functions are *not* the same in [FS; Kra]; they do however, agree with those in [GH] to which we will refer frequently.) Observe that $\partial\Omega = \{ r(p) = 0 \}$. The domain Ω is *strongly pseudoconvex* if it has a defining function r such that, for all $a \in \mathbb{C}^n$,

$$a \neq 0 \text{ and } \sum_i a_i \frac{\partial r}{\partial z_i} = 0 \implies \sum_{i,j} \frac{\partial^2 r}{\partial z_i \partial \bar{z}_j} a_i \bar{a}_j < 0 \quad \text{for } p \in \partial\Omega.$$

This condition depends only on the domain Ω and not on the particular choice of defining function [FS]. It is always possible [FS] to modify the defining function so that, for all $a \in \mathbb{C}^n$,

$$a \neq 0 \implies \sum_{i,j} \frac{\partial^2 r}{\partial z_i \partial \bar{z}_j} a_i \bar{a}_j < 0 \quad \text{for } p \in \bar{\Omega}.$$

Note that this condition holds not only on the boundary of Ω, but at each point of $\bar{\Omega}$. From now on we shall assume that our defining functions satisfy this condition.

PROPOSITION 5.1. *The form*

$$\sum_{ij} h_{ij} dz_i \otimes d\bar{z}_j = -\sum_{ij} \frac{\partial^2 \log(r)}{\partial z_i \partial \bar{z}_j} dz_i \otimes d\bar{z}_j$$

defines a Hermitian metric on Ω. The real part of h_{ij} is a complete Riemannian metric on Ω. Furthermore, $\bar{\Omega}$ is a metric compactification of the complete Riemannian manifold Ω.

Proof. Lemmas 1 and 2 of [GH, Sec. 1]. See also the paper of Donnelly [Don]. □

From now on we will consider the strongly pseudoconvex domain Ω to be equipped with the Hermitian metric defined in this proposition.

We recall the construction of the Dolbeault operator of the Hermitian manifold Ω. According to the decomposition of the complexified cotangent bundle,

$$T_{\mathbb{C}}\Omega = \Lambda^{1,0}\Omega \oplus \Lambda^{0,1}\Omega,$$

there is a decomposition of the exterior algebra bundle and hence of the spaces of smooth compactly supported complex-valued n-forms on Ω:

$$A^n \cong \bigoplus_{p+q=n} A^{p,q},$$

where $A^{p,q}$ is the space of smooth compactly supported forms of type (p,q). Having specified a Hermitian metric on Ω (and so a Riemannian metric and orientation), we see that the space $A^{p,q}$ has a natural inner product. Denote by $\mathcal{A}_h^{p,q}$ the Hilbert space completion. The formal adjoint of $\bar{\partial}$ is denoted $\bar{\partial}^*$. The Dolbeault operator of Ω is

$$\bar{\partial} + \bar{\partial}^* \colon \bigoplus_{q \text{ even}} A^{0,q} \to \bigoplus_{q \text{ odd}} A^{0,q},$$
$$\bar{\partial} + \bar{\partial}^* \colon \bigoplus_{q \text{ odd}} A^{0,q} \to \bigoplus_{q \text{ even}} A^{0,q}.$$

We are interested in the *twisted* Dolbeault operator,

$$D_+ = \bar{\partial} + \bar{\partial}^* \colon \bigoplus_{q \text{ even}} A^{n,q} \to \bigoplus_{q \text{ odd}} A^{n,q},$$
$$D_- = \bar{\partial} + \bar{\partial}^* \colon \bigoplus_{q \text{ odd}} A^{n,q} \to \bigoplus_{q \text{ even}} A^{n,q};$$

as usual, we write

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}.$$

It is well known that the (twisted) Dolbeault operator is a Dirac-type operator, meaning that it is associated to a Clifford module [Roe1; BGV; Gil]. As such, it has finite propagation speed. We are thus in the setup of Section 3.2, and the twisted Dolbeault operator has a class in the relative E-theory group $E_0(\bar{\Omega}, \partial\Omega)$. We shall denote this class, with a slight abuse of notation, by

$$[\bar{\partial}] \in E_0(\bar{\Omega}, \partial\Omega).$$

REMARK. The twisted Dolbeault operator is nothing other than the classical Dolbeault operator with coefficients in the canonical line bundle $\Lambda^{n,0}\Omega$. This line bundle is of course topologically trivial, and its presence amounts to a change in the Hermitian structure of the vector bundle on which D acts. Thus, by the results of Section 3.1, the Dolbeault and twisted Dolbeault operators determine the same relative E-theory class.

The following proposition completes the verification that the twisted Dolbeault operator satisfies the hypotheses of Section 4 [GH; Don].

PROPOSITION 5.2. *Zero is isolated in the spectrum of the twisted Dolbeault operator. Furthermore, its kernel consists of forms of type* $(n, 0)$:

$$\operatorname{kernel}(D) = \operatorname{kernel}(D_+) \subset \mathcal{A}_h^{n,0}.$$

Proof. This follows from Proposition 2 of Section 2 of [GH] as discussed in Section 3 of that paper. See also [Don]. □

The final ingredient necessary to apply the generalized boundary calculation of Section 4 and so obtain our result is the following lemma [GH].

LEMMA 5.3. *The assignment*

$$\varphi \mapsto \varphi 2^{-n/2} dz_1 \ldots dz_n$$

extends to a unitary isomorphism from $L^2(\Omega)$ *(computed using the usual Lebesgue measure on* Ω*) to* $\mathcal{A}_h^{n,0}$. *It maps the Bergman space* $B(\Omega)$ *of holomorphic* L^2*-functions to the space of holomorphic* $(n, 0)$*-forms in* $\mathcal{A}_h^{n,0}$.

Proof. The proof is an exercise in the conventions for the metrics. □

THEOREM 5.4. *The image of the class* $[\bar{\partial}]$ *under the boundary map* $E_0(\bar{\Omega}, \partial\Omega) \to E_{-1}(\partial\Omega)$ *is the class of the Toeplitz extension* $\mathfrak{T}$ *on the Bergman space*:

$$\partial[\bar{\partial}] = [\mathfrak{T}] \in E_{-1}(\partial\Omega).$$

Proof. By the general boundary calculation in Proposition 4.3, the image of $[\bar{\partial}]$ is the class of the Toeplitz extension on the kernel of the twisted Dolbeault operator. By the previous theorem this is the space of holomorphic forms in $\mathcal{A}_h^{n,0}$, which the previous lemma identifies with the Bergman space $B(\Omega)$. Furthermore, this identification actually identifies by conjugation the Toeplitz extensions on the respective spaces. □

REMARK. It is not necessary to prove the existence of the Toeplitz extension on the Bergman space, as this follows from the discussion of Section 4. In fact, the results there, combined with Lemma 5.3, show that for $\varphi, \psi \in C(\bar{\Omega})$ we have

$$T_\varphi T_\psi - T_{\varphi\psi} \in \mathcal{K},$$

where T_φ is the Toeplitz operator

$$B(\Omega) \xrightarrow{\text{multiply by } \varphi} L^2(\Omega) \xrightarrow{\text{project}} B(\Omega).$$

A direct proof of this fact, without appeal to the Dolbeault operator, would consist of identifying the Toeplitz operators as integral operators and carrying out the delicate analysis of the integral kernel on the domain Ω. We have carried out a similar analysis for Toeplitz operators on the Fock space [Gue3]. The analysis is accessible—although already somewhat more complicated for the Poincaré disk—and relies on the methods of Jovović [Jov]. In general, it appears to be quite difficult since it involves the delicate analysis of the Bergman kernel of Ω.

6. Remarks on the Results of Baum–Douglas–Taylor

In this short section we collect a few remarks relating our results to those obtained by Baum, Douglas, and Taylor (hereafter "BDT") in the setting of relative K-homology [BD2; BDT]. In particular, we explain how their results can be duplicated in the setting of relative E-theory by virtue of the abstract boundary calculation of Proposition 4.3 and the groundwork done in Section 3.3 associating relative E-theory classes to operators on manifolds with boundary.

We begin by recalling that a cycle for the relative K-homology group $K_0(\bar{M}, \partial M)$ consists of a graded Hilbert space $H = H_+ \oplus H_-$, together with a representation $C(\bar{M}) \to \mathcal{B}(H)$ as even operators, and a bounded operator $T : H_+ \to H_-$ that satisfies the following assumptions:

(a) T has closed range, and is a partial isometry plus a compact operator;
(b) $\varphi T - T\varphi \in \mathcal{K}(H_+, H_-)$ for all $\varphi \in C(\bar{M})$; and
(c) $\varphi P_\pm \in \mathcal{K}(H_\pm)$ for all $\varphi \in C_0(M)$, where P_+ is the projection onto kernel(T) and P_- that onto kernel(T^*).

The image of $[T] \in K_0(\bar{M}\partial M)$ under the boundary map $K_0(\bar{M}, \partial M) \to K_{-1}(\partial M)$ is the difference, $[\tau_+] - [\tau_-] \in K_{-1}(\partial M)$ of the Toeplitz extensions on the kernels of T and T^*.

Let D_+ be a first-order, elliptic differential operator on a manifold $\bar{M}$ with boundary. Let $\tilde{D}_+$ be an extension of D_+ to a closed unbounded operator satisfying these two conditions:

(i) multiplication by $\varphi \in C_c^\infty(\bar{M})$ preserves domain$(\tilde{D}_+)$; and
(ii) either $\tilde{D}_+\tilde{D}_+^*$ or $\tilde{D}_+^*\tilde{D}_+$ has compact resolvent.

(In the notation of [BDT], our D_+ is D and our $\tilde{D}_+$ is D_B.) In their paper BDT prove that $T = \tilde{D}_+(\tilde{D}_+^*\tilde{D}_+ + 1)^{-1}$ determines an element $[\tilde{D}_+] \in K_0(\bar{M}, \partial M)$. Assume for definiteness that $\tilde{D}_+\tilde{D}_+^*$ has compact resolvent. Then the image of $[D_+]$ under the boundary map is $[\text{kernel}(\tilde{D}_+)] \in K_{-1}(\partial M)$, the Toeplitz extension on the kernel of $\tilde{D}_+$:

$$\partial[\tilde{D}_+] = [\text{Toeplitz extension on kernel}(\tilde{D}_+)] \in K_{-1}(\partial M).$$

We recast these results in the framework of relative E-theory. Begin by observing that assumption (i) shows that, for $\varphi \in C^\infty(\bar{M})$, the commutator $[\varphi, \tilde{D}_+]$ is defined. Arguments as in Section 3 show that it extends to a bounded operator and that multiplication by φ preserves the domain of $\tilde{D}_+^*$. Thus, we can apply the formula of Theorem 3.14 to see that $A = \begin{pmatrix} 0 & \tilde{D}_+^* \\ \tilde{D}_+ & 0 \end{pmatrix}$ determines an element of the relative E-theory $[\tilde{D}_+] \in E_0(\bar{M}, \partial M)$.

Assume for definiteness that $\tilde{D}_+ \tilde{D}_+^*$ has compact resolvent. Then the arguments of Section 3.3 carry over verbatim to show that A is spectrally isolated and commutator compact. Thus the abstract boundary calculation applies. Since $\tilde{D}_+ \tilde{D}_+^*$ has compact resolvent, the kernel of D_+^* is finite-dimensional and contributes nothing to the boundary calculation; hence we obtain

$$\partial[\tilde{D}_+] = [\text{Toeplitz extension on kernel}(\tilde{D}_+)] \in E_{-1}(\partial M).$$

We close this section by recalling two results from BDT, which are recast into the framework of relative E-theory. The first is the calculation of the previous section, but approached from the point of view of a manifold with boundary.

PROPOSITION 6.1. *The boundary of the Dolbeault operator of a strongly pseudoconvex domain* (*now computed with respect to its usual Euclidean structure*) *is the Toeplitz extension on the Bergman space.*

Proof. This is Proposition 4.5 of [BDT]. The self-adjoint extension used to define the relative E-theory element is the one given by $\bar{\partial}$-Neumann conditions. This operator satisfies the assumptions of BDT, and hence also our assumptions. □

REMARK. Notice that this calculation follows directly from our results. The stability results for the E-theory and relative E-theory class of an operator show that the class of the Dolbeault operator is independent of the self-adjoint extension, or even the metric on the underlying manifold.

PROPOSITION 6.2. *The image under the boundary map of the Dirac operator of a* spin^c*-manifold M with boundary is the Dirac operator of the boundary.* (For an explanation of the terminology of Dirac operators and spin^c-manifolds, see [BD2].)

Proof. This is Proposition 4.4 of [BDT] and surrounding discussion. The appropriate self-adjoint extension of the Dirac operator D of M is the extension described in Section 3.3. By the previous discussion we obtain

$$\partial[D] = [\text{Toeplitz extension on kernel}(D_+^{\max})] \in E_{-1}(\partial M).$$

Following BDT, the kernel of $D_+^{\max}$ identifies, up to a finite-dimensional subspace, with the positive spectral space of D_∂, the Dirac operator of the boundary. Further, by conjugation this identification yields an equivalence of *Toeplitz extensions*

$$[\text{kernel}(D_+^{\max})] = [\text{positive spectral space of } D_\partial] \in K_{-1}(\partial).$$

Our result now follows from the fact that unitarily equivalent extensions determine the same element in E-theory. We conclude that

$$\partial[D] = [\text{Toeplitz extension on positive spectral space of } D_\partial] \in E_{-1}(\partial M). \qquad \square$$

REMARK. Another proof of this last result in E-theory is possible, and proceeds along the lines of an earlier proof of Higson [Hig2]. This approach has a more topological and less analytic feel than the one taken here. It was adopted in [Gue1].

7. Appendix

We present a few technical lemmas about relative asymptotic morphisms. They should be viewed as supplements to the material on relative asymptotic morphisms presented in [Gue2].

Recall once again that all C^*-algebras in this paper are assumed separable. Let $C_b(T, B)$ denote the C^*-algebra of continuous B-valued bounded functions on $T = [1, \infty)$ and let $B_\infty = C_b(T, B)/B_\infty$. The set of equivalence classes of asymptotic morphisms $A \to B$ is in bijective correspondence with the set of $*$-homomorphisms $A \to B_\infty$. Under this correspondence, the classes of relative asymptotic morphisms $A \rhd I \to B \rhd J$ correspond to the $*$-homomorphisms of pairs $A \rhd I \to B_\infty \rhd J_\infty$. Denote by $\hat{\varphi}\colon A \to B_\infty$ the $*$-homomorphism defined by $\{\varphi_t\}\colon A \to B$.

LEMMA 7.1. *Let A be a nuclear C^*-algebra, and let $B \rhd J$ and $C \rhd K$ be pairs of C^*-algebras. Let $\{\varphi\}\colon A \to C$ and $\{\psi\}\colon B \to C$ be asymptotic morphisms such that, for all $a \in A$,*

(i) $[\varphi_t(a), \psi_t(b)] \to 0$ *for all $b \in B$, and*
(ii) $\varphi_t(a)\psi_t(b) \in K$ *for all $b \in J$.*

Then there exists a relative asymptotic morphism $\{\theta_t\}\colon A \otimes (B \rhd J) \to C \rhd K$ such that

$$\theta_t(a \otimes b) - \varphi_t(a)\psi_t(b) \to 0 \quad \textit{for all } \ a \in A,\ b \in B.$$

Proof. The proof is a simple adaptation of that of Lemma 5 of [CH2], once one is familiar with the notation. Denote by $\hat{\varphi}\colon A \to C_\infty$ the $*$-homomorphism associated to $\{\varphi_t\}$ and likewise for $\hat{\psi}$. By the universal property of the tensor product and our first assumption, the linear map

$$a \odot b \mapsto \hat{\varphi}(a)\hat{\psi}(b) : A \odot B \to C_\infty$$

extends uniquely to a $*$-homomorphism $A \otimes B \to C_\infty$. Further, the composition $A \otimes J \hookrightarrow A \otimes B \to C_\infty$ is the unique extension of $A \odot J \to K_\infty$ and hence maps into K_∞. Any relative asymptotic morphism $\{\theta_t\}$ associated to the resulting $*$-homomorphism of pairs $A \otimes (B \rhd J) \to C_\infty \rhd K_\infty$ satisfies our requirements. $\square$

REMARK. The nuclearity of A is used to ensure that $A \otimes J$ is an ideal of $A \otimes B$. In the proof of the next lemma we also use the fact (see [WO]) that $A \otimes B/A \otimes J \cong$

$A \otimes (B/J)$. The universal property is that of the maximal tensor product, although by the nuclearity of A the maximal and minimal tensor products coincide.

In the proof of the following lemma, we use the simple fact that an asymptotic morphism $\{\varphi_t\}\colon A \to B$ is equivalent to a continuous family of $*$-homomorphisms if and only if the associated $*$-homomorphism $\hat{\varphi}\colon A \to B_\infty$ lifts to a $*$-homomorphism $\tilde{\varphi}\colon A \to C_b(T, B)$.

LEMMA 7.2. *Let A, $B \rhd J$ and $C \rhd K$ be as in the previous lemma. Let $\{\varphi_t\}\colon A \to C$ and $\{\psi_t\}\colon B \to C$ be continuous families of $*$-homomorphisms satisfying the conditions of the previous lemma and the further condition*

$$[\varphi_t(a), \psi_t(b)] \in K \quad \textit{for all } a \in A \textit{ and } b \in B.$$

Then there exists a compact asymptotic morphism $\{\theta_t\}\colon A \otimes (B \rhd J) \to C \rhd K$ satisfying the conclusion of the previous lemma.

Proof. Let $p\colon C \to C/K$ be the projection. The assignment

$$a \odot b \mapsto p(\tilde{\varphi}(a)\tilde{\psi}(b)) : A \odot B \to C_b(T, C/K) \quad \text{for all } a \in A,\ b \in B$$

extends to a unique $*$-homomorphism α. Arguing as in the proof of the previous lemma, we see that α maps $A \otimes J \to 0$. Thus α descends uniquely to a $*$-homomorphism $\bar{\alpha}\colon A \otimes (B/J) \to C_b(T, C/K)$.

Let $\{\theta_t\}$ be as in the previous lemma, with quotient asymptotic morphism $\{\bar{\theta}_t\}$. Let $\pi : C_b(T, C/K) \to (C/K)_\infty$ be the projection. The proof is completed by showing that $\pi \circ \bar{\alpha} = \hat{\bar{\theta}}$. Notice that $p \circ \hat{\theta}\colon A \otimes B \to (C/K)_\infty$ maps $A \otimes J \to 0$ and, by definition, $\hat{\bar{\theta}}$ is its unique factorization to a $*$-homomorphism $A \otimes (B/J) \to (C/K)_\infty$. It therefore suffices to show $\pi \circ \alpha = p \circ \hat{\theta}$, which follows from the functoriality of π with respect to $*$-homomorphisms. $\square$

We close this appendix with a lemma relating the definition of compact asymptotic morphisms given in this paper to that given in [Gue2]. For the purposes of this lemma, we use the term *strongly compact* for a relative asymptotic morphism $\{\varphi_t\}\colon A \rhd I \to B \rhd J$ for which

$$\varphi_t(aa') - \varphi_t(a)\varphi_t(a') \in J, \qquad \varphi_t(a^*) - \varphi_t(a)^* \in J,$$
$$\text{and} \quad \varphi_t(a + \lambda a') - \varphi_t(a) - \lambda\varphi_t(a') \in J$$

for all $a, a' \in A$, $\lambda \in \mathbb{C}$, and $t \geq 1$. This is the definition from [Gue2].

LEMMA 7.3. *A relative asymptotic morphism is equivalent to a strongly compact asymptotic morphism if and only if it is compact.*

Proof. Clearly, a strongly compact asymptotic morphism is compact. Further, since the quotient construction is well-defined on equivalence classes, a relative asymptotic morphism that is equivalent to a compact asymptotic morphism is also compact. This proves one implication.

It remains to show that a compact asymptotic morphism is equivalent to a strongly compact one. Let $\{\varphi_t\}$ be a compact asymptotic morphism. By definition this means that the $\{\bar{\varphi}_t\}$ is equivalent to a continuous family of $*$-homomorphisms, $\{\alpha_t\}$. Let $q\colon A \to A/I$ be the projection so that $\{\alpha_t \circ q\}$ is a continuous family of $*$-homomorphisms $A \to B/J$, each mapping $I \to 0$. By definition of the quotient, $\{p \circ \varphi_t\}$ is equivalent to $\{\bar{\varphi}_t \circ q\}$, which is equivalent to $\{\alpha_t \circ q\}$.

Let $\{\beta_t\} = \{p \circ \varphi_t - \alpha_t \circ q\}$ so that $\{\beta_t\}\colon A \to B/J[1, \infty)$. Finally define $\{\psi_t\} = \{\varphi_t - s \circ \beta_t\}$, where $s\colon B/J \to B$ is a continuous section of p for which $s(0) = 0$. Since $\{\psi_t\}\colon A \to C_b(T, B)$ and is equivalent to $\{\varphi_t\}$, it is an asymptotic morphism. Because each $\alpha_t \circ q$ maps $I \to 0$ and $\{\varphi_t\}$ is a relative asymptotic morphism, so is $\{\psi_t\}$. One easily checks that $\{\psi_t\}$ is strongly compact. $\square$

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