Möbius Invariant Spaces on the Unit Ball

MARCO M. PELOSO

In this paper we study spaces of holomorphic functions on the unit ball B in $\mathbf{C}^n$ that are invariant under composition with automorphisms. These spaces are called Möbius invariant, and were introduced by Arazy, Fisher, and Peetre [AFP1]. Previously, Arazy and Fisher [AF1] had proved that on the unit disc D in $\mathbf{C}$ there exists a unique Möbius invariant Hilbert space of holomorphic functions. This turns out to be the classical Dirichlet space $\mathfrak{D}$. Also, Arazy, Fisher, and Peetre constructed a space on the unit disc that is minimal in the class of Möbius invariant Banach spaces. Moreover, in [AFP1] it was proved that, on the unit disc, the minimal space $\mathfrak{M}$ can be identified with the 1-Besov space B_1.

For $n>1$, in [Z1] Zhu proved that there exists a unique Möbius invariant Hilbert space on the unit ball. However, he was not able to find a characterization of this space that extended the Dirichlet space to higher dimensions. This same result was obtained by Peetre, but never published. More recently, Arazy and Fisher [AF2] proved that on any bounded symmetric domain there exists a unique Möbius invariant Hilbert space of holomorphic functions. Again this description is in terms of the power expansion of the holomorphic functions, and a more explicit characterization seems to be desirable. Again in the case $n>1$, Arazy, Fisher, Janson, and Peetre [AFJP] and independently Zhu [Z2] have proved that, for $p>2n$, where n is the dimension of the unit ball, the diagonal Besov spaces $B_p=B_p^{n/p}$ are Möbius invariant.

In this paper we study the Möbius invariant spaces on the unit ball B. We construct a space $\mathfrak{M}$ analogous to the space on the unit disc that we prove to be minimal in the class of Möbius invariant spaces. Moreover, we prove that the space $\mathfrak{M}$ can be identified with the 1-Besov space B_1. As a consequence we obtain that, for $1\le p<\infty$, the Besov spaces B_p are Möbius invariant. Moreover, we prove that the 2-Besov space B_2 is the unique Hilbert space of holomorphic functions that is Möbius invariant, and we give an explicit description of the invariant inner product. Finally, among other properties of the invariant inner product, we prove that the dual of $\mathfrak{M}$ is the Bloch space $\mathcal{B}$, with equality of norms.

The paper is organized as follows. In Section 1 we give the basic definitions and introduce the Möbius invariant spaces. In Section 2 we construct

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the minimal space $\mathfrak{M}$. In Section 3 we introduce the (analytic) Besov spaces B_p, $0<p<\infty$. We prove that B_p can be obtained as a projection of a weighted L^p space. Moreover, we describe the atomic decomposition of these spaces, as obtained by Coifman and Rochberg [CR]. In Section 4 we prove that the minimal space $\mathfrak{M}$ can be identified with the Besov space B_1. As a consequence we obtain that the Besov spaces are Möbius invariant for $1\le p<\infty$. In Section 5 we show that the 2-Besov space is the unique Möbius invariant Hilbert space. Using a different approach, Arazy [A] has recently described the invariant inner product in a similar way. Using the expression of the invariant inner product, we prove that the dual of the minimal Banach space $\mathfrak{M}$ is the Bloch space $\mathcal{B}$.

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1. Definitions and Basic Facts

Throughout this paper we denote by B the unit ball in $\mathbf{C}^n$. For $z, w \in \mathbf{C}^n$ we write the inner product as

$$z \cdot \bar{w} = \sum_{i=1}^{n} z_i \bar{w}_i .$$

The space of holomorphic functions on B will be denoted by $\mathcal{H}(B)$ and the group of automorphisms of B by $\operatorname{Aut} B$. The group $\operatorname{Aut} B$, also called the Möbius group, consists of all biholomorphic self-maps of B onto itself. The group $\operatorname{Aut} B$ can be described as follows (see [R]). For any $\zeta \in B$ define φ_ζ by

$$\varphi_\zeta = \frac{\zeta - P_\zeta z - (1-|z|^2)^{1/2} Q_\zeta z}{1 - z\cdot\bar{\zeta}}, \tag{1}$$

where P_ζ is the orthogonal projection onto the subspace generated by ζ and $Q_\zeta = I - P_\zeta$. Then $\varphi_\zeta \in \operatorname{Aut} B$ and φ_ζ is an involution that interchanges ζ with the origin. Moreover,

$$\operatorname{Aut} B = \{\varphi_\zeta \circ U : \zeta \in B,\ U \in \mathcal{U}\},$$

whre $\mathcal{U}$ is the space of unitary transformations of $\mathbf{C}^n$.

The normalized Lebesgue measure on B will be denoted by dV. In the case when $n=1$, we denote the unit disc by D and the normalized area measure by dA. For any $n \ge 1$, the *invariant volume form* is

$$d\Sigma(z) = \frac{dV(z)}{(1-|z|^2)^{n+1}},$$

(see [R]).

The gradient of a holomorphic function f will be denoted by ∂f; that is, $\partial f = (\partial_1 f, \dots, \partial_n f)$, where

$$\partial_j f = \frac{\partial f}{\partial z_j}.$$

We will also use the following notation. We write

$$\partial^\alpha f(z) = \frac{\partial^\alpha f}{\partial z^\alpha}(z),$$

where α is a multi-index. Also, we set

$$\partial^m f(z) = \left(\frac{\partial^\alpha f}{\partial z^\alpha}(z)\right)_{|\alpha|=m}.$$

The radial derivative of $f \in \mathcal{H}(B)$ is defined to be

$$Rf(z) = z \cdot \partial f(z) = \sum_{i=1}^{n} z_i \partial_i f(z).$$

If $\varphi_z \in \operatorname{Aut} B$ is defined as in (1), the invariant derivative of $f \in \mathcal{H}(B)$ is defined as

$$\tilde{D}f(z) = |\partial(f \circ \varphi_z)(0)|.$$

We now introduce the diagonal analytic Besov spaces.

DEFINITION 1.1. Let B be the unit ball in $\mathbf{C}^n$, $0 < p \leq \infty$ and let s be any real number. Moreover, let m be a nonnegative integer, $m > s$. We define the *diagonal Besov spaces* B_p^s of holomorphic functions by

$$B_p^s = \left\{ f \in \mathcal{H}(B) \colon (1-|z|^2)^{m-s} |R^m f(z)| \in L^p\left(\frac{\mathrm{dV}(z)}{1-|z|^2}\right)\right\}.$$

REMARK 1.2. It is well known that the definition is independent of m. It is also well known that one can replace the expression $|R^m f(z)|$ with $|\partial^m f(z)|$. The reader may consult [BB] and [AFJP] as reference for these results.

We will deal particularly with one family of Besov spaces: the ones corresponding to the value $s = n/p$. In this particular case we isolate the weight so that the invariant volume form appears explicitly. Then we have the following.

DEFINITION 1.3. For $0 < p \leq \infty$ we define the analytic Besov spaces B_p as $B_p = B_p^{n/p}$. Explicitly: Let m be any integer satisfying $mp > n$. Define the space B_p as

$$B_p = \{ f \in \mathcal{H}(B) \colon (1-|z|^2)^m |R^m f(z)| \in L^p(\mathrm{d}\Sigma)\}.$$

DEFINITION 1.4. Let $s > -1$. We introduce the reproducing kernels $\mathcal{K}_s = \mathcal{K}_s(z, w)$ defined by

$$\mathcal{K}_s(z, w) = \gamma_s \frac{(1-|w|^2)^s}{(1 - z \cdot \bar{w})^{n+1+s}},$$

where

$$\gamma_s = \frac{\Gamma(n+s+1)}{\Gamma(n+1)\Gamma(s+1)}.$$

It is well known (see [R]) that the kernels $\mathcal{K}_s$ reproduce the holomorphic functions. (Indeed, they are the reproducing kernels for the Bergman spaces $A^{2,s}$; see Remark 1.9.)

We introduce a space of functions: the Bloch space. When we define the Möbius invariant spaces, we will restrict ourself to Bloch functions. We will explain shortly afterwards how this is not a too restrictive assumption.

DEFINITION 1.5. Let $f \in \mathcal{H}(B)$. Let $\tilde{D}f$ be the module of its covariant derivative. We say that f is a *Bloch function* if

$$\rho_{\mathcal{B}}(f) \equiv \sup_{z \in B} \tilde{D}f(z) < \infty.$$

Moreover, we define the *Bloch norm* $\|f\|_{\mathcal{B}}$ of f as

$$\|f\|_{\mathcal{B}} = \sup_{z \in B} \tilde{D}f(z) + |f(0)|.$$

We define the *little Bloch space* $\mathcal{B}_0$ to be the subspace of $\mathcal{B}$ for which

$$\lim_{|z| \to 1^-} \tilde{D}f(z) = 0.$$

It is easy to see (cf. [T1]) that $\mathcal{B}$ is a Banach space with the above norm, and that the seminorm satisfies $\rho_{\mathcal{B}}(f \circ \varphi) = \rho_{\mathcal{B}}(f)$ for all $f \in \mathcal{B}$ and all $\varphi \in \operatorname{Aut} B$.

It is well known that the Bloch space $\mathcal{B}$ is maximal in a very large class of function spaces that are preserved by composition with automorphisms (see [RT], [T2], and [AFP1]).

We now introduce the Möbius invariant spaces.

DEFINITION 1.6. Let X be a linear space of analytic functions on B endowed with a seminorm $\rho: X \to [0, \infty)$. Suppose that the following properties hold.

(1) X is complete on the topology generated by ρ and embeds continuously in the Bloch space $\mathcal{B}$; that is, there exists a positive constant C such that, for all $f \in X$,

$$\sup_{z \in B} \tilde{D}f(z) \leq C\rho(f).$$

(2) For all $f \in X$ and $\varphi \in \operatorname{Aut} B$, $f \circ \varphi \in X$.

(3) There exists a constant $c > 0$ such that, for all $f \in X$ and $\varphi \in \operatorname{Aut} B$,

$$\rho(f \circ \varphi) \leq c\rho(f).$$

(4) The group action is continuous. That is, for each fixed $f \in X$, the mapping $C_f: \operatorname{Aut} B \to X$ defined by $C_f(\varphi) = f \circ \varphi$ is continuous.

If X satisfies conditions (1) through (4) then it is called a *Möbius invariant space.* Moreover, if the constant c in (3) equals 1 then X is called a *strict* Möbius invariant space.

REMARK 1.7. (a) Condition (3) in the definition requires that the composition operators C_φ, $\varphi \in \operatorname{Aut} B$,

$$C_\varphi : X \to X$$

defined by

$$C_\varphi(f) = f \circ \varphi$$

are (well-defined and) uniformly bounded in the (semi)norm of X. If we define a new seminorm ρ' on X by setting

$$\rho'(f) = \sup\{\rho(f \circ \varphi) : \varphi \in \operatorname{Aut} B\},$$

we obtain a seminorm equivalent to ρ. Such a new seminorm ρ' satisfies the condition

$$\rho'(f \circ \varphi) = \rho'(f) \quad \text{for all } f \in X,\ \varphi \in \operatorname{Aut} B.$$

Hence every Möbius invariant space can be made into a strict Möbius invariant one.

(b) As a consequence of Rubel and Timoney's theorem [RT; T2], condition (1) is equivalent to requiring the existence of a nonzero continuous linear functional on X that is also continuous in compact-open topology. Such linear functionals have been called "decent" in the literature. In particular, it suffices to require that any point evaluation of derivatives of any order is continuous on X.

EXAMPLES 1.8. (a) For $1 < p < \infty$ the Besov spaces $B_p(D)$ on the unit disc D are such that

$$B_p = \left\{ f \in \mathcal{H}(D) : \int_D (1-|z|^2)^p |f'(z)|^p \, d\Sigma(z) < \infty \right\}.$$

Recall that since $n = 1$,

$$d\Sigma(z) = \frac{dA(z)}{(1-|z|^2)^2}$$

and that $(1-|z|^2)|f'(z)|$ is the invariant derivative $\tilde{D}f(z)$ of f. Hence B_p is a (strict) Möbius invariant space. Notice that for $p = 2$, the space $B_p(D)$ reduces to the Dirichlet space

$$\mathfrak{D} \equiv \left\{ f \in \mathcal{H}(D) : \int_D |f'(z)|^2 \, dA(z) < \infty \right\}$$

of analytic functions on the unit disc of finite area. The quotient space $\mathfrak{D}/\mathbf{C}$ is a Hilbert space. We have already mentioned that $\mathfrak{D}/\mathbf{C}$ is the unique Hilbert space which is Möbius invariant in the sense of Definition 1.6.

(b) Let $A(B)$ be the *ball algebra*, that is, the space of holomorphic functions on B continuous up to the boundary. Then $A(B)$ is a (strict) Möbius invariant space with the supnorm.

(c) Let $\mathcal{B}$ be Bloch space. The seminorm $\rho_{\mathcal{B}}$,

$$\rho_{\mathcal{B}}(f) = \sup\{\tilde{D}f(z) : z \in B\},$$

is (strict) Möbius invariant. But $\mathcal{B}$ does not satisfy condition (3) in Definition 1.6; that is, the group action is not continuous. However, the group action *is* continuous in the weak*-topology.

(d) The little Bloch space $\mathcal{B}_0$ is a (strict) Möbius invariant space. The proof of the continuity of the group action is elementary. Just notice that if $f \in \mathcal{B}_0$ and f is nonconstant, then there exists $z_0 \in \mathcal{B}$ that realizes the seminorm. That is,

$$\sup_{z \in B} \tilde{D}f(z) = \tilde{D}f(z_0).$$

(e) The classical $H^p(B)$ spaces do not satisfy condition (3) in Definition 1.6 for $1 \le p < \infty$. Indeed, an easy calculation shows that, if φ_a is defined as in equation (1), then the composition operator C_{φ_a} (see Remark 1.7(a)) has norm $(1-|a|^2)^{-1/p}$, which is unbounded as $a \to S$. By contrast, the space $H^\infty(B)$ does not satisfy condition (4) in Definition 1.6.

REMARK 1.9. Let us consider the diagonal Besov spaces B_p^s as defined in Definition 1.1. Notice that for $s<0$ and $0<p<\infty$ we obtain the classical *weighted Bergman spaces* on the ball $A^{p,\nu}(B)$, where $\nu = -s-1 > -1$. Explicitly,

$$A^{p,\nu}(B) = \left\{ f \in \mathcal{H}(B) : \int_B |f(z)|^p (1-|z|^2)^\nu \, dV(z) < \infty \right\}.$$

On the Bergman spaces $A^{p,\nu}$ is possible to define a weighted action

$$f \mapsto (\det \varphi')^\beta (f \circ \varphi),$$

where $\beta = (1-\nu/(n+1))^{2/p}$. It would then be interesting to investigate the present theory under this approach, extending it to the whole scale of weighted Besov spaces.

We now present some properties of the Möbius invariant spaces. Proposition 1.10 is an immediate consequence of the definition; nonetheless, it furnishes a key tool in studying Möbius invariant spaces.

PROPOSITION 1.10. *Let X be a Möbius invariant space and let μ be a finite measure on* $\operatorname{Aut} B$. *Then the integral operator*

$$(T_\mu f)(z) = \int_{\operatorname{Aut} B} f(\varphi(z)) \, d\mu(\varphi)$$

maps X into X and

$$\rho(T_\mu f) \le \|\mu\| \rho(f).$$

The integral converges in the topology of X.

Proof. This follows immediately from the properties of vector-valued integrals. The continuity of the group action on X implies that $f(\varphi)$ is a continuous function on $\operatorname{Aut} B$. □

The next proposition has been used rather explicitly by some authors. For $n=1$ it was proved by Arazy, Fisher and Peetre [AFP1]. For $n \geq 1$, partial results were obtained in [Z2] and [AFJP]. Our proof follows the one on the disc by Arazy, Fisher, and Peetre.

We introduce here some more notation that will be used later on.

DEFINITION 1.11. For $j=1,\dots,n$ we define the *coordinate functions* v_j on B as

$$v_j(z)=z_j.$$

PROPOSITION 1.12. *Let X be a Möbius invariant space. Suppose that X contains a nonconstant function f. Then X contains all polynomials and these are dense in X. Moreover,*

$$\rho(z^\alpha)=O(|\alpha|) \quad as \ |\alpha|\to\infty.$$

Proof. Here we consider only the case $n>1$; the case $n=1$ is in [AFP1, Prop. 2]. Let $\mathcal{V}$ be the subgroup of the unitary matrices $\mathcal{U}$ that are diagonal. Thus, $\mathcal{V}$ is defined by

$$\mathcal{V}=\left\{U\in\mathcal{U}: U=\begin{pmatrix} e^{i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_n}\end{pmatrix}, \theta=(\theta_1,\dots,\theta_n)\in\mathbf{R}^n\right\}. \tag{2}$$

For a multi-index α and $\theta=(\theta_1,\dots,\theta_n)\in\mathbf{R}^n$, write

$$\alpha\cdot\theta=\sum_{i=1}^n \alpha_i\theta_i.$$

Given a multi-index β, define the measure $\mu=\mu_\beta$ on $\operatorname{Aut} B$ by the formula

$$\int_{\operatorname{Aut} B} f(\varphi)\,d\mu_\beta(\varphi)=\frac{1}{(2\pi)^n}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} f(\theta_1,\dots,\theta_n)e^{-i\beta\cdot\theta}\,d\theta_n\cdots d\theta_1,$$

where we identify $\theta\in\mathbf{R}^n$ with $U\in\mathcal{V}$ as in (2). Since X is a Möbius invariant space, by Proposition 1.10 it follows that for any $f\in X$ the function

$$T_{\mu_\beta}f(z)=\frac{1}{(2\pi)^n}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} f(e^{i\theta_1}z_1,\dots,e^{i\theta_n}z_n)e^{i\beta\cdot\theta}\,d\theta_n\cdots d\theta_1$$

belongs to X. Let $f(z)=\sum_\alpha a_\alpha z^\alpha$ be the power series expansion of f. Then

$$\begin{aligned} T_{\mu_\beta}f(z)&=\sum_\alpha a_\alpha z^\alpha \prod_{j=1}^n \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\alpha_j\theta_j}e^{-i\beta_j\theta_j}\,d\theta_j \\ &=a_\beta z^\beta. \end{aligned}$$

Hence $a_\beta\neq 0$ implies $z^\beta\in X$. Let $\mathcal{H}_k$ be the space of all holomorphic polynomials of degree k. Since X is unitarily invariant and $\{z^\beta\circ U\}_{U\in\mathcal{U}}$ span $\mathcal{H}_{|\beta|}$ (see [R, 12.2.8]), we have that

$$\mathcal{H}_{|\beta|}\subset X.$$

In particular, $v_j^{|\beta|}\circ\varphi\in X$ for all $\varphi\in \operatorname{Aut} B$, $j=1,\dots,n$ (recall Definition (1.11). Therefore

$$\left(\frac{r-z_j}{1-rz_j}\right)^{|\beta|}\in X \quad \text{for } -1<r<1,\ j=1,\dots,n.$$

Now, arguing as in [AFP1, Prop. 2], we obtain that X contains all polynomials.

The proof that the polynomials are dense in X is the same as in the case $n=1$. It suffices to notice that, if $f\in\mathcal{H}(B)$ and $f(z)=\sum_0^\infty F_j(z)$ is its homogeneous expansion, then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta}z)e^{-ik\theta}\,d\theta=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_j e^{ij\theta}F_j(z)\right)e^{-ik\theta}\,d\theta=F_k(z)$$

(here $e^{i\theta}z$ indicates the scalar product). Set

$$\sigma_N(\theta)=\sum_{j=-N}^{N}\left(1-\frac{|j|}{N+1}\right)e^{ij\theta},$$

the Nth Fejer kernel, and

$$\sigma_N(f)(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta}z)e^{-ik\theta}\sigma_N(\theta)\,d\theta.$$

Using standard arguments of approximation of the identity, one shows that

$$\rho(\sigma_N(f)-f)\to 0 \quad \text{as } N\to\infty.$$

Finally, we wish to estimate the size of z^α in the seminorm ρ. By [AFP1, Prop. 1] it follows that, for $j=1,\dots,n$,

$$\rho(z_j^m)=O(m) \quad \text{as } m\to\infty.$$

By the theory of spherical harmonics (see [R, 12.2.5]) we know that for any m there exists a function $H_m(z,\eta)$ defined on $B\times S$ such that for all α, $|\alpha|=m$,

$$\begin{aligned} z^\alpha &= \int_S \eta^\alpha H_m(z,\eta)\,\mathrm{d}\sigma(\eta)\\ &= \int_{\mathcal{U}} (U^{-1}e_1)^\alpha H_m(Uz,e_1)\,\mathrm{d}U, \end{aligned}$$

where $\mathrm{d}U$ is the normalized Haar measure on the unitary group $\mathcal{U}$ and e_1 is the unit vector $(1,0,\dots,0)$ (see [R, 1.4.7]). Since X is Möbius invariant, the mapping $U\mapsto H_m(U,e_1)$ is continuous from $\mathcal{U}$ into X. Thus, by Proposition 1.10,

$$\rho(z^\alpha)\le\rho(H_m(\cdot,e_1)).$$

By [R, 12.2.6] we have that

$$H_m(z,e_1)=\int_{\mathcal{U}_{e_1}}(v_1^m\circ U)(z)\,\mathrm{d}U,$$

where $\mathcal{U}_{e_1}$ is the subgroup of $\mathcal{U}$ that fixes e_1, with measure dU normalized so that the subgroup has measure 1, and v_1 is the coordinate function introduced in Definition 1.11. Then

$$\rho(z^\alpha) \le \rho(H_m(\cdot, e_1)) \le \rho(v_j^m) = O(m).$$

This gives the desired conclusion. □

2. The Minimal Space

In this section we construct the space $\mathfrak{M}$ that turns out to be the smallest Möbius invariant space, in the sense that it is contained in every other Möbius invariant space.

For $a \in B$ let $\varphi_a \in \operatorname{Aut} B$ as defined in (1). Notice that $\varphi_a \to a$ uniformly on compact subsets of B as $|a| \to 1^-$. Thus we identify φ_a with a if $|a| = 1$. Recall that, for $j = 1, \ldots, n$, v_j are the coordinate functions as defined in Definition 1.11.

DEFINITION 2.1. With the above convention define

$$\mathfrak{I} = \{\tau : \bar{B} \to \mathbf{C} : \tau = v_j \circ \psi \text{ for } \psi \in \operatorname{Aut} B \text{ and } j = 1, \ldots, n\}.$$

Let $\mathfrak{M}$ be the subspace of $\mathcal{H}(B)$ defined by

$$\mathfrak{M} = \left\{ f : f = \sum_1^\infty c_i \tau_i \text{ with } \tau_i \in \mathfrak{I},\ c_i \in \mathbf{C},\ \sum_1^\infty |c_i| < \infty \right\}.$$

We give $\mathfrak{M}$ the norm

$$\|f\|_{\mathfrak{M}} = \inf\left\{ \sum_1^\infty |c_i| : f = \sum_1^\infty c_i \tau_i \right\}.$$

REMARK 2.2. The following are elementary.

(a) For all $\tau \in \mathfrak{I}$ we have $\|\tau\|_{\mathfrak{M}} = 1$.
(b) The space $\mathfrak{M}$ is complete in this norm.
(c) The space $\mathfrak{M}$ continuously embeds into the little Bloch space $\mathcal{B}_0$; that is, $\mathfrak{M} \subseteq \mathcal{B}_0$ and for each $f \in \mathfrak{M}$ we have

$$\|f\|_{\mathcal{B}} \le \|f\|_{\mathfrak{M}}.$$

(d) For every $f \in \mathfrak{M}$ and $f \in \operatorname{Aut} B$ we have that

$$\|f \circ \varphi\|_{\mathfrak{M}} = \|f\|_{\mathfrak{M}}.$$

Therefore, in order to conclude that $\mathfrak{M}$ is a Möbius invariant space, we need to show that the group action is continuous (see Definition 1.6 (4)). This is our next task.

LEMMA 2.3. *Let $j = 1, \ldots, n$ and $k = 1, 2, \ldots$. Then the functions z_j^k satisfy*

$$\|z_j^k\|_{\mathfrak{M}} = O(k) \quad \text{as } k \to \infty.$$

Proof. Obviously, the functions $z_1, \dots, z_n$ all belong to $\mathfrak{M}$. For $\lambda \in D$ let $\varphi_\lambda \in \operatorname{Aut} B$ be defined by

$$\varphi_\lambda(z) = \left(\frac{\lambda - z_1}{1-\bar{\lambda} z_1}, -\frac{(1-|\lambda|^2)^{1/2} z_2}{1-\bar{\lambda} z_1}, \dots, -\frac{(1-|\lambda|^2)^{1/2} z_n}{1-\bar{\lambda} z_1}\right).$$

Set $\sigma_\lambda = v_1 \circ \varphi_\lambda$ (recall Definition 1.11). Applying [AFP1, Thm. 6] we obtain that, for suitable λ_j with $|\lambda_j| \le 1$,

$$v_1^{k+1} = \sum_1^\infty a_j \sigma_{\lambda_j}$$

and

$$\begin{aligned} \inf\left\{\sum_1^\infty |a_j| : v_1^{k+1} = \sum_1^\infty a_j \sigma_{\lambda_j}\right\} &\le (k+2)\left(1+\frac{2}{k}\right)^{k/2} \\ &= O(k) \quad \text{as } k \to \infty. \end{aligned}$$

This gives the desired result. □

LEMMA 2.4. *For each multi-index α, the function $z^\alpha \in \mathfrak{M}$ and*

$$\|z^\alpha\|_{\mathfrak{M}} = O(|\alpha|) \quad \textit{as } |\alpha| \to \infty.$$

Proof. Recall that $\mathcal{H}_k$ is the space of all holomorphic homogeneous polynomials of degree k. We want to show that, for each α with $|\alpha| = k$, the action $U \mapsto z^\alpha \circ U$ of $\mathcal{U}$ into $\mathfrak{M}$ is continuous.

Let $U_m = \{(u_{jj}^m)_{j=1}^n\}$ be a sequence of unitary diagonal matrices converging to the identity. Fix the function $f_\alpha(z) = z^\alpha$ in $\mathcal{H}_k$. Then

$$\begin{aligned} \|f_\alpha \circ U_m - f_\alpha\|_{\mathfrak{M}} &= \left\| \prod_{j=1}^n (u_{jj}^m)^{\alpha_j} z_j^{\alpha_j} - z^\alpha \right\|_{\mathfrak{M}} \\ &\le \sum_{j=1}^n |1-(u_{jj}^m)^{\alpha_j}| \cdot \|z^\alpha\|_{\mathfrak{M}}, \end{aligned}$$

and the right-hand side tends to 0 as $m \to \infty$.

If U_m are not diagonal matrices, we can find unitary matrices V_m such that $V_m U_m V_m^*$ are diagonal. Then

$$\|f_\alpha \circ U_m - f_\alpha\|_{\mathfrak{M}} = \|g_\alpha \circ V_m U_m V_m^* - g_\alpha\|_{\mathfrak{M}},$$

where $g_\alpha = f_\alpha \circ V_m^*$ is a function of the variable $w = V_m z$. Thus the unitary group acts continuously on $\mathfrak{M}$. Now, as in the proof of Proposition 1.12,

$$z^\alpha = \int_{\mathcal{U}} (U^{-1} e_1)^\alpha H_k(Uz, e_1)\, dU$$

with $\|H_k(\cdot, e_1)\|_{\mathfrak{M}} \le \|v_1^k\|_{\mathfrak{M}}$. Then, by Proposition 1.10 it follows that

$$\|z^\alpha\|_{\mathfrak{M}} = O(|\alpha|) \quad \text{as } |\alpha| \to \infty. \qquad \square$$

THEOREM 2.5. *The space $\mathfrak{M}$ is a Möbius invariant space. Moreover, $\mathfrak{M}$ is minimal in the sense that if X is any Möbius invariant space that contains* *a nonconstant function then there exists a positive constant $c>0$ such that, for all $f\in\mathfrak{M}$,*

$$\|f\|_X\le c\|f\|_{\mathfrak{M}}.$$

Proof. In order to prove that $\mathfrak{M}$ is Möbius invariant, we need only prove the continuity of the group action. Notice that it suffices to show that $f\circ\varphi^m\to f$ in $\mathfrak{M}$ as $\varphi^m\to\mathrm{id}$ in $\operatorname{Aut}B$. Notice that $-\varphi_a\to\mathrm{id}$ as $a\to 0$. We first show that if $a\to 0$ in B then

$$v_1\circ(-\varphi_a)\to v_1 \text{ in } \mathfrak{M},$$

where

$$\varphi_a(z)=\frac{a-P_az-(1-|a|^2)^{1/2}Q_az}{1-z\cdot\bar a}$$

$$=\frac{1}{1-z\cdot\bar a}\left(a_j-\frac{z\cdot\bar a}{|a|^2}a_j-(1-|a|^2)^{1/2}\left(z_j-\frac{z\cdot\bar a}{|a|^2}a_j\right)\right)_{j=1}^{n}.$$

We have

$$v_1\circ(-\varphi_a)(z)-v_1(z)$$

$$=\frac{1}{1-z\cdot\bar a}$$

$$\times\left(-a_1+(1-|a|^2)^{1/2}z_1-z_1+z\cdot\bar a\left(\frac{a_1}{|a|^2}-\frac{(1-|a|^2)^{1/2}}{|a|^2}a_1+z_1\right)\right) \tag{3}$$

$$=\sum_0^\infty(z\cdot\bar a)^k(-a_1-z_1(1-(1-|a|^2)^{1/2}))$$

$$+\sum_0^\infty(z\cdot\bar a)^{k+1}\left(a_1\frac{1-(1-|a|^2)^{1/2}}{|a|^2}+z_1\right).$$

Let $e_1=(1,0,\dots,0)$ and U_a be any unitary matrix such that $U_a(|a|e_1)=a$. Then

$$\|(z\cdot\bar a)^k\|_{\mathfrak{M}}=|a|^k\|v_1^k\|_{\mathfrak{M}}. \tag{4}$$

Hence (4), together with the estimate in Lemma 2.3, gives

$$\left\|\sum_0^\infty(z\cdot\bar a)^k\right\|_{\mathfrak{M}}\le C\sum_1^\infty k|a|^k. \tag{5}$$

Arguing as before we obtain

$$\|z_1(z\cdot\bar a)^k\|_{\mathfrak{M}}\le C(k+1)|a|^k. \tag{6}$$

These estimates also yield

$$\left\|\sum_0^\infty(z\cdot\bar a)^{k+1}\right\|_{\mathfrak{M}}\le C\sum_0^\infty(k+1)|a|^{k+1} \tag{7}$$

and

$$\left\| \sum_0^\infty z_1(z\cdot\bar{a})^{k+1} \right\|_{\mathfrak{M}} \le C\sum_0^\infty (k+2)|a|^{k+2}. \tag{8}$$

Therefore, collecting the estimates (4) through (8) and using the expansion in (3), we find

$$\begin{aligned}
&\|v_1\circ(-\varphi_a(z)) - v_1\|_{\mathfrak{M}} \\
&\quad\le |a|\cdot\left\| \sum_{k=0}^\infty (z\cdot\bar{a})^k \right\|_{\mathfrak{M}} + (1-(1-|a|^2)^{1/2})\left\| \sum_{k=0}^\infty z_1(z\cdot\bar{a})^k \right\|_{\mathfrak{M}} \\
&\qquad + \left| |a|\frac{1-(1-|a|^2)^{1/2}}{|a|^2} \right| \left\| \sum_0^\infty (z\cdot\bar{a})^{k+1} \right\|_{\mathfrak{M}} + \left\| \sum_{k=0}^\infty z_1(z\cdot\bar{a})^{k+1} \right\|_{\mathfrak{M}} \\
&\quad\le C|a|\sum_1^\infty k|a|^k + (1-(1-|a|^2)^{1/2})\sum_1^\infty (k+1)|a|^{k+1} \\
&\qquad + |a|\sum_1^\infty (k+1)|a|^{k+1} + \sum_1^\infty (k+2)|a|^{k+2} \\
&\quad = O(|a|) \quad \text{as } |a|\to 0.
\end{aligned}$$

Thus we have proved that, for $j=1,\dots,n$,

$$\|v_j\circ(-\varphi_a) - v_j\|_{\mathfrak{M}} \to 0 \quad \text{as } a\to 0.$$

This easily implies that

$$\|f\circ\varphi^m - f\|_{\mathfrak{M}} \to 0 \quad \textit{as } \varphi^m \to \mathrm{id}.$$

Hence $\mathfrak{M}$ is a Möbius invariant space.

The minimality of $\mathfrak{M}$ as a Möbius invariant space follows at once. Indeed, let X be any Möbius invariant space that contains a nonconstant function. By Proposition 1.12 it follows that X contains all the polynomials. In particular, X contains the functions $v_1,\dots,v_n$. By Möbius invariance, X contains $\mathfrak{I}$ as defined in Definition 2.1. Let $f\in\mathfrak{M}$, $f=\sum_1^\infty c_i\tau_i$, $\tau_i\in\mathfrak{I}$. Let $M=\max\{\|v_j\|_X : j=1,\dots,n\}$, where $\|\cdot\|_X$ is the Möbius seminorm on X. Then

$$\begin{aligned}
\|f\|_X &= \left\| \sum_1^\infty c_i\tau_i \right\|_X \\
&\le M\sum_1^\infty |c_i|.
\end{aligned}$$

Taking the infimum of the right-hand side over all representations of f in $\mathfrak{M}$, we obtain that

$$\|f\|_X \le M\|f\|_{\mathfrak{M}}.$$

This concludes the proof. □

3. Analytic Besov Spaces

In this section we collect some results on the Besov spaces B_p or, more generally, on the weighted Besov spaces B_p^ν on the unit ball. After proving a simple fact about the equivalence of several norms on such spaces, we identify the weighted Besov space B_p^ν with a quotient of the L^p space $L^p((1-|z|^2)^s dV)$. This extends a result of Zhu on the unit disc (see [Z3]). Finally, we describe the atomic decomposition of the Besov spaces, as obtained by Coifman and Rochberg (see [CR]).

Recall that for $f \in \mathcal{H}(B)$ we write

$$\partial f(z) = (\partial_j f(z))_{j=1,\dots,n},$$

and that

$$\partial^\alpha f(z) = \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(z) \quad (\alpha \text{ a multi-index}).$$

Also, for any positive integer m we write

$$\partial^m f(z) = (\partial^\alpha f(z))_{|\alpha|=m}$$

and

$$|\partial^m f(z)| = \sum_{|\alpha|=m} |\partial^\alpha f(z)|.$$

Recall that in Definition 1.1 we defined the analytic Besov spaces B_p^ν by

$$B_p^\nu = \left\{ f \in \mathcal{H}(B) : (1-|z|^2)^{m-\nu} |R^m f(z)| \in L^p\left(\frac{dV(z)}{1-|z|^2}\right)\right\},$$

and in the case $\nu = n/p$ we defined (see Definition 1.3) B_p as

$$B_p = \{ f \in \mathcal{H}(B) : (1-|z|^2)^m |R^m f(z)| \in L^p(d\Sigma)\}.$$

Our next goal is to prove that we can replace the differential operators ∂^m and R^m with other differential operators in the definition of the norm of B_p^ν. The next proposition furnishes a simple but particularly useful criterion for when this can be done. First we have two definitions.

DEFINITION 3.1. Let $\mathfrak{D}$ be a linear first-order partial differential operator in $\partial_1, \dots, \partial_n$ with coefficients that are C^∞ up to the boundary; that is,

$$\mathfrak{D} = \sum_{j=1}^n d_j(z) \frac{\partial}{\partial z_j},$$

where d_j are functions C^∞ up to the boundary. We say that $\mathfrak{D}$ is never tangential on B if, for all $z \in \bar{B}$ and $z \neq 0$,

$$\mathfrak{D}(|z|^2)(z) \neq 0$$

(that is, if the projection of $\mathfrak{D}$ onto the normal direction is never 0).

DEFINITION 3.2. Let $0 < p \le \infty$. Let m be any positive integer such that $m > s$. Define the spaces $B_{\mathfrak{D}_{p,s}}$ by

$$B_{\mathfrak{D}_{p,s}} = \left\{ f \in \mathcal{H}(B) : \int_B [(1-|z|^2)^{m-s} |\mathfrak{D}^m f(z)|]^p \frac{dV(z)}{(1-|z|^2)} < \infty \right\}.$$

Moreover, set

$$\|f\|_{B_{\mathfrak{D}_{p,s}}}=\left(\int_B[(1-|z|^2)|\mathfrak{D}^m f(z)|]^p\,\frac{dV(z)}{(1-|z|^2)}\right)^{1/p}+\sum_{|\alpha|<m}|\partial^\alpha f(0)|.$$

PROPOSITION 3.3. *Let $0<p\leq\infty$, s be a real number, and $\mathfrak{D}$ be a first-order differential operator which is never tangential. Then there exist two constants $c_1, c_2>0$, depending only on p, s, m, n, and $\mathfrak{D}$, such that for all $f\in\mathcal{H}(B)$ we have*

$$\begin{aligned}\int_B(1-|z|^2)^s|R^m f(z)|^p\,dV(z)&\leq c_1\int_B(1-|z|^2)^s|\mathfrak{D}^m f(z)|^p\,dV(z)+\sum_{|\alpha|<m}|\partial^\alpha f(0)|\\&\leq c_2\int_B(1-|z|^2)^s|R^m f(z)|^p\,dV(z)+\sum_{|\alpha|<m}|\partial^\alpha f(0)|.\end{aligned}$$

Notice that it is part of the statement that the finiteness of one right-hand side implies the finiteness of the left-hand side.

Proof. The case $p=\infty$ is an easy consequence of the arguments in [T1] and [Z4]. The case $p<\infty$ in contained in Theorem 1.1 of [BS]. □

COROLLARY 3.4. *Let $0<p\leq\infty$ and let $\mathfrak{D}$ be a first-order differential operator which is never tangential. With the above notation there exist two constants $c_1, c_2>0$, depending only on p, n, and $\mathfrak{D}$, such that for all $f\in\mathcal{H}(B)$ we have*

$$\|f\|_{B_p}\leq c_1\|f\|_{B_{\mathfrak{D}_{p,n/p}}}\leq c_2\|f\|_{B_p}.$$

Proof. It suffices to set $s=mp-n-1$ for any positive integer m satisfying $mp>n$, in Proposition 3.3. □

We now introduce some differential operators. These operators are modifications of the radial derivative, and behave particularly nicely with respect to the reproducing kernels $\mathcal{K}_s$ (as defined in Definition 1.4). They also give rise to fractional integrations since they can be defined for noninteger values, too.

DEFINITION 3.5. Let $s>-1$ and μ be a real number. We define a linear operator $\mathcal{R}_s^\mu$ on $L^1((1-|z|^2)^s\,dV)$ by

$$\mathcal{R}_s^\mu f(z)=\gamma_s\int_B\frac{(1-|w|^2)^s f(w)}{(1-z\cdot\bar{w})^{n+1+s+\mu}}\,dV(w).$$

Recall that $\gamma_s=\Gamma(n+s+1)/\Gamma(n+1)\Gamma(s+1)$ (see Definition 1.4).

REMARK 3.6. (a) Notice that for $f\in\mathcal{H}(B)$

$$((n+1)^{-1}R+I)f(z)=\mathcal{R}_0^1 f(w),$$

and for m a positive integer

$$\begin{aligned}\mathcal{R}_s^m f&=((n+1+s+m-1)^{-1}R+I)\mathcal{R}_s^{m-1}f\\&=\gamma_s\sum_{k=0}^{\infty}(k(n+1+s+m-1)^{-1}+1)\cdots(k(n+1+s)^{-1}+1)F_k,\end{aligned}$$

where $f=\sum_{k=0}^{\infty} F_k$ is the homogeneous expansion of f (and R is the radial derivative).

(b) For all $s>-1$ and $\mu, \rho \in \mathbf{R}$ with $s+\rho>-1$, it is easy to see that

$$\mathcal{R}_{s+\rho}^{\mu}\mathcal{R}_s^{\rho}=\mathcal{R}_s^{\mu+\rho}.$$

(c) Let $\mathcal{K}(z,w)$ be the Bergman kernel and let $\mathcal{R}_s^{\mu}$ act on the z-variable. Then

$$\mathcal{R}_s^{\mu}(\mathcal{K}(\cdot,w)^{1+s/(n+1)})=\mathcal{K}(\cdot,w)^{1+\mu/(n+1)+s/(n+1)}.$$

PROPOSITION 3.7. *Let $0<p<\infty$ and $t\ge 0$. Then for $\mu>t$ and $s>-1$, the expression*

$$\left(\int_B |\mathcal{R}_s^{\mu} f(z)|^p (1-|z|^2)^{(\mu-t)p-1}\,dV(z)\right)^{1/p}$$

defines an equivalent norm on B_p^t.

Proof. For the case $0<p\le 1$, see [W]. Let $p>1$. If μ is an integer we have nothing to prove. For μ nonintegral let m be the positive integer such that $m>t$ and $|\mu-m|<1$. Write $\mathcal{R}_s^{\mu}=\mathcal{R}_{m+s}^{\mu-m}\mathcal{R}_s^{m}$. Then the integral in the statement is less than or equal to a constant times

$$\int_B ((1-|z|^2)^{\mu-t}|\mathcal{R}_{m+s}^{\mu-m}\mathcal{R}_s^{m} f(z)|)^p (1-|z|^2)^{(m-t)p-1}\,dV(z).$$

It follows from [Z5] that $(1-|z|^2)^{(\mu-m)}\mathcal{R}_{m+s}^{\mu-m}$ is a bounded linear operator on $L^p((1-|z|^2)^{(m-t)p-1}\,dV)$. This argument works both ways. □

Coifman and Rochberg [CR] have proved that the weighted Bergman spaces $A^{p,\nu}$ $(0<p<\infty,\ \nu>-1)$ admit an atomic decomposition. Precisely:

$$f(z)=\sum c_i \frac{(1-|\zeta^i|^2)^{[(n+1)(\rho+1)+\nu]/p}}{(1-z\cdot\bar{\zeta}^i)^{[(n+1)(\rho+2)+2\nu]/p}},$$

where $\{\zeta^i\}$ form a lattice,

$$\rho>\left(1+\frac{\nu}{n+1}\right)\max(-1,p-2),$$

and

$$\|f\|_{A^{p,\nu}}^p \approx \sum |c_i|^p.$$

We now illustrate how to obtain the corresponding atomic decomposition for the weighted Besov spaces. The case $0<p<1$ can also be found in [W].

THEOREM 3.8 [CR]. *Let $0<p<\infty$ and $t\in\mathbf{R}$. Let*

$$\beta>\begin{cases}\max((\frac{n}{p}-t)(p-1),0) & \text{if } t<0,\\ \max(\frac{n}{p}(p-1),0,t-\frac{n}{p}) & \text{if } t\ge 0.\end{cases}$$

Then there exists $\theta_0=\theta_0(p,\beta,n)$ such that if the points $\{\zeta^i\}$ form a θ-lattice, $0<\theta<\theta_0$, then the following conditions hold.

(1) *If $f \in B_p^t$ then there exist numbers $\{c_i\}$ such that*

$$f(z) = \sum_1^\infty c_i \frac{(1-|\zeta^i|^2)^\beta}{(1-z\cdot\bar{\zeta}^i)^{\beta+n/p-t}}$$

and

$$\sum_i^\infty |c_i|^p \leq C\|f\|_{B_p^t}^p.$$

(2) *If $\sum_i^\infty |c_i|^p < \infty$ and f is defined as in* (1), *then $f \in B_p^t$ and*

$$\|f\|_{B_p^t}^p \leq C \sum_i^\infty |c_i|^p.$$

Proof. If $t<0$ we can choose $m=0$ in the definition of the norm of the Besov space. Then this is just the Coifman and Rochberg theorem.

If $t \geq 0$, let m be a real number such that $m>t$. Let s be such that $m+s>-1$. It is easy to see that $\mathcal{R}_{s+m}^{-m}$ is an isomorphism of $A^{p,(m-t)-1}$ onto B_p^t and $\mathcal{R}_{s+2m}^m$ is its inverse. Set $\nu=(m-t)p-1$. Let $\rho > (1+\nu/(n+1))\max(-1, p-2)$ and let $\{\zeta^i\}$ be a θ-lattice for θ small enough. Then, for $f \in B_p^t$,

$$\mathcal{R}_{s+2m}^m f(z) = \sum_1^\infty c_i \frac{(1-|\zeta^i|^2)^{((n+1)/p)\rho+n/p+m-t}}{(1-z\cdot\bar{\zeta}^i)^{((n+1)/p)\rho+2(n/p+m-t)}}.$$

Let

$$\beta = \frac{n}{p} + m - t + \frac{n+1}{p}\rho.$$

Since $\rho > (1+\nu/(n+1))\max(-1,p-2)$ and $\nu=(m-t)p-1$, it follows that

$$\begin{aligned} \beta &> \left(m+t+\frac{n}{p}\right)(1+\max(-1,p-2)) \\ &= \left(m-t+\frac{n}{p}\right)\max(0,p-1). \end{aligned} \tag{9}$$

Choose s so that $m+s>-1$ and

$$m+s+n+1 = \frac{n+1}{p}\rho + 2\left(\frac{n}{p}+m-t\right).$$

Because of our choice of β, this is always possible. Next, if

$$\beta + \frac{n}{p} - t > 0 \tag{10}$$

then we have that

$$\mathcal{R}_{m+s}^{-m} \frac{1}{(1-z\cdot\bar{\zeta})^{\beta+n/p-t+m}} = \frac{1}{(1-z\cdot\bar{\zeta})^{\beta+n/p-t}}.$$

By Remark 3.6(a) and (b) we have

$$\begin{aligned} f(z) &= c\mathcal{R}_{s+m}^{-m}\mathcal{R}_{s+2m}^m f(z) \\ &= c\sum_1^\infty c_i \frac{(1-|\zeta^i|^2)^\beta}{(1-z\cdot\bar{\zeta}^i)^{\beta+n/p-t}}. \end{aligned}$$

Since $m > t$ was arbitrary, from (9) we have that

$$\beta > \max\left(\frac{n}{p}(p-1), 0\right). \tag{11}$$

Finally, (10) and (11) give the range for β.

The norm estimates follow easily. □

COROLLARY 3.9. *Let $s > -1$ and $\mu + s > -1$. Then*

$$\mathcal{R}_s^\mu : B_p^\nu \to B_p^{\nu-\mu}$$

is a continuous isomorphism onto.

Thus, if $\mu < 0$ then $\mathcal{R}_s^\mu$ is a fractional integration, and if $\mu > 0$ then it is a fractional derivation.

Proof. Simply use the atomic decomposition and compute. □

We conclude this section by extending a result of Zhu to the weighted Besov spaces on the unit ball. We prove that the Besov spaces B_p^ν ($1 \le p \le \infty$, $\nu \in \mathbf{R}$) can be seen as quotient spaces of the weighted L^p spaces. We introduce some more notation. For $s \in \mathbf{R}$, let dV_s denote the measure $(1-|z|^2)^s\, dV$. Moreover, we have the following definition.

DEFINITION 3.10. Let $s > -1$ and $\mu > 0$. On $A^{1,s}$ define the operators

$$\begin{aligned} V_{\mu,s} g(z) &= (1-|z|^2)^\mu \gamma_s \int_B \frac{(1-|w|^2)^s g(w)}{(1-z\cdot\bar{w})^{n+1+s+\mu}}\, dV(w) \\ &= (1-|z|^2)^\mu \mathcal{R}_s^\mu g(z). \end{aligned}$$

Also, when $\mu = 0$ we write the operators $\mathcal{R}_s^0$ as $\mathcal{P}_s$. That is, for $s > -1$ we have

$$\mathcal{P}_s f(z) = \gamma_s \int_B \frac{(1-|w|^2)^s}{(1-z\cdot\bar{w})^{n+1+s}} f(w)\, dV(w).$$

Now we are ready to state our theorem. Similar results, but with different techniques, are proved in [BB].

THEOREM 3.11. *Let $1 \le p \le \infty$. Then, for any $s > -1$,*

$$\mathcal{P}_s : L^p(dV_{-(\nu p+1)}) \to B_p^\nu$$

is a continuous projection onto. Moreover, if $t > \nu$ then

$$V_{t,s} : B_p \to L^p(dV_{-(p\nu+1)})$$

is a continuous embedding and

$$\mathcal{P}_s V_{t,s} = \gamma_{t+s} I.$$

COROLLARY 3.12. *For $1 \le p < \infty$, the Besov spaces B_p can be realized as quotient spaces of $L^p(d\Sigma)$.*

Proof of the theorem. Let $f \in L^p(dV_{-(\nu p+1)})$. Since for each fixed z $(|1-z\cdot\bar{w}|)^{-(n+1+s)}$ is bounded, $\mathcal{P}_s f(z)$ is well defined. If f has support away from the boundary then the estimate is trivial. Fix the size θ of a lattice as in Theorem 3.8. Then it suffices to check the boundedness of $\mathcal{P}_s$ for characteristic functions of balls in the Bergman metric with radius θ. Let $E \equiv E(\zeta,\theta)$ be the ball of center ζ and radius θ. Let $f = d\chi_E$, where χ_E is the characteristic function of the ball $E(\zeta,\theta)$ and $d=(1-|\zeta|^2)^{\nu-n/p}$, so that f has unit norm. Notice that, for $t>\nu$,

$$|R^t \mathcal{P}_s f(z)| \le c\frac{d(1-|\zeta|^2)^{n+1+s}}{(1-z\cdot\bar{\zeta})^{n+1+s+t}}.$$

Then

$$\|f\|^p_{B^\nu_p} = d^p(1-|\zeta|^2)^{(n+1+s)p}\int_B \frac{(1-|z|^2)^{(t-\nu)p-1}}{|1-z\cdot\bar{\zeta}|^{(n+1+s+t)p}}\,dV(z)$$

$$\le (1-|\zeta|^2)^{(s+\nu)p+(n+1)(p-1)}\int_B \frac{(1-|z|^2)^{(t-\nu)p-1}}{|1-z\cdot\bar{\zeta}|^{(n+1+s+t)p}}\,dV(z) \tag{12}$$

$$\le C,$$

where we have applied the Forelli–Rudin estimate (see [R, Thm. 1.4.10]). Then it follows that $\mathcal{P}_s$ is bounded from $L^p(dV_{-(\nu p+1)})$ into B^ν_p.

Conversely, let $f \in B^\nu_p$. Let $t>\nu$ and let

$$V_{t,s}f(z) = (1-|z|^2)^t \mathcal{R}^t_s f(z).$$

We have

$$\int_B (1-|z|^2)^{-\nu p-1}|V_{t,s}f(z)|^p\,dV(z) = \|f\|^p_{B^\nu_p}. \tag{13}$$

Again, notice that $V_{t,s}f=0$ implies $f=0$ and that

$$\mathcal{P}_s V_{m,s} = \frac{1}{\gamma_{m+s}} I.$$

Then $V_{m,s}$ is a continuous embedding, and we are done. □

4. Möbius Invariance of the Besov Spaces

In this section we prove two of the main results. First, we identify the minimal space $\mathfrak{M}$ with the 1-Besov space B_1. As a consequence of this and of the interpolation of Besov spaces B_p, $1\le p\le\infty$, we then prove that these are Möbius invariant according to Definition 1.6. In fact, we need only prove that the composition operators induced by the automorphisms are uniformly bounded in the norm (condition (3)).

THEOREM 4.1. *There exists a positive constant c such, that for all $f \in \mathfrak{M}$,*

$$\frac{1}{c}\|f\|_{\mathfrak{M}} \le \|f\|_{B_1} \le c\|f\|_{\mathfrak{M}};$$

that is, B_1 and $\mathfrak{M}$ can be identified as spaces.

Proof. First of all we want to show that there exists a positive constant c, depending only on the dimension n, such that for all $\tau \in \mathfrak{J}$ (see Definition 2.1)

$$\|\tau\|_{B_1} \le c.$$

Let λ be any complex number $|\lambda|<1$, and let $\varphi_\lambda \in \operatorname{Aut} B$ be defined by

$$\varphi_\lambda(z) = \left(\frac{\lambda - z_1}{1-\bar\lambda z_1}, -\frac{(1-|\lambda|^2)^{1/2} z_2}{1-\bar\lambda z_1}, \dots, -\frac{(1-|\lambda|^2)^{1/2} z_n}{1-\bar\lambda z_1}\right).$$

Set $\tau = v_1 \circ \varphi_\lambda$. Then

$$|\partial^{n+1}\tau(z)| = \left|(n+1)!\,\bar\lambda^n \frac{1-|\lambda|^2}{(1-\bar\lambda z_1)^{n+2}}\right|.$$

Hence,

$$\int_B |\partial^{n+1}\tau(z)|\,dV(z) \le c_n \int_B \frac{1-|\lambda|^2}{|1-\bar\lambda z_1|^{n+1}}\,dV(z)$$
$$\le c_n$$

by [R, 1.4.10]. Now let $k=2,\dots,n$. Put $\tau = v_k \circ \varphi_\lambda$. Then

(14)
$$\begin{aligned} |\partial^{n+1}\tau(z)| &= |\partial_1^{n+1}\tau(z)| + |\partial_1^n \partial_k \tau(z)| \\ &= \left|(n+1)!\,\bar\lambda^{n+1}\frac{(1-|\lambda|^2)^{1/2} z^k}{(1-\bar\lambda z_1)^{n+2}}\right| + \left|n!\,\bar\lambda^n \frac{(1-|\lambda|^2)^{1/2}}{(1-\bar\lambda z_1)^{n+1}}\right|. \end{aligned}$$

Notice that $|z_k| < (1-|z_1|^2)^{1/2}$. Therefore,

(15)
$$\left|\frac{(1-|\lambda|^2)^{1/2} z_k}{(1-\bar\lambda z_1)^{n+2}}\right| \le \sqrt{2}\,\frac{(1-|\lambda|^2)^{1/2}}{(1-\bar\lambda z_1)^{n+1+1/2}}.$$

Hence, from (14) and (15) and [R, 1.4.10] again, we obtain that

(16)
$$\begin{aligned} \int_B |\partial^{n+1}\tau(z)|\,dV(z) &\le c_n \int_B \frac{(1-|\lambda|^2)^{1/2}}{|1-\bar\lambda z_1|^{n+1+1/2}}\,dV(z) \\ &\le c_n. \end{aligned}$$

Next notice that, if U is any unitary transformation and $\varphi_\lambda \in \operatorname{Aut} B$ is as before, then for $j=1,\dots,n$ we have

$$\begin{aligned} \|v_j \circ (U \circ \varphi_\lambda)\|_{B_1} &= \int_B |\partial^{n+1}(v_j \circ (U\circ\varphi_\lambda))(z)|\,dV(z) + \sum_{|\alpha|\le n} |\partial^\alpha (v_j \circ (U \circ \varphi_\lambda))(0)| \\ &\le n \sum_{k=1}^{n} \int_B |\partial^{n+1}(v_k \circ \varphi_\lambda)(z)\,dV(z) + \sum_{|\alpha|\le n} |\partial^\alpha (v_k\circ\varphi_\lambda)(0)| \\ &\le c_n. \end{aligned}$$

Furthermore, for $f \in B_1$ and $U \in \mathfrak{U}$ we have

$$|\partial^{n+1}(f\circ U)(z)| \le c_n |\partial^{n+1} f(Uz)|.$$

Therefore,

$$\|f\circ U\|_{B_1}=\int_B|\partial^{n+1}(f\circ U)(z)|\,\mathrm{dV}(z)+\sum_{|\alpha|\le n}|\partial^\alpha(f\circ U)(0)|$$
$$\le c_n\|f\|_{B_1}.$$

Next, notice that any automorphism $\varphi\in\operatorname{Aut}B$ can be obtained as a composition

$$V\circ\varphi_\lambda\circ U,$$

where φ_λ is as above and U and V are unitary transformations. Hence, for any $\tau\in\mathfrak{J}$,

$$\|\tau\|_{B_1}\le c_n,$$

where c_n is a constant depending only on n.

Now let f be any function in $\mathfrak{M}$, and let $f=\sum_1^\infty c_i\tau_i$ be any representation of f. Then

$$\|f\|_{B_1}=\left\|\sum_1^\infty c_i\tau_i\right\|_{B_1}\le c_n\sum_1^\infty|c_i|. \tag{17}$$

Taking the infimum of the right-hand side of (17) over all representations of f in $\mathfrak{M}$, we obtain that

$$\|f\|_{B_1}\le c_n\|f\|_{\mathfrak{M}}.$$

Conversely, let $f\in B_1$. We want to show that $f\in\mathfrak{M}$ and that

$$\|f\|_{\mathfrak{M}}\le c\|f\|_{B_1}$$

for some constant c independent of f. Since $f\in B_1$, we have that $\mathcal{R}^{n+1}f\in L^1(\mathrm{dV})$. Moreover, by Remark 3.6 it follows that

$$f(z)=\int_B\frac{(1-|w|^2)^{n+1}\mathcal{R}^{n+1}f(w)}{(1-z\cdot\bar w)^{n+1}}\,\mathrm{dV}(w).$$

From [R, 2.2.2] it follows that

$$(1-\varphi_w(z)\cdot\bar w))^{n+1}=\left(\frac{1-|w|^2}{1-z\cdot\bar w}\right)^{n+1}$$

Consider the mapping $h:B\to\mathfrak{M}$ defined by $h(\eta)=h_\eta$, where $h_\eta(\xi)\equiv(1-\xi\cdot\bar\eta)^{n+1}$. It is easy to see that h is continuous from B into $\mathfrak{M}$. Then the mapping

$$\begin{aligned}H:B\times\operatorname{Aut}B&\to\mathfrak{M}\\(\eta,\psi)&\mapsto h_\eta\circ\psi\end{aligned}$$

is continuous. Then by Proposition 1.10 we obtain that, for all finite Borel measures $d\tilde\nu$ on $B\times\operatorname{Aut}B$, the function

$$\int_{B\times\operatorname{Aut}B}H(\eta,\psi)\,d\tilde\nu(\eta,\psi)$$

belongs to $\mathfrak{M}$. Identify φ_w with $w\in B$. Set

$$d\tilde\nu(\eta,\varphi_w)\equiv d\tilde\nu(\eta,w)=d\delta_w(\eta)\mathcal{R}^{n+1}f(w)\,\mathrm{dV}(w),$$

where δ_w is the Dirac delta at w. Then

$$\begin{aligned}\int_{B\times \mathrm{Aut}\,B} H(\eta,\varphi_w)\,d\tilde{\nu}(\eta,\varphi_w) &= \int_{B\times B}(1-\varphi_w\cdot\bar{\eta})^{n+1}\,d\delta_w(\eta)\,\mathcal{R}^{n+1}f(w)\,\mathrm{dV}(w)\\ &= \int_B (1-\varphi_w\cdot\bar{w})^{n+1}\mathcal{R}^{n+1}f(w)\,\mathrm{dV}(w)\\ &= f.\end{aligned}$$

Moreover,

$$\begin{aligned}\|f\|_{\mathfrak{M}} &\le \sup_{(\eta,w)\in B\times B}\|H(\eta,\varphi_w)\|_{\mathfrak{M}}\|\tilde{\nu}\|\\ &\le \sup_{\eta\in B}\|h_\eta\|_{\mathfrak{M}}\|f\|_{B_1}\\ &\le c\|f\|_{B_1}.\end{aligned}$$

This concludes the proof. □

COROLLARY 4.2. *The* 1-*Besov space* B_1 *is a Möbius invariant space* (*in the sense of Definition 1.6*).

Our next goal is to interpolate between B_1 and B_∞. We use our Theorem 3.11 and complex interpolation (see [BL] for notation). We actually just refer to Zhu's proof in the 1-dimensional case (see [Z3, Thm. 5).

THEOREM 4.3. *Let* $1\le p_0<p<p_1\le\infty$ *and* $1/p=(1-\theta)/p_0+\theta/p_1$ *with* $0<\theta<1$. *Then*

$$[B_p, B_{p_1}]_\theta \cong B_p.$$

Proof. The very same argument as in [Z3, Thm. 5] applies here. □

COROLLARY 4.4. *For* $1\le p<\infty$, *the* B_p *spaces are Möbius invariant.*

Proof. This follows at once. We actually interpolate on functions modulo constants. Notice that, for f holomorphic across the boundary and $\varphi\in \mathrm{Aut}\,B$,

$$\|f\circ\varphi\|_{\mathcal{B}} = \|f\|_{\mathcal{B}}$$

and

$$\|f\circ\varphi\|_{B_1}\le c\|f\|_{B_1}.$$

By interpolation for functions modulo constants,

$$\|f\circ\varphi\|_{B_p}\le c\|f\|_{B_p}$$

for $1\le p<\infty$, where c is independent of φ. □

5. The Invariant Inner Product and the Invariant Hilbert Space

It has been known for some time that there exists a unique Möbius invariant space. Peetre proved uniqueness in an unpublished note. Years later, Zhu

[Z1] reproved uniqueness and gave a power series description of the invariant inner product. In this section we show how the invariant inner product can be realized in a closed form on the 2-Besov space B_2. Consequently B_2 is the unique Möbius invariant Hilbert space. We also prove that the invariant inner product can be used to realize the dualities $\mathfrak{M}^* = \mathcal{B}/\mathbf{C}$ and $(\mathcal{B}_0/\mathbf{C})^* = \mathfrak{M}$. Such dualities give equality of norms, hence showing the naturalness of the Möbius invariant pairing of duality.

THEOREM 5.1 [Z1]. *Let H be a Hilbert space of holomorphic functions on the unit ball B. Suppose the polynomials are dense in H and that H has a Möbius invariant inner product, that is, $f \circ \varphi \in H$ for all $f \in H$, $\varphi \in \operatorname{Aut} B$, and*

$$\langle f \circ \varphi, g \circ \varphi \rangle = \langle f, g \rangle$$

for all $f, g \in H$, $\varphi \in \operatorname{Aut} B$.

Then H can be identified with

$$\left\{ f \in \mathcal{H}(B) \colon \text{if } f(z) = \sum_\alpha a_\alpha z^\alpha \text{ then } \sum |\alpha_\alpha|^2 \frac{\alpha! \, |\alpha|}{|\alpha|!} < \infty \right\},$$

and if $f = \sum_\alpha a_\alpha z^\alpha$ and $g = \sum_\beta b_\beta z^\beta \in H$ then

$$\langle f, g \rangle \equiv \sum_\alpha a_\alpha \bar{b}_\alpha \frac{\alpha! \, |\alpha|}{|\alpha|!}.$$

In the remainder of this paper, $\langle \cdot, \cdot \rangle$ will always indicate the invariant inner product defined above.

We now introduce one more variation of the radial derivative R. This new operator $\tilde{R}$ will define an equivalent norm on B_2 and at the same time be compatible with the invariant inner product.

DEFINITION 5.2. Define the differential operators $\tilde{R}^k$ for $k = 0, 1, 2, \dots$ by setting $\tilde{R}^0 = I$, $\tilde{R}^1 = R$, and for $k > 1$,

$$\tilde{R}^k = ((k-1)^{-1} R + I) \tilde{R}^{k-1}.$$

REMARK 5.3. (a) It is clear that if $f \in \mathcal{H}(B)$ with $f = \sum_{j=0}^\infty F_j$ its homogeneous expansion, then

$$\tilde{R}^k f = \sum_{j=1}^\infty \binom{k-1+j}{j} F_j.$$

(b) By Proposition 3.3 it follows that the norm on B_2 can be realized by setting

$$\|f\|_{B_2} = \left(\int_B ((1-|z|^2)^m |\tilde{R}^m f(z)|)^2 \, d\Sigma(z) \right)^{1/2} + |f(0)|$$

for any $m > n/2$. Moreover, it is well known that in the case of holomorphic functions it suffices to integrate outside the ball of radius r_0, for any $r_0 < 1$. As a consequence, the norm

$$\left(\int_B \left(\frac{(1-|z|^2)^n}{|z|^n}\right)^2 |\tilde{R}^n f(z)|^2 \, d\Sigma(z)\right)^{1/2} + |f(0)|$$

is equivalent to the one on B_2 (notice that $\tilde{R}^n f$ has a zero of first order at the origin, so there is no integrability problem there).

DEFINITION 5.4. Define an inner product $\langle\cdot,\cdot\rangle_2$ in B_2 by setting

$$\begin{aligned}\langle f,g\rangle_2 &= \int_B \frac{(1-|z|^2)^n}{|z|^n}\tilde{R}^n f(z)\frac{(1-|z|^2)^n}{|z|^n}\overline{\tilde{R}^n g(z)}\, d\Sigma(z)\\ &= \int_B \frac{(1-|z|^2)^{n-1}}{|z|^{2n}}\tilde{R}^n f(z)\overline{\tilde{R}^n g(z)}\, dV(z).\end{aligned}$$

THEOREM 5.5. *The 2-Besov space B_2 is the unique Möbius invariant Hilbert space and*

$$n\langle f,g\rangle_2 = \langle f,g\rangle$$

for all $f,g\in B_2$.

An immediate corollary is the following.

COROLLARY 5.6. *The inner product $\langle\, ,\rangle_2$ is Möbius invariant. Explicitly, for all $f,g\in B_2$ and all $\varphi\in \operatorname{Aut} B$ we have*

$$\langle f\circ\varphi, g\circ\varphi\rangle_2 = \langle f,g\rangle_2.$$

Proof of the theorem. Let $f,g\in\mathcal{H}(B)$, $f(z)=\sum_\alpha a_\alpha z^\alpha$, and $g(z)=\sum_\beta b_\beta z^\beta$. Then

$$\begin{aligned}\langle f,g\rangle_2 &= \int_B (1-|z|^2)^{n-1}\tilde{R}^n f(z)\overline{\tilde{R}^n g(z)}\frac{dV(z)}{|z|^{2n}}\\ &= \int_B (1-|z|^2)^{n-1}\sum_\alpha \binom{n-1+|\alpha|}{|\alpha|} a_\alpha z^\alpha \sum_\beta \binom{n-1+|\beta|}{|\beta|}\bar{b}_\beta \bar{z}^\beta \frac{dV(z)}{|z|^{2n}}\\ &= \sum_{|\alpha|\ge 1}\binom{n-1+|\alpha|}{|\alpha|}^2 a_\alpha \bar{b}_\alpha \frac{1}{2n}\int_0^1 (1-r^2)^{n-1} r^{2|\alpha|} r^{-1}\int_S |\zeta^\alpha|^2\, d\sigma(\zeta)\, dr\\ &= \frac{1}{n}\sum_{|\alpha|\ge 1}((n-1+|\alpha|)\cdots|\alpha|)^2 a_\alpha \bar{b}_\alpha \frac{\alpha!\,(|\alpha|-1)!}{((n-1+|\alpha|)!)^2}\\ &= \frac{1}{n}\sum_\alpha a_\alpha \bar{b}_\alpha \frac{\alpha!\,|\alpha|}{|\alpha|!}\\ &= \frac{1}{n}\langle f,g\rangle.\end{aligned}$$

□

DEFINITION 5.7. The *Möbius invariant pairing* on the unit ball is given by

$$\langle f,g\rangle = \lim_{r\to 1^-}\sum_\alpha r^{2|\alpha|} a_\alpha \bar{b}_\alpha \frac{\alpha!\,|\alpha|}{|\alpha|!},$$

where $f(z)=\Sigma_\alpha a_\alpha z^\alpha$ and $g(z)=\Sigma_\beta g_\beta z^\beta$. Also, by Definition 5.4 we have that

$$\langle f,g\rangle = n\lim_{r\to 1^-}\int_{|z|<r}(1-|z|^2)^{2n}\tilde{R}^n f(z)\overline{\tilde{R}^n g(z)}\frac{d\Sigma(z)}{|z|^{2n}}.$$

Notice that in general the invariant pairing must be taken as a limit.

In the rest of the paper, we will prove the following dualities.

(A) $\mathfrak{M}^*=\mathcal{B}/\mathbf{C}$.

(B) $(\mathcal{B}_0/\mathbf{C})^*=\mathfrak{M}$.

(C) For $p>1$, let p' be its conjugate exponent, $p'=p/(p-1)$. Then $B_p^*=B_{p'}$.

The dualities (A) and (B) have been proved by Arazy, Fisher, and Peetre [AFP1] in the case $n=1$. The duality in (C) is well known, even for $n>1$. The novelty here is in proving (A) and (B) for $n>1$, obtaining isometric equality in (A), and using the invariant pairing in all (A), (B), and (C) dualities.

REMARK 5.8. The reproducing kernel for the invariant Hilbert space is

$$K_2(z,w)=\log\frac{1}{(1-z\cdot\bar{w})}.$$

Indeed, the functions $\{z^\alpha\}_{|\alpha|>0}$ form an orthogonal basis. Since $\|z^\alpha\|^2=\alpha!/(|\alpha|-1)!$, it follows that

$$\begin{aligned}K_2(z,w)&=\sum_{|\alpha|>0}\frac{z^\alpha}{\|z^\alpha\|}\cdot\frac{\bar{w}^\alpha}{\|w^\alpha\|}\\&=\sum_1^\infty\frac{1}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}z^\alpha\bar{w}^\alpha\\&=\sum_1^\infty\frac{1}{k}(z\cdot\bar{w})^k\\&=\log(1-z\cdot\bar{w})^{-1}.\end{aligned}$$

We now prove two lemmas that will be needed to derive some new reproducing formulas.

LEMMA 5.9. *Let K_2 be as in Remark 5.8 and let $\tilde{R}^k$ be as in Definition 5.4 and act in the z-variables. Then, for $k=1,2,\dots,$*

$$\tilde{R}^k K_2(z,w)=\frac{1-(1-z\cdot\bar{w})^k}{(1-z\cdot\bar{w})^k}.$$

Proof. An easy induction argument shows that

$$\tilde{R}^k\log(1-z\cdot\bar{w})^{-1}=\frac{1}{(1-z\cdot\bar{w})^k}-1. \qquad\square$$

LEMMA 5.10. *Let $\varphi_\lambda\in\operatorname{Aut}B$ be as in the proof of Theorem 4.1. Let $\tau=v_1\circ\varphi_\lambda$ and $\sigma_j=v_j\circ\varphi_\lambda$ for $j=2,\dots,n$. Then, for $k=1,2,\dots,$ we have*

$$\tilde{R}^k \tau(z) = \frac{k(|\lambda|^2 - 1)z_1}{(1-\bar{\lambda}z_1)^{k+1}}$$

and

$$\tilde{R}^k \sigma_j(z) = \frac{-k(1-|\lambda|^2)^{1/2} z_j}{(1-\bar{\lambda}z_1)^{k+1}}.$$

Proof. These are just straightforward computations. □

PROPOSITION 5.11. *For all $f \in B_2$, the following reproducing formulas hold:*

$$f(z) = \int_B \frac{1-(1-z\cdot\bar{w})^n}{(1-z\cdot\bar{w})^n} \tilde{R}^n f(w) \frac{(1-|w|^2)^{n-1}}{|w|^{2n}} \, \mathrm{dV}(w)$$

and

$$\frac{\partial f}{\partial z_j}(z) = n \int_B \frac{1}{(1-z\cdot\bar{w})^{n+1}} \tilde{R}^n f(w) \frac{(1-|w|^2)^{n-1}}{|w|^{2n}} \, \mathrm{dV}(w).$$

Proof. The first statement is immediate from Definition 5.4 and Lemma 5.9. The second follows from the first. □

We are almost ready to prove our final result. We need one more definition.

DEFINITION 5.12. Let X be a Möbius invariant space. We say that X^* is the dual space of X if X^* is the set of linear functionals L on X such that

$$|L(f)| \le c\rho(f),$$

where c is independent of f.

THEOREM 5.13. *We have the following dualities:*

(A) $\mathfrak{M}^* = \mathcal{B}/\mathbf{C}$;

(B) $(\mathcal{B}_0/\mathbf{C})^* = \mathfrak{M}$.

Proof. Let $\varphi_{\lambda e_1} \equiv \varphi_\lambda \in \operatorname{Aut} B$ be as in the proof of Theorem 4.1, and let $g \in \mathcal{B}$ with $g(0) = 0$. By Lemma 5.10 and Proposition 5.11 we have

$$\begin{aligned} \langle v_1 \circ \varphi_\lambda, g\rangle &= n \int_B \frac{(|\lambda|^2-1)\bar{z}_1}{(1-\bar{\lambda}z_1)^{n+1}} \tilde{R}^n g(z) \frac{(1-|z|^2)^{n-1}}{|z|^{2n}} \, \mathrm{dV}(z) \\ &= (|\lambda|^2 - 1) \frac{\partial g}{\partial z_1}(\lambda e_1), \end{aligned} \tag{18}$$

and, for $j = 2, \ldots, n$,

$$\langle v_j \circ \varphi_\lambda, g\rangle = -(1-|\lambda|^2)^{1/2} \frac{\partial g}{\partial z_j}(\lambda e_1) \tag{19}$$

Recall that $\tilde{D}g(\zeta) = |\partial(g \circ \varphi_\zeta)(0)|$. By [R, 2.2.2],

$$\varphi'_\zeta(0) = -(1-|\zeta|^2)P_\zeta - (1-|\zeta|^2)^{1/2} Q_\zeta,$$

where P_ζ is the orthogonal projection onto the subspace generated by ζ, and $Q_\zeta = I - P_\zeta$. Then (18) and (19) give

$$|(\langle v_j\circ\varphi_{\lambda e_1}, g\rangle)_{j=1}^n| = \tilde{D}g(\lambda e_1).$$

Now let $\zeta\in B$ and $U_\zeta\in\mathcal{U}$ be such that $U_\zeta\zeta=\lambda e_1$, $|\lambda|<1$. Then

$$\begin{aligned}\tilde{D}g(\zeta) &= \tilde{D}(g\circ U_\zeta)(\lambda e_1)\\ &= |(\langle v_j\circ\varphi_\lambda, g\circ U_\zeta\rangle)_{j=1}^n|\\ &= |(\langle v_j\circ U_\zeta\circ\varphi_\zeta, g\rangle)_{j=1}^n|\\ &= |U_\zeta(\langle v_j\circ\varphi_\zeta, g\rangle)_{j=1}^n)\\ &= |(\langle v_j\circ\varphi_\zeta, g\rangle)_{j=1}^n|.\end{aligned}$$

Thus, for $g\in\mathcal{B}$, $g(0)=0$, we have

$$\sup_{\tau\in\mathfrak{J}}|\langle\tau, g\rangle|\le\|g\|_{\mathcal{B}}. \tag{20}$$

Next we want to show that actually equality holds in (20). Let $U=(u_{ij})\in\mathcal{U}$ be such that

$$U(\langle v_j\circ\varphi_\zeta, g\rangle)_{j=1}^n = |\langle v_j\circ\varphi_\zeta, g\rangle|e_1.$$

Then

$$\begin{aligned}\tilde{D}g(\zeta) &= |U(\langle v_j\circ\varphi_\zeta, g\rangle)_{j=1}^n|\\ &= \left|\left\langle\sum_j u_{1j}v_j\circ\varphi_\zeta, g\right\rangle\right|\\ &= |\langle v_1\circ U\circ\varphi_\zeta, g\rangle|.\end{aligned}$$

Thus we have shown that, for all $f\in\mathcal{B}$,

$$\sup_{\tau\in\mathfrak{J}}|\langle\tau, f\rangle| = \|f\|_{\mathcal{B}}.$$

Since $\mathfrak{M}$ distinguishes the polynomials, using the atomic decomposition of $\mathfrak{M}$ we obtain that

$$\mathcal{B}/\mathbf{C}\subseteq\mathfrak{M}^* \tag{21}$$

and

$$\mathfrak{M}\subseteq(\mathcal{B}_0/\mathbf{C})^*,$$

where the first inclusion (21) is an isometry.

In order to prove the reverse inclusion in (21), let $L\in\mathfrak{M}^*$. Since we are working on functions modulo constants, suppose that $L(1)=0$. Set $L(z^\alpha)=(\alpha!|\alpha|/|\alpha|!)\bar{f}_\alpha$ and define $f(z)=\sum_\alpha f_\alpha z^\alpha$. Since

$$|f_\alpha| = \frac{(|\alpha|-1)!}{\alpha!}|L(z^\alpha)| \le \frac{(|\alpha|-1)!}{\alpha!}\|L\|_{\mathfrak{M}^*}\|z^\alpha\|_{\mathfrak{M}} \le C\frac{|\alpha|!}{\alpha!}\|L\|_{\mathfrak{M}^*},$$

it follows that f is analytic in B. Next, let φ_λ be as before, with $\tau=v_1\circ\varphi_\lambda$. Then

$$\tau(z)=\tau(z_1)=(|\lambda|^2-1)\sum_1^\infty\bar{\lambda}^{k-1}z_1^k.$$

Therefore,

$$\overline{L(\tau)} = (|\lambda|^2-1)\sum_1^\infty \lambda^{k-1}\overline{L(z_1)^k}$$
$$= (|\lambda|^2-1)\sum_1^\infty \bar{\lambda}^{k-1} f_{ke_1}$$
$$= (|\lambda|^2-1)\partial_1 f(\lambda e_1).$$

Now let $\sigma(z) = v_j \circ \varphi_\lambda$, $j = 2, \dots, n$. Then

$$\overline{L(\sigma)} = (1-|\lambda|^2)^{1/2}\overline{L(z^2 \sum \bar{\lambda}^k z_1^k)}$$
$$= (1-|\lambda|^2)^{1/2} \sum \lambda^k \overline{L(z_2 z_1^k)}$$
$$= (1-|\lambda|^2)^{1/2} \sum \lambda^k \frac{k!\,(k+1)}{(k+1)!} f_{ke_1+e_2}$$
$$= (1-|\lambda|^2)^{1/2}\partial_1 f(\lambda e_1).$$

Thus,

$$|(L(v_j \circ \varphi_\lambda))_{j=1}^n| = \tilde{D}f(\lambda e_1).$$

By rotation invariance it follows that, for $\zeta \in B$,

$$|(L(v_j \circ \varphi_\zeta))_{j=1}^n| = \tilde{D}f(\zeta).$$

Hence,

$$\|L\|_{\mathfrak{M}^*} \geq \|f\|_{\mathcal{B}}.$$

On the other hand,

$$\|L\|_{\mathfrak{M}^*} \leq \sup_{|\zeta|\leq 1;\, j=1,\dots,n} |L(v_j \circ \varphi_\zeta)|$$
$$\leq \sup_{|\zeta|\leq 1} |\tilde{D}f(\zeta)|$$
$$\leq \|f\|_{\mathcal{B}}.$$

Let $p(z) = \sum_{|\alpha|\leq N} c_\alpha z^\alpha$ be a polynomial. Then

$$\langle p, f\rangle = \sum_{|\alpha|\leq N} c_\alpha \frac{\alpha!\,|\alpha|}{|\alpha|!}\bar{f}_\alpha$$
$$= \sum_{|\alpha|\leq N} c_\alpha L(z^\alpha)$$
$$= L(p).$$

Thus, $\mathfrak{M}^* = \mathcal{B}/\mathbf{C}$ isometrically. Finally, the same argument as in [AFP1] shows that $\mathfrak{M} = (\mathcal{B}_0/\mathbf{C})^*$. □

PROPOSITION 5.14. *For p and p' conjugate exponents, we have*

$$(B_p)^* = B_{p'}.$$

Proof. We use the invariant pairing of duality. Let $f \in B_p$, $g \in B_{p'}$. Write

$$n-1 = n - \frac{n+1}{p} + n - \frac{n+1}{p'}$$

and apply Hölder's inequality. □

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Dipartimento di Matematica
Politecnico di Torino
10129 Torino
Italy