Isomorphic Classification of Cartesian Products of Power Series Spaces

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Introduction

Here we obtain a complete isomorphic classification of the Cartesian products of the kind $E_0(a) \times E_\infty(b)$, where $E_0(a)$ is a finite power series space and $E_\infty(b)$ is an infinite power series space. In the case where at least one of the Cartesian factors is a Schwartz space, such a classification is known by the results of the third author obtained in [10; 12] by using the theory of Fredholm operators (see in [12] the modification of Douady's lemma and Theorem 1).

If both Cartesian factors are non-Schwartz spaces then the approach used in [10; 12] is not applicable (at least in the form developed there). We used in this case the method of generalized linear topological invariants developed in [11; 13; 14] as a generalization of the classical invariants [1; 3; 4; 8] (and initiated by [5; 6]—see the survey [15] for more details). The method of generalized linear topological invariants is always applicable—see Theorem 1, where we obtain necessary conditions for the isomorphism of Cartesian products of power series spaces. However, in the case when both the Cartesian factors are Schwartz spaces, it turns out that the methods developed in [10; 12] give stronger results; for details see [9], where the two methods are compared in this case. We used the same invariant characteristics that were considered in [9]. Our results are announced without proofs in [2].

Preliminaries

Recall that if $A = (a_{ip})_{i \in I, p \in N}$ is a matrix of real numbers such that $0 \le a_{ip} \le a_{i,p+1}$ for each p and for each index i in the countable set I, then *the Köthe space* $K(A)$, defined by the matrix A, is the Fréchet space of all sequences $x = (x_i)$ of scalars such that $|x|_p := \sum_{i \in I} |x_i| a_{ip} < \infty$, $p \in N$, with the topology generated by the system of seminorms $\{|\cdot|_p : p \in N\}$. The Cartesian product

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$K(A)\times K(B)$ of the Köthe spaces $K(A)$ and $K(B)$, where $A=(a_{ip})$ and $B=(b_{ip})$ $(i,p\in N)$, is naturally isomorphic to the space $K(C)$ and where $C=(c_{ip})$, with $c_{ip}=a_{kp}$ if $i=2k-1$ and $c_{ip}=b_{kp}$ if $i=2k$. For any sequence $a=(a_k)$ of positive real numbers, the Köthe spaces

$$E_0(a)=K\left(\exp\left(-\frac{1}{p}a_k\right)\right) \quad \text{and} \quad E_\infty(a)=K(\exp(pa_k))$$

are called, respectively, finite and infinite power series spaces. They are Schwartz spaces if and only if $a_k\to\infty$. The sequences $a,\tilde{a}$ of positive numbers are called *weakly equivalent* (we write $a_i\asymp\tilde{a}_i$) if

$$\exists C>0\colon \frac{1}{C}a_i\le\tilde{a}_i\le Ca_i.$$

For any set B we denote by $|B|$ the number of elements in B if it is finite and the symbol ∞ if B is infinite.

Suppose $X=K(a_{ip})_{i\in I}$ and $Y=K(b_{jp})_{j\in J}$ are Köthe spaces. An operator $T\colon X\to Y$ is called *quasidiagonal* if there exist a function $\varphi\colon I\to J$ and constants r_i, $i\in I$, such that

$$Te_i=r_i\tilde{e}_{\varphi(i)}, \quad i\in I,$$

where (e_i) and $(\tilde{e}_j)$ are the canonical bases in the spaces X and Y. We denote respectively by $X\overset{\mathrm{qd}}{\hookrightarrow}Y$ and $X\overset{\mathrm{qd}}{\simeq}Y$ a quasidiagonal isomorphic imbedding and a quasidiagonal isomorphism.

The next statement is well known (see e.g. [12]).

LEMMA 1. *If for Köthe spaces X and Y there are quasidiagonal imbeddings $X\overset{\mathrm{qd}}{\hookrightarrow}Y$ and $Y\overset{\mathrm{qd}}{\hookrightarrow}X$, then $X\overset{\mathrm{qd}}{\simeq}Y$.*

Proof. If the quasidiagonal imbeddings $X\overset{\mathrm{qd}}{\hookrightarrow}Y$ and $Y\overset{\mathrm{qd}}{\hookrightarrow}X$ are defined respectively by (r_i), $\varphi\colon I\to J$, and (ρ_j), $\psi\colon J\to I$, then by the theorem of Kantor and Bernstein there exist complementary subsets $I_1,I_2\subset I$ and $J_1,J_2\subset J$ such that $\varphi(I_1)=J_1$ and $\psi(J_2)=I_2$. Putting $Te_i=\gamma_i\tilde{e}_{g(i)}$, where $\gamma_i=r_i$ and $g(i)=\varphi(i)$ for $i\in I_1$ and where $\gamma_i=\rho^{-1}_{\psi^{-1}(i)}$ and $g(i)=\psi^{-1}(i)$ for $I\in I_2$, we obtain a quasidiagonal isomorphism T between X and Y. $\square$

LEMMA 2. *If $a=(a_k)$ and $\tilde{a}=(\tilde{a}_k)$ are sequences of positive numbers satisfying*

$$\exists M,\ \forall t\ge\tau,\quad |\{k\colon\tau\le a_k\le t\}|\le|\{k\colon\tau/M\le\tilde{a}_k\le Mt\}|, \tag{1}$$

then there exists an injection $\varphi\colon N\to N$ such that

$$\frac{1}{M^2}a_k\le\tilde{a}_{\varphi(k)}\le M^2a_k \quad \forall k\in N. \tag{2}$$

This statement is proved in [6] by using the Hall–König theorem. An alternative direct proof is given in the survey [15].

COROLLARY. *If $a = (a_k)$ and $\tilde{a} = (\tilde{a}_k)$ are sequences of positive numbers satisfying* (1), *then $E_0(a)$ can be imbedded quasidiagonally into $E_0(\tilde{a})$ and $E_\infty(a)$ can be imbedded quasidiagonally into $E_\infty(\tilde{a})$.*

In the case where a is a bounded sequence, the situation is trivial. Namely, we have the following lemma.

LEMMA 3. *If a sequence a of positive numbers is bounded, then*

$$E_0(a) \overset{\mathrm{qd}}{\simeq} l^1, \qquad E_\infty(a) \overset{\mathrm{qd}}{\simeq} l^1.$$

Invariant Characteristics

We give a short description of the invariants used here. For more details concerning the general theory of linear topological invariants we refer to [15].

Suppose E is a linear space, U and V are absolutely convex sets in E, and $\mathcal{E}_V$ is the set of all finite-dimensional subspaces of E that are spanned on elements of V. We set

$$\beta(V, U) = \sup\{\dim L : L \in \mathcal{E}_V,\ L \cap U \subset V\}.$$

It is obvious that

$$\tilde{V} \subset V,\ U \subset \tilde{U} \ \Rightarrow\ \beta(\tilde{V}, \tilde{U}) \le \beta(V, U);$$

of course, if T is an injective linear operator defined on E then

$$\beta(T(V), T(U)) = \beta(V, U).$$

Let E be a Köthe space and let A be the set of all sequences with positive terms. For any $a, b \in A$ we set

$$a \cdot b = (a_i b_i), \quad a^\alpha = (a_i^\alpha), \quad a \wedge b = (\min(a_i, b_i)), \quad a \vee b = (\max(a_i, b_i)).$$

Also, for any $x = (x_i) \in E$ and $a \in A$ we put

$$\|x\|_a = \sum_i |x_i| a_i, \qquad B_a = \{x \in E : \|x\|_a < 1\}.$$

It is easy to see that

$$B_{a \vee b} \subset B_a \cap B_b \subset 2B_{a \vee b}, \qquad B_{a \wedge b} = \operatorname{conv}(B_a \cup B_b). \tag{3}$$

LEMMA 4. *If $a, b \in A$ then*

$$\beta(B_a, B_b) = |\{i : a_i / b_i \le 1\}|.$$

Proof. Set

$$J = \{i : a_i \le b_i\}, \qquad Px = \sum_{i \in J} x_i e_i,$$

and let M be the linear span of the vectors $\{e_i,\ i \in J\}$. Then for $x \in M$ it is obvious that $\|x\|_a \le \|x\|_b$, hence $M \cap B_b \subset B_a$ and $\beta(B_a, B_b) \ge \dim M = |J|$.

Conversely, suppose L is a finite-dimensional subspace in X satisfying $L \cap B_b \subset B_a$ (i.e., $\|x\|_a \le \|x\|_b$ for all $x \in L$). If $\dim L > |J|$ then there exists an element $x \in L$, $x \ne 0$, such that $Px = 0$. But then $x_i = 0$ for $i \in J$ and $a_i > b_i$ for $i \notin J$, yielding $\|x\|_a > \|x\|_b$, a contradiction. Hence $\beta(B_a, B_b) = |J|$.

For convenience we put $B_a^\alpha B_b^{1-\alpha} = B_{a^\alpha b^{1-\alpha}}$. It is well known that sets of the kind $B_a^\alpha B_b^{1-\alpha}$ have a natural interpolation property. We formulate this property in an appropriate form in the next lemma.

LEMMA 5. *Suppose E and $\tilde{E}$ are Köthe spaces, (e_i) and $(\tilde{e}_j)$ are their canonical bases, and $T\colon E \to \tilde{E}$ is a linear operator. If $a, b, \tilde{a}, \tilde{b} \in A$ and*

$$T(B_a) \subset B_{\tilde{a}} \quad \textit{and} \quad T(B_b) \subset B_{\tilde{b}},$$

then for any $\alpha \in (0,1)$ we have

$$T(B_a^\alpha B_b^{1-\alpha}) \subset B_{\tilde{a}}^\alpha B_{\tilde{b}}^{1-\alpha}.$$

Proof. Set

$$Te_i = \sum_j t_{ij} \tilde{e}_j, \quad i = 1, 2, \dots;$$

then, since $\|Tx\|_{\tilde{a}} \le \|x\|_a$ and $\|Tx\|_{\tilde{b}} \le \|x\|_b$, for any i we have

$$\|Te_i\|_{\tilde{a}} = \sum_j |t_{ij}| \tilde{a}_j \le \|e_i\|_a = a_i, \qquad \|Te_i\|_{\tilde{b}} = \sum_j |t_{ij}| \tilde{b}_j \le \|e_i\|_b = b_i.$$

By the Hölder inequality it follows that

$$\|Te_i\|_{\tilde{a}^\alpha \tilde{b}^{1-\alpha}} = \sum_j |t_{ij}| \tilde{a}_j^\alpha \tilde{b}_j^{1-\alpha} \le \Big(\sum_j |t_{ij}| \tilde{a}_j\Big)^\alpha \Big(\sum_j |t_{ij}| \tilde{b}_j\Big)^{1-\alpha} \le a_i^\alpha b_i^{1-\alpha},$$

hence

$$\|Tx\|_{\tilde{a}^\alpha \tilde{b}^{1-\alpha}} \le \sum_i |x_i| \, \|Te_i\|_{\tilde{a}^\alpha \tilde{b}^{1-\alpha}} \le \sum_i |x_i| a_i^\alpha b_i^{1-\alpha} = \|x\|_{a^\alpha b^{1-\alpha}}. \qquad \square$$

If $E = K(a_{ip})$ is a Köthe space and $U_p = \{x \in E : |x|_p = \sum_i |x_i| a_{ip} < 1\}$, $p = 1, 2, \dots$, are the corresponding unit balls, then $U_p = B_{a_p}$ where $a_p = (a_{ip})$. Moreover, we write $U_p^\alpha U_q^{1-\alpha}$ instead of $B_{a_p}^\alpha B_{a_q}^{1-\alpha}$.

Main Results

THEOREM 1. *If $E_0(a) \times E_\infty(b) \simeq E_0(\tilde{a}) \times E_\infty(\tilde{b})$, then the following relations hold:*

$$\exists M, \tau_0 > 0\colon |\{i : \tau \le a_i \le t\}| \le |\{i : \tau/M \le \tilde{a}_i \le Mt\}|, \quad \tau \ge \tau_0; \tag{4}$$

$$\exists M, \tau_0 > 0\colon |\{i : \tau \le b_i \le t\}| \le |\{i : \tau/M \le \tilde{b}_i \le Mt\}|, \quad \tau \ge \tau_0. \tag{5}$$

Proof. The Cartesian products $E_0(a) \times E_\infty(b)$ and $E_0(\tilde{a}) \times E_\infty(\tilde{b})$ are naturally isomorphic to the Köthe spaces $X = K(c_{ip})$ and $Y = K(d_{ip})$, where

$$c_{ip} = \begin{cases} \exp(-a_k/p), & i = 2k-1, \\ \exp(pb_k), & i = 2k; \end{cases} \qquad d_{ip} = \begin{cases} \exp(-\tilde{a}_k/p), & i = 2k-1, \\ \exp(p\tilde{b}_k), & i = 2k. \end{cases}$$

Suppose now that X and Y are isomorphic and that $T: X \to Y$ is an isomorphism. Let (U_p) and (V_p) be respectively the systems of unit balls in X and Y. For convenience we write $V \prec W$ if $V \subset \text{const}\, W$. We choose indices

$$p_2 < p < p_1 < q_2 < q < q_1 < r_2 < r < r_1 < s_2 < s < s_1, \quad 2p_1 < q_2, \ 2q_1 < r_2,$$

such that

$$V_{p_2} \succ T(U_p) \succ V_{p_1} \succ V_{q_2} \succ T(U_q) \succ V_{q_1} \succ V_{r_2} \succ T(U_r) \succ V_{r_1} \succ V_{s_2} \succ T(U_s) \succ V_{s_1}.$$

By Lemma 5 and the elementary properties of β it then follows that, for some constant $c > 0$,

$$\begin{aligned} &\beta(U_q \cap e^t U_s, \operatorname{conv}(U_q \cup U_p^{1/2} U_r^{1/2} \cup e^\tau U_r)) \\ &\qquad \le \beta(c V_{q_2} \cap e^t V_{s_2}, \operatorname{conv}(V_{q_1} \cup V_{p_1}^{1/2} V_{r_1}^{1/2} \cup e^\tau V_{r_1})); \qquad (6) \\ &\beta(U_p^{1/2} U_r^{1/2} \cap e^t U_r \cap U_q, \operatorname{conv}(U_q \cup e^\tau U_s)) \\ &\qquad \le \beta(c V_{p_2}^{1/2} V_{r_2}^{1/2} \cap e^t V_{r_2} \cap V_{q_2}, \operatorname{conv}(V_{q_1} \cup e^\tau V_{s_1})). \qquad (7) \end{aligned}$$

Estimating the left-hand sides of (6) and (7) from below and the right-hand sides from above by using (3), Lemma 4, and the elementary properties of β, we obtain

$$\begin{aligned} &\left| \left\{ i : \frac{\max(c_{iq}, e^{-t} c_{is})}{\min(c_{iq}, c_{ip}^{1/2} c_{ir}^{1/2}, e^{-\tau} c_{ir})} \le 1 \right\} \right| \\ &\qquad \le \left| \left\{ i : \frac{\max(d_{iq_2}, e^{-t} d_{is_2})}{\min(d_{iq_1}, d_{ip_1}^{1/2} d_{ir_1}^{1/2}, e^{-\tau} d_{ir_1})} \le 2c \right\} \right|; \qquad (8) \\ &\left| \left\{ i : \frac{\max(c_{ip}^{1/2} c_{ir}^{1/2}, e^{-t} c_{ir}, c_{iq})}{\min(c_{iq}, e^{-\tau} c_{is})} \le 1 \right\} \right| \\ &\qquad \le \left| \left\{ i : \frac{\max(d_{ip_2}^{1/2} d_{ir_2}^{1/2}, e^{-t} c_{ir_2}, c_{iq_2})}{\min(d_{iq_1}, e^{-\tau} d_{is_1})} \le 4c \right\} \right|. \qquad (9) \end{aligned}$$

It follows that

$$\begin{aligned} &\left| \left\{ i : \frac{c_{iq}}{c_{ip}^{1/2} c_{ir}^{1/2}} \le 1, \frac{c_{iq}}{e^{-\tau} c_{ir}} \le 1, \frac{e^{-t} c_{is}}{c_{iq}} \le 1 \right\} \right| \\ &\qquad \le \left| \left\{ i : \frac{d_{iq_2}}{d_{ip_1}^{1/2} d_{ir_1}^{1/2}} \le 2c, \frac{d_{iq_2}}{e^{-\tau} d_{ir_1}} \le 2c, \frac{e^{-t} d_{is_2}}{d_{iq_1}} \le 2c \right\} \right|; \qquad (10) \\ &\left| \left\{ i : \frac{c_{ip}^{1/2} c_{ir}^{1/2}}{c_{iq}} \le 1, \frac{c_{iq}}{e^{-\tau} c_{is}} \le 1, \frac{e^{-t} c_{ir}}{c_{iq}} \le 1 \right\} \right| \\ &\qquad \le \left| \left\{ i : \frac{d_{ip_2}^{1/2} d_{ir_2}^{1/2}}{d_{iq_1}} \le 4c, \frac{d_{iq_2}}{e^{-\tau} d_{is_1}} \le 4c, \frac{e^{-t} d_{ir_2}}{d_{iq_1}} \le 4c \right\} \right|. \qquad (11) \end{aligned}$$

Namely, the left-hand side of (10) (respectively (11)) is equal to the left-hand side of (8) (respectively (9)), while the right-hand sides of (10) and (11) are respectively greater than or equal to the right-hand sides of (8) and (9).

Further, we show that (10) implies the relation (5). The first inequality in the left-hand side of (10) is $c_{iq} \le c_{ip}^{1/2} c_{ir}^{1/2}$. For the odd indices $i = 2k-1$ this is equivalent to the inequality $(-1/q + 1/2p + 1/2r)a_k \le 0$, which is impossible because $q > 2p$. For the even indices $i = 2k$ it is equivalent to $(2q - p - r)b_k \le 0$, which is always true because $r > 2q$. Therefore the left-hand side of (10) equals

$$\left|\left\{k : \frac{\tau}{r-q} \le b_k \le \frac{t}{s-q}\right\}\right|. \tag{12}$$

Let us consider now the right-hand side of (10). The first inequality there is $d_{iq_2} \le 2c d_{ip_1}^{1/2} d_{ir_1}^{1/2}$. For the odd indices $i = 2k-1$ this is equivalent to the inequality

$$\tilde{a}_k \le \tau_1 := \frac{\log 2c}{-1/q_2 + 1/2p_1 + 1/2r_1}.$$

In this case the other two inequalities imply

$$\frac{\tau - \log 2c}{1/q_2 - 1/r_1} \le \tilde{a}_k \le \frac{t + \log 2c}{1/q_1 - 1/s_2}.$$

Hence, for $\tau > \tau_2 := \tau_1(1/q_2 - 1/r_1) + \log 2c$, the triple of inequalities in the right-hand side of (10) does not hold for odd indices.

For the even indices $i = 2k$, the first inequality in the right-hand side of (10) is equivalent to the inequality $(2q_2 - p_1 - r_1)\tilde{b}_k \le \log 2c$, which is true always because $r_1 > 2q_2$ (we can assume without loss of generality that $c > 1$). Thus for $\tau > \tau_2$ the right-hand side of (10) equals the expression

$$\left|\left\{k : \frac{\tau - \log 2c}{r_1 - q_2} \le \tilde{b}_k \le \frac{t + \log 2c}{s_2 - q_1}\right\}\right|. \tag{13}$$

Since for $\tau > \tau_2$ the expression (12) is less than the expression (13), there exist a constant $M > 0$ and a $\tau_0 > \tau_2$ such that the relation (5) holds. In an analogous way, (11) implies (4). □

Analyzing the relations (4) and (5) in the non-Schwartz case and using the results of [12], we obtain the following theorem.

THEOREM 2. *If $X = E_0(a) \times E_\infty(b)$ and $Y = E_0(\tilde{a}) \times E_\infty(\tilde{b})$, then the following conditions are equivalent*:

(i) $X \simeq Y$;

(ii) $X \overset{\text{qd}}{\simeq} Y$;

(iii) *either X, Y are Schwartz spaces and there exists an integer s and permutations of indices σ, ν such that*

$$\tilde{a}_k \asymp a_{\sigma(k)+s}, \qquad \tilde{b}_k \asymp b_{\nu(k)-s}; \tag{14}$$

or X, Y are non-Schwartz spaces and the relations (4) *and* (5) *together with the symmetric relations* (*obtained by interchanging the roles of a, b and $\tilde{a}, \tilde{b}$*) *hold.*

Proof. It is trivial that (ii) $\Rightarrow$ (i). Let us show that (i) implies (iii). If $X \simeq Y$ and X, Y are Schwartz spaces, then of course $E_0(a)$, $E_0(\tilde{a})$, $E_\infty(b)$, $E_\infty(\tilde{b})$ are also Schwartz spaces (hence each of the sequences $a, b, \tilde{a}, \tilde{b}$ tends to ∞). In this case, by [12, Thm. 1] there is an integer s such that

$$E_0(\tilde{a}) \simeq E_0(a)^{(s)} \quad \text{and} \quad E_\infty(\tilde{b}) \simeq E_\infty(b)^{(-s)},$$

where $E_0(a)^{(s)}$ denotes when $s>0$ an arbitrary subspace of codimension s and when $s<0$ an arbitrary space of the kind $E_0(a)\times L$, $\dim L = -s$. Since

$$E_0(a)^{(s)} \simeq E_0((a_{k+s})_{k=1}^{\infty}) \quad \text{and} \quad E_\infty(b)^{(-s)} \simeq E_\infty((b_{k-s})_{k=1}^{\infty})$$

(where $a_{-1}, \dots, a_{-|s|}$ or $b_{-1}, \dots, b_{-|s|}$ are arbitrary numbers), we obtain

$$E_0(\tilde{a}) \simeq E_0((a_{k+s})_{k=1}^{\infty}), \qquad E_\infty(\tilde{b}) \simeq E_\infty((b_{k-s})_{k=1}^{\infty}).$$

Let us assume for a moment that the sequences $a, b, \tilde{a}, \tilde{b}$ are increasing. Then by [4, Prop. 18] it follows that

$$\tilde{a}_k \asymp a_{k+s}, \qquad \tilde{b}_k \asymp b_{k-s}.$$

In the general case we rearrange $a, b, \tilde{a}, \tilde{b}$ in increasing order and obtain (14) by the same argument.

If X, Y are non-Schwartz spaces then (i) implies (iii) by Theorem 1.

We now prove that (iii) $\Rightarrow$ (ii). If X and Y are Schwartz spaces then the relations (14) hold. Suppose $s>0$ (the case $s<0$ can be treated in an analogous way). Then

$$E_0(a)\times E_\infty(b) \overset{\text{qd}}{\simeq} E_0((a_{\sigma(k)+s}))\times E_\infty((b_{\nu(k)}))\times L,$$

where $\dim L = s$. On the other hand, by (14) we have

$$E_0(\tilde{a}) \overset{\text{qd}}{\simeq} E_0((a_{\sigma(k)+s})), \qquad E_\infty(\tilde{b}) \overset{\text{qd}}{\simeq} E_\infty((b_{\nu(k)}))\times L,$$

hence $X \overset{\text{qd}}{\simeq} Y$.

If X and Y are non-Schwartz spaces then (iii) means that the relations (4) and (5) (together with the corresponding symmetric relations) hold. We suppose that the constant M in the relations (4) and (5) is the same and denote by a', a'', b', b'', $\tilde{a}'$, $\tilde{a}''$, $\tilde{b}'$, $\tilde{b}''$ the sequences, consisting of the terms of the sequences $a, b, \tilde{a}, \tilde{b}$ that satisfy respectively

$$a_k > \tau_0, \quad a_k \le \tau_0, \quad b_k > \tau_0, \quad b_k \le \tau_0,$$

$$\tilde{a}_k > \tau_0/M, \quad \tilde{a}_k \le \tau_0/M, \quad \tilde{b}_k > \tau_0/M, \quad \tilde{b}_k \le \tau_0/M.$$

Since Y is a non-Schwartz space, we can assume without loss of generality that at least one of the sequences $\tilde{a}''$, $\tilde{b}''$ is infinite (in the opposite case one can take a greater τ_0). Then we have

$$X \overset{\text{qd}}{\simeq} E_0(a')\times E_\infty(b')\times E_0(a'')\times E_\infty(b'');$$

$$Y \overset{\text{qd}}{\simeq} E_0(\tilde{a}')\times E_\infty(\tilde{b}')\times E_0(\tilde{a}'')\times E_\infty(\tilde{b}'').$$

By (4) and (5) and Lemma 2, there exist quasidiagonal imbeddings

$$E_0(a') \overset{\text{qd}}{\hookrightarrow} E_0(\tilde{a}'), \qquad E_\infty(b') \overset{\text{qd}}{\hookrightarrow} E_\infty(\tilde{b}').$$

On the other hand, by Lemma 3 the space $E_0(\tilde{a}'') \times E_\infty(\tilde{b}'')$ is quasidiagonally isomorphic to l^1, while the space $E_0(a'') \times E_\infty(b'')$ is either quasidiagonally isomorphic to l^1 or is finite-dimensional. Hence there exists a quasidiagonal imbedding

$$E_0(a'') \times E_\infty(b'') \overset{\text{qd}}{\hookrightarrow} E_0(\tilde{a}'') \times E_\infty(\tilde{b}'').$$

The relations (4) and (5) therefore imply the existence of a quasidiagonal imbedding $X \overset{\text{qd}}{\hookrightarrow} Y$. Analogously, the relations symmetric to (4) and (5) imply the existence of a quasidiagonal imbedding $Y \overset{\text{qd}}{\hookrightarrow} X$, so by Lemma 1 it follows that $X \overset{\text{qd}}{\simeq} Y$. This proves the theorem. □

Note that if X and Y are Schwartz spaces then we can obtain by Theorem 1 only the following weaker result (cf. [9]):

$$X \simeq Y \Rightarrow \exists s_1, s_2 \in Z \colon \tilde{a}_k \asymp a_{\sigma(k)+s_1}, \ \tilde{b}_k \asymp b_{\nu(k)+s_2}.$$

The condition $s_1 + s_2 = 0$ was obtained in [12] by using Riesz theory. In the non-Schwartz case the necessary conditions obtained in Theorem 1 turn out to be sufficient, so in that case the relations (4) and (5) together with the symmetric relations give us a criterion for isomorphism (cf. the criterion of Mityagin [6; 7] for isomorphism of power series spaces).

In particular, we have the following result.

THEOREM 3. *If $E_0(a) \times E_\infty(b) \simeq E_0(\tilde{a}) \times E_\infty(\tilde{b})$ and each of the sequences $a, b, \tilde{a}, \tilde{b}$ does not tend to infinity, then $E_0(a) \simeq E_0(\tilde{a})$ and $E_\infty(b) \simeq E_\infty(\tilde{b})$.*

It seems that in this case it is not possible to apply the method developed in [12]. In connection with these remarks the following questions arise.

QUESTION 1. Is it possible to modify the method from [12] in order to derive thereby the result of Theorem 3?

QUESTION 2. Is it possible to consider stronger linear topological invariants and obtain the condition $s_1 + s_2 = 0$ without using Riesz theory?

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