On the Steenrod Homology Theory
of Compact Spaces

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1. Introduction

For compact metric spaces, Steenrod [31] defined homology groups based on "regular cycles." On the category $\mathbf{A}_{CM}$ of compact metric pairs the Steenrod homology theory H_* has many good properties. It satisfies the seven Eilenberg–Steenrod axioms; for a wide variety of coefficient groups it is isomorphic to the Čech homology theory; it is exact under every exact coefficient sequence; and it is proved in [25] that H_* has the following modified form of the continuity property. Let

$$(X_1, A_1) \leftarrow (X_2, A_2) \leftarrow \cdots$$

be an inverse sequence of compact metric pairs with the inverse limit (X, A). Then there is an exact sequence

$$0 \rightarrow \varprojlim_i{}^{(1)} H_{n+1}(X_i, A_i, G) \rightarrow H_n(X, A, G) \rightarrow \varprojlim_i H_n(X_i, A_i, G) \rightarrow 0$$

for every integer n and coefficient group G, where $\varprojlim^{(1)}$ is the first derived functor of $\varprojlim$.

The Steenrod homology has important applications in geometric topology and operator theory [9; 14]. The final argument in favour of the Steenrod homology is the well-known Steenrod duality theorem:

If A is an arbitrary closed subset of the sphere S^{n+1}, then for $0 < q < n$ the Steenrod homology group $H_q(A, G)$ is isomorphic to the Čech cohomology group $\check{H}^{n-q}(S^{n+1} \backslash A, G)$.

Note that Sitnikov [27] defined the Steenrod homology groups for metric spaces and extended the Steenrod duality theorem to arbitrary subsets of the sphere S^{n+1}.

The axiomatic characterization of the Steenrod homology theory on the category $\mathbf{A}_{CM}$ of compact metric pairs was obtained by Milnor [25]. Milnor characterized the Steenrod homology theory with the seven Eilenberg–Steenrod axioms together with the invariance axiom under a relative homeomorphism and the cluster axiom. Skljarenko [28] also obtained a characterization

of the Steenrod homology theory on the category $\mathbf{A}_{CM}$ of compact metric pairs.

In 1936, for locally compact spaces Kolmogorov [18] defined homology groups based on "functions of sets." These homology groups are closely connected with the Steenrod homology groups. In fact, in the case of a compact coefficient group, Chogoshvili [8] obtained an isomorphism of the Kolmogorov homology groups with the Čech homology groups (and therefore with the Steenrod homology groups for compact metric spaces) by using his exact homology theory constructed in [7; 8] for arbitrary topological spaces and based on "decompositions." Moreover, it is proved by Mdzinarishvili [24] that for an arbitrary coefficient group (and not only for a compact coefficient group) the construction of homology groups proposed by Kolmogorov in [18] gives homology groups isomorphic to the Steenrod homology groups on the category of compact metric spaces. Therefore, an exact homology theory isomorphic to the Steenrod homology theory was discovered by Kolmogorov (see the comment of Chogoshvili on the Kolmogorov homology theory in [19, pp. 405–411]). Later, many other exact homology theories were constructed on the category of compact pairs [25; 4; 12; 21; 20; 29; 23] which coincide with the Steenrod homology theory on the subcategory of compact metric pairs, except the Borel–Moore homology theory for arbitrary coefficient groups (it is so only for all finitely generated coefficient groups; see [28]). The isomorphism of some of such theories was proved in [5; 21; 20; 29]. The first uniqueness theorem for exact homology theories on the category $\mathbf{A}_C$ of compact Hausdorff pairs was obtained by Berikashvili in [2; 3]. All the above-mentioned exact homology theories satisfy the axiomatic characterization given by Berikashvili. This uniquely defined homology theory on the category $\mathbf{A}_C$ will be called the Steenrod homology theory. In [3] it is proved that if a homology theory on the category $\mathbf{A}_C$ satisfies the seven Eilenberg–Steenrod axioms and the following three axioms, then it is naturally isomorphic to the Chogoshvili homology theory [7; 8]. The new axioms are as follows.

AXIOM A. *For every compact pair* $(X, A) \in \mathbf{A}_C$, *the projection* $(X, A) \to$ $(X/A, *)$ *induces an isomorphism* $H_*(X, A) \xrightarrow{\approx} H_*(X/A, *)$.

AXIOM B. *For every integer n and every inverse system of finite clusters of n-spheres* $\{(S_\alpha^n, \pi_\beta^\alpha)\}$, *where the map* π_β^α *carries each n-sphere either into the point $*$ or homeomorphically onto an n-sphere, there is a natural isomorphism*

$$H_*(\varprojlim_\alpha \{(S_\alpha^n, *), \pi_\beta^\alpha\}) \to \varprojlim_\alpha \{H_*(S_\alpha^n, *), (\pi_\beta^\alpha)_*\}.$$

To formulate Axiom C, consider the compact filtered space $|N(X)|$ and the canonical map $\omega : |N(X)| \to X$ which are associated with a given compact Hausdorff space X (see [3]). The space $|N(X)|$ is the inverse limit of the inverse system $\{|N_\alpha(X)|, \pi_\beta^\alpha\}$ of finite polyhedra, where each $|N_\alpha(X)|$ is the

realization of the nerve $N_\alpha(X)$ of the closed covering of X induced by a finite decomposition of the space X, and where π_β^α is the natural map $|N_\beta(X)| \to |N_\alpha(X)|$. Let $N_\alpha^p(X)$ be the p-dimensional skeleton of $N_\alpha(X)$.

AXIOM C. *For every compact Hausdorff space X, the homomorphism*

$$\varinjlim_{p} H_*(\varprojlim_{\alpha} \{|N_\alpha^p(X)|, \pi_\beta^\alpha\}) \to H_*(X)$$

induced by the map ω is an isomorphism.

In [13] we obtained a different characterization of the Steenrod homology theory on the category A_C of compact Hausdorff pairs. To the seven Eilenberg–Steenrod axioms we added Milnor's modified form of the continuity axiom.

SEMI-CONTINUITY AXIOM. *Let (X, A) be the inverse limit of an inverse system $\{(K_\alpha, L_\alpha), \pi_\beta^\alpha\}$ of finite polyhedra. Then there is a functorial exact sequence*

$$0 \to \varprojlim_{\alpha}{}^{(1)} H_{n+1}(K_\alpha, L_\alpha) \to H_n(X, A) \to \varprojlim_{\alpha} H_n(K_\alpha, L_\alpha) \to 0$$

for every integer n.

The object of this paper is to give (in Section 3) a new axiomatic characterization of the Steenrod homology theory on the category A_C of compact Hausdorff pairs and continuous maps, a characterization which uses the Eilenberg–Steenrod axioms, the continuity property, and the exactness property under exact coefficient sequences.

The novelty of our approach is that we consider (in Section 2) the homology theory as a bifunctor theory defined on the product of the categories of spaces and coefficient groups. The properties of the homology theory under coefficient groups can therefore be used to charaterize exact bifunctor homology theories. The characterization is simpler and is obtained together with Eilenberg–Steenrod axioms in the classical terms of exactness and continuity. Moreover, it is analogous to the well-known Eilenberg–Steenrod characterization of the Čech cohomology on the category of compact pairs. For the bifunctor homology theory we investigate also the Vietoris–Begle theorem to characterize the Steenrod and Čech bifunctor homology theories with the Vietoris property and the partial continuity property.

2. Exact Bifunctor Homology Theory

Let A denote an admissible category of pairs of spaces and continuous maps. Let G be an abelian category. It will be said that H_* is a bifunctor homology theory on the category A with coefficients in the abelian category G if, for every $n \in Z$, we have (1) a covariant bifunctor $H_n(X, A, G)$: $A \times G \to G$, and

(2) a boundary operator $\Delta_n: H_n(X, A, G) \to H_{n-1}(A, G)$ for every $(X, A) \in$ **A** and $G \in$ **G** which have the following properties:

(a) for every $G \in$ **G** the sequence $(H_*(-, G), \Delta_*)$ is a connected sequence of functors on **A**; and

(b) for every morphism $f: G \to G'$ of **G**, $(X, A) \in$ **A**, and $n \in Z$, the diagram

$$H_n(X, A, G) \xrightarrow{\Delta_n} H_{n-1}(A, G)$$
$$H_n(X,A,f) \downarrow \qquad\qquad \downarrow H_{n-1}(A,f)$$
$$H_n(X, A, G') \xrightarrow{\Delta_n} H_{n-1}(A, G')$$

is commutative.

The singular homology is a bifunctor homology theory with coefficients in the category of abelian groups on the categories of finite CW-complexes, of CW-complexes, and of arbitrary spaces. The Čech homology theory is a bifunctor homology theory with coefficients in the category of abelian groups on the category of compact pairs. The Steenrod homology theory is a bifunctor homology theory with coefficients in the category of abelian groups on the category of compact metric pairs. For the category **G** of coefficients we can also take the category of Λ-modules and the category of compact abelian groups.

A bifunctor homology theory H_* with coefficients in **G** satisfies the axioms of homotopy and excision if H_* satisfies these axioms for every $G \in$ **G**. It will be said that H_* satisfies the axiom of exactness (partial exactness) if H_* is an exact (partially exact) homology theory for every $G \in$ **G** and if, furthermore, we have a boundary operator $d_n: H_n(X, A, G_2) \to H_{n-1}(X, A, G_1)$ for every exact sequence $0 \to G_1 \to G \to G_2 \to 0$ in **G** and $(X, A) \in$ **A** which have the following properties:

(c) for every $(X, A) \in$ **A** the sequence $(H_*(X, A, -), d_*)$ is a connected sequence of functors on **G**;

(d) for every map $\varphi: (X, A) \to (Y, B)$ of **A**, exact sequence $0 \to G_1 \to G \to G_2 \to 0$ in **G**, and $n \in Z$, the diagram

$$H_n(X, A, G_2) \xrightarrow{d_n} H_{n-1}(X, A, G_1)$$
$$H_n(\varphi, G_2) \downarrow \qquad\qquad \downarrow H_{n-1}(\varphi, G_1)$$
$$H_n(Y, B, G_2) \xrightarrow{d_n} H_{n-1}(Y, B, G_1)$$

is commutative; and

(e) for every exact sequence $0 \to G_1 \to G \to G_2 \to 0$ in **G** and $(X, A) \in$ **A**, there is a long exact (partially exact) sequence

$$\cdots \to H_{n+1}(X, A, G_2) \xrightarrow{d_{n+1}} H_n(X, A, G_1) \to H_n(X, A, G)$$
$$\to H_n(X, A, G_2) \xrightarrow{d_n} H_{n-1}(X, A, G_1) \to \cdots.$$

The singular and Steenrod bifunctor homology theories are exact bifunctor homology theories. The Čech bifunctor homology theory with coefficients

in the category of compact abelian groups is an exact bifunctor homology theory on the category of compact pairs.

We shall say that H_* satisfies the axiom of dimension if H_* satisfies the dimension axiom for every $G \in \mathbf{G}$ and if, furthermore, for every point $*$ and $G \in \mathbf{G}$ we have an isomorphism $H_0(*, G) \xrightarrow{\approx} G$ compatible with respect to 1-point spaces and such that, for every morphism $f: G \to G'$ of $\mathbf{G}$, the diagram

$$
\begin{array}{ccc}
H_0(*, G) & \xrightarrow{\approx} & G \\
\downarrow & & \downarrow f \\
H_0(*, G') & \xrightarrow{\approx} & G'
\end{array}
$$

is commutative. A bifunctor homology theory H_* satisfies the Eilenberg–Steenrod axioms if H_* satisfies the axioms of homotopy, excision, exactness, and dimension.

We shall say that the bifunctor homology theory H_* is continuous on the category of compact pairs if, whenever a compact pair (X, A) is the inverse limit of an inverse system $\{(K_\alpha, L_\alpha), \pi_\beta^\alpha\}$ of pairs (K_α, L_α) of finite polyhedra, then the natural morphism

$$
H_*(X, A, G) \to \varprojlim_\alpha \{H_*(K_\alpha, L_\alpha, G), (\pi_\beta^\alpha)_*\}
$$

is an isomorphism for every $G \in \mathbf{G}$. H_* will be called partially continuous on the category of compact pairs if the continuity property is satisfied for inverse systems $\{(K_\alpha, L_\alpha), \pi_\beta^\alpha\}$ of pairs of finite polyhedra with simplicial maps π_β^α. On the category $\mathbf{A}_{CM}$ the continuity and partial continuity properties mean that these properties are satisfied for inverse sequences of pairs of finite polyhedra. The notion of partial continuity was introduced by Kaul [15] on the category of compact metric pairs for a fixed coefficient group.

The notion and properties of a bifunctor cohomology theory H^* on $\mathbf{A}$ with coefficients in $\mathbf{G}$ are defined analogously.

On the category $\mathbf{A}_p$ of paracompact pairs we shall now define an exact bifunctor homology theory $\bar{H}_*$ which is given in [12] and which is needed in Section 3 to prove the uniqueness Theorem 1.

Let (X, A) be a paracompact pair of spaces. Consider the set $\overline{\mathrm{Cov}}(X, A) = \{(U_\alpha, V_\alpha)\}$ of all locally finite multiplicative open coverings of (X, A) with the following order: $(U_\beta, V_\beta) \gg (U_\alpha, V_\alpha)$ if U_β and V_β are refinements of U_α and V_α (resp.) and if, furthermore, $u_\alpha \cap u_\beta \in U_\beta$ when $u_\alpha \in U_\alpha$, $u_\beta \in U_\beta$, and $u_\alpha \cap u_\beta \neq \emptyset$. The set $\overline{\mathrm{Cov}}(X, A)$ becomes a partially ordered directed set; it was introduced by Alexandrov in [1]. If $(U_\beta, V_\beta) \gg (U_\alpha, V_\alpha)$ then there is a canonical map $p_\beta^\alpha: (N(U_\beta), N(V_\beta)) \to (N(U_\alpha), N(V_\alpha))$ of their nerves by letting $p_\beta^\alpha(u_\beta) = \bigcap_{u_\alpha \supset u_\beta} u_\alpha$. It is obvious that p_α^α is an identity map for every (U_α, V_α) and that

$$
p_\gamma^\beta p_\beta^\alpha = p_\gamma^\alpha \quad \text{for} \quad (U_\gamma, V_\gamma) \gg (U_\beta, V_\beta) \gg (U_\alpha, V_\alpha).
$$

For every $(U_\alpha, V_\alpha) \in \overline{\mathrm{Cov}}(X, A)$, consider now the oriented co-chain complex (or the ordered co-chain complex) $C^*(N(U_\alpha), N(V_\alpha), Z)$ of the nerve

$(N(U_\alpha), N(V_\alpha))$ with integer coefficients. Let $C^*(X, A, Z)$ be the direct limit of the direct system $\{C^*(N(U_\alpha), N(V_\alpha), Z), q_\alpha^\beta\}$ of co-chain complexes, where q_α^β is induced by the map p_β^α. Take a projective resolution P of $C^*(X, A, Z)$ and finally, for every abelian group G, consider the chain complex

$$\bar{C}_*(X, A, G) = \operatorname{Hom}(P, G).$$

Then, by definition,

$$\bar{H}_*(X, A, G) = H_* \operatorname{Hom}(P, G);$$

it does not depend on the projective resolution P. A map $(X, A) \to (Y, B)$ of the category $\mathbf{A}_p$ induces a map $\overline{\operatorname{Cov}}(Y, B) \to \overline{\operatorname{Cov}}(X, A)$ and therefore a homomorphism $\bar{H}_*(X, A, G) \to \bar{H}_*(Y, B, G)$. The sequence of chain complexes

$$0 \to \bar{C}_*(A, G) \to \bar{C}_*(X, G) \to \bar{C}_*(X, A, G) \to 0$$

is exact, and therefore $\bar{H}_*$ is an exact homology theory for every G. For every integer n we have the functorial exact sequence

(1) $0 \to \operatorname{Ext}(\check{H}^{n+1}(X, A, Z), G) \to \bar{H}_n(X, A, G) \to \operatorname{Hom}(\check{H}^n(X, A, Z), G) \to 0.$

It follows easily that $\bar{H}_*$ satisfies the axioms of homotopy, excision, and dimension, since $\check{H}^*$ has these properties. Note that we have a negative homology group $\bar{H}_{-1}(X, A, G) \approx \operatorname{Ext}(\check{H}^0(X, A, Z), G)$.

For every paracompact pair (X, A) and every exact sequence $0 \to G_1 \to G \to G_2 \to 0$ of coefficient groups, we have the exact sequence of chain complexes

$$0 \to \bar{C}_*(X, A, G_1) \to \bar{C}_*(X, A, G) \to \bar{C}_*(X, A, G_2) \to 0.$$

Thus the homology theory $\bar{H}_*$ is also exact under exact coefficient sequences. Therefore $\bar{H}_*$ is an exact bifunctor homology theory on the category $\mathbf{A}_p$ of paracompact pairs which satisfies the Eilenberg–Steenrod axioms.

If (X, A) is a compact pair, then we have a natural direct construction of $\bar{H}_*(X, A, G)$ which does not use the hyperhomology. In fact, we may consider only the set $\overline{\operatorname{Cov}}_f(X, A)$ of finite multiplicative open coverings of (X, A) with the same order. Let $C_*(X, A, G)$ be the inverse limit of the inverse system

$$\{C_*(N(U_\alpha), N(V_\alpha), G), (p_\beta^\alpha)_*, (U_\alpha, V_\alpha) \in \overline{\operatorname{Cov}}_f(X, A)\}$$

of oriented chain complexes with coefficients in the abelian group G. It is clear that we have the isomorphism $C_*(X, A, G) \approx \operatorname{Hom}(C^*(X, A, Z), G)$. We claim that for every n the abelian group $C^n(X, A, Z)$ is free. To show this, consider a directed system $\{M_\alpha, s_\alpha^\beta\}$ of free Λ-modules M_α with finite base F_α. Assume that this directed system satisfies the following conditions: there exist maps $z_\beta^\alpha: F_\beta \to F_\alpha$ $(\beta > \alpha)$ which give an inverse system $\{F_\alpha, z_\beta^\alpha\}$ of finite sets such that for every pair $\beta > \alpha$ and $g_\alpha \in F_\alpha$ we have $s_\alpha^\beta(g_\alpha) = \sum_i g_\beta^i$, where $z_\beta^\alpha(g_\beta^i) = g_\alpha$ and $s_\alpha^\beta(g_\alpha) = 0$ if the set $(z_\beta^\alpha)^{-1}(g_\alpha)$ is empty. Kaup and Keane proved in [16] that the direct limit of such direct systems of finitely

generated free Λ-modules is free over Λ. Moreover, it is noted in [3] that the result of Kaup and Keane is also true when the condition $s_\alpha^\beta(g_\alpha) = \sum_i g_\beta^i$ is extended to the condition $s_\alpha^\beta(g_\alpha) = \sum_i \epsilon_i g_\beta^i$, where $\epsilon_i = \pm 1$. It is easy to see that the direct system

$$\{C^n(N(U_\alpha), N(V_\alpha), Z), q_\alpha^\beta, (U_\alpha, V_\alpha) \in \overline{\text{Cov}}_f(X, A)\}$$

of finitely generated free abelian groups satisfies these conditions and that therefore its direct limit $C^n(X, A, Z)$ is a free abelian group, as claimed (there exist other constructions of free co-chains for exact homology; see [7; 8; 25; 3; 30]). From this it follows that there is a natural isomorphism

$$\bar{H}_*(X, A, G) \approx H_*(C_*(X, A, G))$$

for every compact pair (X, A) and abelian group G. It also follows that $\bar{H}_{-1}(X, A, G) = 0$ for every compact pair (X, A). If I is an infinitely divisible abelian group, then for every compact pair (X, A) we have natural isomorphisms

$$\bar{H}_*(X, A, I) \approx \text{Hom}(\check{H}^*(X, A, Z), I) \approx \check{H}_*(X, A, I).$$

Therefore the homology theory $\bar{H}_*$ has the continuity property on the category of compact pairs for every infinitely divisible coefficient group. Finally, on the category $\mathbf{A}_{CM}$ of compact metric pairs, from the exact sequence (1) it follows easily that $\bar{H}_*$ satisfies the invariance axiom under relative homeomorphisms and the Milnor cluster axiom, since $\check{H}^*$ satisfies these axioms. Thus $\bar{H}_*$ is isomorphic to the Steenrod homology theory on the category $\mathbf{A}_{CM}$. Note that the homology theories defined in [18; 25; 4; 7; 8; 24; 21; 29; 23] are also exact bifunctor homology theories on the category of compact pairs with coefficients in the category of abelian groups.

3. Uniqueness Theorems

The main result of this section is the characterization of exact bifunctor homology theories on the category $\mathbf{A}_C$ of compact Hausdorff pairs.

THEOREM 1. *There exists one and only one exact bifunctor homology theory on the category* $\mathbf{A}_C$ *of compact Hausdorff pairs with coefficients in the category of abelian groups* (*up to natural equivalence*) *which satisfies the axioms of homotopy, excision, dimension, and continuity for every infinitely divisible group.*

Proof. Existence. The exact bifunctor homology theory constructed in Section 2 satisfies all conditions of Theorem 1.

Uniqueness. It will be proved that a bifunctor homology theory H_* which satisfies the conditions of the theorem is naturally isomorphic to our concrete bifunctor homology theory $\bar{H}_*$ (see Section 2).

It is easy to see that H_* satisfies the axioms of homotopy, excision, and dimension: Consider an injective resolution of the abelian group

(2) $0 \to G \to I \to J \to 0;$

the assertion then follows from exactness and from the axioms of homotopy, excision, and dimension for the infinitely divisible groups I and J. In particular, we obtain in this way isomorphisms for the inclusion maps and projection map:

$$i_0, i_1 \colon (X, A) \to (X \times [0, 1], A \times [0, 1]),$$

$$p \colon (X \times [0, 1], A \times [0, 1]) \to (X, A),$$

and the homotopy axiom follows immediately. Moreover, if I is an infinitely divisible abelian group, then it follows from the well-known Eilenberg-Steenrod theorem [10] that $H_*(-, I)$ is naturally isomorphic to the Čech homology theory $\check{H}_*(-, I)$.

Let (X, A) be a pair of compact Hausdorff spaces. Consider $\overline{\mathrm{Cov}}_f(X, A)$, the set of finite multiplicative open coverings of (X, A). For every $(U_\alpha, V_\alpha) \in \overline{\mathrm{Cov}}_f(X, A)$ let (K_α, L_α) be the realization of the nerve (N_α, Q_α) of the covering (U_α, V_α). We obtain an inverse system $\{(K_\alpha, L_\alpha), |p_\beta^\alpha|\}$ of pairs of finite polyhedra and simplicial maps, where p_β^α is defined in Section 2. Let $(K, L) = \varprojlim_\alpha \{(K_\alpha, L_\alpha), |p_\beta^\alpha|\}$. We define a continuous map $\vartheta \colon K \to X$ as follows: let $z = \{z_\alpha\} \in K$ and $z_\alpha \in K_\alpha = |N_\alpha|$; let $|\sigma_\alpha|$ be the smallest simplex in K_α containing z_α and let u_α be the carrier of σ_α. Since X is a compact Hausdorff space, the intersection $\bigcap_\alpha u_\alpha$ is a single point of X and we set $\vartheta(z) = \bigcap_\alpha u_\alpha$. The map ϑ is onto and $\vartheta^{-1}(A) = L$. Such spaces K associated with a given compact space X were used in [15; 5; 2] to characterize homology theories. Note that for any $x \in X$ we have $\vartheta^{-1}(x) = \varprojlim_\alpha |\bar{\sigma}_\alpha(x)|$, where $\bar{\sigma}_\alpha(x)$ denotes the simplex of N_α corresponding to all members of the covering U_α containing x. Moreover, for every α, the diagram

$$
\begin{array}{ccc}
(K, L) & \xrightarrow{\;p_\alpha\;} & \\
{\scriptstyle \vartheta} \downarrow & \searrow & (K_\alpha, L_\alpha) \\
(X, A) & \xrightarrow{\;q_\alpha\;} &
\end{array}
$$

is commutative up to homotopy, where p_α is the projection and q_α is a canonical map. Thus, for every infinitely divisible group I, the diagram

$$
\begin{array}{ccc}
H_*(K, L, I) & \xrightarrow{\;H_*(p_\alpha, I)\;} & \\
{\scriptstyle H_*(\vartheta, I)} \downarrow & \searrow & H_*(K_\alpha, L_\alpha, I) \\
H_*(X, A, I) & \xrightarrow{\;H_*(q_\alpha, I)\;} &
\end{array}
$$

is commutative, where $\{H_*(p_\alpha, I)\}$ give an isomorphism by the continuity axiom and $\{H_*(q_\alpha, I)\}$ give also an isomorphism by the isomorphism $H_*(X, A, I) \approx \check{H}_*(X, A, I)$ (because the inverse system $\{(K_\alpha, L_\alpha), |p_\beta^\alpha|\}$ is associated with (X, A); see [26]). Therefore, $H_*(\vartheta, I)$ is an isomorphism for every infinitely divisible abelian group I. It follows from this, and from the commutativity of the following diagram with exact rows,

$$\cdots \to H_{n+1}(K,L,I) \to H_{n+1}(K,L,J) \to H_n(K,L,G) \to H_n(K,L,I) \to H_n(K,L,J) \to \cdots$$

$$\downarrow \approx \qquad \downarrow \approx \qquad \downarrow H_n(\vartheta,G) \qquad \downarrow \approx \qquad \downarrow \approx$$

$$\cdots \to H_{n+1}(X,A,I) \to H_{n+1}(X,A,J) \to H_n(X,A,G) \to H_n(X,A,I) \to H_n(X,A,J) \to \cdots,$$

which is obtained for the exact coefficient sequence (2), that $H_*(\vartheta, G)$ is an isomorphism for every abelian group G. Let K^p be the inverse limit of the inverse system $\{K_\alpha^p, |p_\beta^\alpha|\}$, where K_α^p denotes the p-dimensional skeleton of the polyhedron K_α. It will be proved that, for every abelian group G, the following equalities hold:

(3)
$$H_p(K^n \cup L, K^{n-1} \cup L, G) = 0 \quad \text{if } p \neq n,$$
$$H_p(K^n \cup L, L, G) = 0 \quad \text{if } p > n.$$

For the pair $(K^n \cup L, K^{n-1} \cup L)$ and for the exact sequence (2), we have the exact sequence

(4)
$$\cdots \to H_{p+1}(K^n \cup L, K^{n-1} \cup L, I) \to H_{p+1}(K^n \cup L, K^{n-1} \cup L, J)$$
$$\to H_p(K^n \cup L, K^{n-1} \cup L, G) \to H_p(K^n \cup L, K^{n-1} \cup L, I)$$
$$\to H_p(K^n \cup L, K^{n-1} \cup L, J) \to \cdots.$$

It is easy to see that $H_p(K^n \cup L, K^{n-1} \cup L, I) = 0$ for every infinitely divisible group I if $p \neq n$, since $H_*(-, I)$ is isomorphic to the Čech homology theory with coefficients in I. Therefore, it follows from (4) that

$$H_p(K^n \cup L, K^{n-1} \cup L, G) = 0 \quad \text{if } p > n \text{ or } p < n-1.$$

To prove that $H_{n-1}(K^n \cup L, K^{n-1} \cup L, G)$ is also trivial we consider the following commutative diagram with exact rows:

(5)
$$0 \to H_n(K^n \cup L, K^{n-1} \cup L, G) \to H_n(K^n \cup L, K^{n-1} \cup L, I) \to H_n(K^n \cup L, K^{n-1} \cup L, J)$$

$$\downarrow \qquad\qquad\qquad \downarrow \approx \qquad\qquad\qquad \downarrow \approx$$

$$\varprojlim_\alpha \Pi_{\sigma_{i,\alpha}^n} H_0(*, G) \to \varprojlim_\alpha \Pi_{\sigma_{i,\alpha}^n} H_0(*, I) \to \varprojlim_\alpha \Pi_{\sigma_{i,\alpha}^n} H_0(*, J)$$

$$\downarrow \approx \qquad\qquad\qquad \downarrow \approx \qquad\qquad\qquad \downarrow \approx$$

$$\varprojlim_\alpha \mathrm{Hom}(C^n(N_\alpha, Q_\alpha), G) \to \varprojlim_\alpha \mathrm{Hom}(C^n(N_\alpha, Q_\alpha), I) \to \varprojlim_\alpha \mathrm{Hom}(C^n(N_\alpha, Q_\alpha), J)$$

$$\searrow$$

$$\varprojlim_\alpha^{(1)} \mathrm{Hom}(C^n(N_\alpha, Q_\alpha), G),$$

where $\sigma_{i,\alpha}^n$ is an oriented n-simplex of $K_\alpha \setminus L_\alpha$ and the vertical arrows are isomorphisms. Since the abelian group $\varinjlim_\alpha C^n(N_\alpha, Q_\alpha, Z)$ is free (see Section 2), it is known [11] that in this case we have $\varprojlim_\alpha^{(1)} \mathrm{Hom}(C^n(N_\alpha, Q_\alpha, Z), G) = 0$. Thus, from the diagram (5) we obtain that the homomorphism

$$H_n(K^n \cup L, K^{n-1} \cup L, I) \to H_n(K^n \cup L, K^{n-1} \cup L, J)$$

is surjective and hence $H_{n-1}(K^n \cup L, K^{n-1} \cup L, G) = 0$ from (4). The triviality of the group $H_p(K^n \cup L, L, G)$ if $p > n$ for every abelian group G follows from the exact sequence

$$\cdots \to H_{p+1}(K^n \cup L, L, J) \to H_p(K^n \cup L, L, G) \to H_p(K^n \cup L, L, I) \to \cdots$$

and from the fact that $H_p(K^n \cup L, L, I)$ is trivial if $p > n$ for every infinitely divisible group I.

Consider now the chain complex

$$\Gamma_*(X, A, G) = \{H_p(K^p \cup L, K^{p-1} \cup L, G), \bar{\Delta}_p\},$$

where $\bar{\Delta}_p = \sigma_{p-1}\Delta_p$, $\Delta_p \colon H_p(K^p \cup L, K^{p-1} \cup L, G) \to H_{p-1}(K^{p-1} \cup L, L, G)$, and $\sigma_{p-1} \colon H_{p-1}(K^{p-1} \cup L, L, G) \to H_{p-1}(K^{p-1} \cup L, K^{p-2} \cup L, G)$ are homomorphisms in the exact sequences of homology groups for triples $(K^j \cup L, K^{j-1} \cup L, L)$. It is well known that there is a natural homomorphism

$$H_p \Gamma_*(X, A, G) \to H_p(K^n \cup L, L, G)$$

if $n > p$, which is an isomorphism if (3) holds. In fact, for every p we have the exact sequence

$$0 \to H_p(K^p \cup L, L, G) \xrightarrow{\sigma_p} H_p(K^p \cup L, K^{p-1} \cup L, G) \xrightarrow{\Delta_p} H_{p-1}(K^{p-1} \cup L, L, G),$$

which induces the isomorphism $\mathrm{Ker}(\sigma_{p-1}\Delta_p) \xrightarrow{\approx} H_p(K^p \cup L, L, G)$. The exact sequence

$$H_{p+1}(K^{p+1} \cup L, K^p \cup L, G) \to H_p(K^p \cup L, L, G) \to H_p(K^{p+1} \cup L, L, G) \to 0$$

shows that the composition

$$\mathrm{Ker}(\sigma_{p-1}\Delta_p) \to H_p(K^p \cup L, L, G) \to H_p(K^{p+1} \cup L, L, G)$$

induces a natural isomorphism $H_p \Gamma_*(X, A, G) \to H_p(K^{p+1} \cup L, L, G)$. Therefore we have the isomorphism $H_p \Gamma_*(X, A, G) \to H_p(K^{p+r} \cup L, L, G)$ for all $r \geq 1$ since, from $H_p(K^n \cup L, K^{n-1} \cup L, G) = 0$ if $n \neq p$, it follows that

$$H_p(K^{p+1} \cup L, L, G) \xrightarrow{\approx} H_p(K^{p+r} \cup L, L, G) \quad \text{for every } r > 1.$$

From diagram (5) it also follows that for every n we have an isomorphism $H_n(K^n \cup L, K^{n-1} \cup L, G) \to \varprojlim_\alpha \prod_{\sigma_{i,\alpha}^n} H_0(*, G)$. The composition

$$H_n(K^n \cup L, K^{n-1} \cup L, G) \to \varprojlim_\alpha \prod_{\sigma_{i,\alpha}^n} H_0(*, G) \to \varprojlim_\alpha \mathrm{Hom}(C^n(N_\alpha, Q_\alpha), G)$$

becomes an isomorphism and commutes with the boundary operators of the chain complexes $\Gamma_*(X, A, G)$ and $C_*(X, A, G)$. Therefore we obtain a natural isomorphism of chain complexes $\Gamma_*(X, A, G) \xrightarrow{\approx} \bar{C}_*(X, A, G)$.

Finally, consider the commutative diagram with exact rows

$$H_{p+1}(K^n \cup L, L, I) \to H_{p+1}(K^n \cup L, L, J) \to H_p(K^n \cup L, L, G) \to H_p(K^n \cup L, L, I) \to H_p(K^n \cup L, L, J)$$

(6) $\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$

$$H_{p+1}(K, L, I) \quad \to \quad H_{p+1}(K, L, J) \quad \to \quad H_p(K, L, G) \quad \to \quad H_p(K, L, I) \quad \to \quad H_p(K, L, J).$$

For every infinitely divisible group I, from the continuity of H_* we have the isomorphism $H_*(K, L, I) \xrightarrow{\approx} \varprojlim_\alpha H_*(K_\alpha, L_\alpha, I)$ and therefore, if $p \leq n$, we have the natural isomorphism $H_p(K^{n+1} \cup L, L, I) \approx H_p(K, L, I)$ for every

infinitely divisible group I. Thus, from diagram (6) we obtain that the homomorphism $H_p(K^n \cup L, L, G) \to H_p(K, L, G)$ is an isomorphism for every abelian group G if $n \geq p + 2$.

We have obtained a sequence of isomorphisms

$$\bar{H}_p(X, A, G) \overset{\approx}{\to} H_p \Gamma_*(X, A, G) \overset{\approx}{\to} H_p(K, L, G) \overset{\approx}{\to} H_p(X, A, G).$$

The composition of these isomorphisms establishes a natural isomorphism $\alpha_p(X, A, G) \colon \bar{H}_p(X, A, G) \overset{\approx}{\to} H_p(X, A, G)$ for every abelian group G and all p. A map $f \colon (X, A) \to (X', A')$ of compact Hausdorff pairs induces the commutative diagram

$$
\begin{array}{ccc}
(K, L) & \overset{\vartheta}{\longrightarrow} & (X, A) \\
\downarrow & & \downarrow f \\
(K', L') & \overset{\vartheta'}{\longrightarrow} & (X', A').
\end{array}
$$

Hence isomorphisms $\alpha_p(-, G)$ $(p \geq 0)$ are compatible with maps of the category A_C. It is easy to see that they are also compatible with the boundary operators Δ_p and d_p, and with homomorphisms of the category of abelian groups. This completes the proof of Theorem 1. $\square$

All exact homology theories constructed in [18; 25; 7; 8; 12; 24; 21; 29; 23] give exact bifunctor homology theories which satisfy the conditions of Theorem 1 and are therefore isomorphic on the category of compact Hausdorff pairs.

It is an immediate and trivial consequence of the well-known uniqueness theorem of Eilenberg–Steenrod for the Čech cohomology theory [10] that if an exact bifunctor cohomology theory H^* on the category A_C of compact Hausforff pairs with coefficients in the category of abelian groups satisfies the conditions of Theorem 1, then H^* is naturally isomorphic to the Čech bifunctor cohomology theory with coefficients in the category of abelian groups. Therefore, on the category of compact Hausdorff pairs, the characterization of the Steenrod bifunctor homology theory is completely analogous to the characterization of the Čech bifunctor cohomology theory.

COROLLARY 2.   *A bifunctor homology theory on the category of compact Hausdorff pairs is isomorphic to the Steenrod bifunctor homology theory if and only if it is an exact bifunctor homology theory which is naturally isomorphic to the Čech homology theory for every infinitely divisible abelian group.*

On the category A_{CM} of compact metric pairs this characterization of the Steenrod bifunctor homology theory is an easy consequence of Milnor's axiomatic characterization.

In order to characterize the Steenrod bifunctor homology theory with the partial continuity property, we shall need the Vietoris property for the bifunctor homology theory. The Vietoris–Begle theorem for the Borel–Moore

homology theory is proved in [6] when the coefficient group is the group of integers. On the category of compact metric pairs, the Vietoris–Begle theorem for the Steenrod homology theory with an arbitrary fixed coefficient group is obtained in [17] when the notion of the Vietoris map is formulated in terms of open coverings. From this it follows that the Vietoris–Begle theorem for the Steenrod homology theory is known when the coefficient group is finitely generated.

Let H_* be a bifunctor homology theory on an admissible category $\mathbf{A}$ of pairs of spaces and continuous maps with coefficients in an abelian category $\mathbf{G}$. A map $f: (X, A) \to (Y, B)$ of the category $\mathbf{A}$ is called a Vietoris map for the bifunctor homology theory H_* if (1) f is surjective; (2) $f^{-1}(B) = A$; and (3) for any $y \in Y$ and every $G \in \mathbf{G}$, the reduced homology $\tilde{H}_*(f^{-1}(y), G)$ is trivial.

For the Steenrod or Čech bifunctor homology theory on the category of compact pairs, a surjective map $f: (X, A) \to (Y, B)$ such that $f^{-1}(B) = A$ is a Vietoris map if the corresponding reduced homology of $f^{-1}(y)$ is trivial for any $y \in Y$ and for the coefficient groups Q and Q/Z. In fact, if

$$\bar{\tilde{H}}_*(f^{-1}(y), Q) = \bar{\tilde{H}}_*(f^{-1}(y), Q/Z) = 0,$$

then from the exactness of $\bar{H}_*$ we deduce $\bar{\tilde{H}}_*(f^{-1}(y), Z) = 0$ for any $y \in Y$. From the exact sequence (1) it follows that

$$\mathrm{Hom}(\check{\tilde{H}}^n(f^{-1}(y), Z), Z) = \mathrm{Ext}(\check{\tilde{H}}^n(f^{-1}(y), Z), Z) = 0$$

for every $n \geq 0$. It has been shown (see [6, §V, Prop. 13.7]) that this implies

$$\check{\tilde{H}}^n(f^{-1}(y), Z) = 0 \quad \text{for every } n \geq 0.$$

Thus from (1) we have $\bar{\tilde{H}}_n(f^{-1}(y), G) = 0$ for every n and for every abelian group G. If

$$\check{\tilde{H}}_*(f^{-1}(y), Q) = \check{\tilde{H}}_*(f^{-1}(y), Q/Z) = 0,$$

then

$$\bar{\tilde{H}}_n(f^{-1}(y), Q) = \bar{\tilde{H}}_n(f^{-1}(y), Q/Z) = 0$$

for every n. Therefore $\bar{\tilde{H}}_*(f^{-1}(y), G) = 0$. But there is a canonical epimorphism

$$\bar{\tilde{H}}_*(f^{-1}(y), G) \to \check{\tilde{H}}_*(f^{-1}(y), G).$$

Thus $\check{\tilde{H}}_*(f^{-1}(y), G)$ is trivial for every abelian group G. We see also that on the category of compact pairs a Vietoris map for the Steenrod bifunctor homology theory is a Vietoris map for the Čech bifunctor homology theory and conversely.

We shall say that a bifunctor homology theory H_* has the Vietoris property if for every Vietoris map $f: (X, A) \to (Y, B)$ of the category $\mathbf{A}$ the induced morphism $H_*(f, G): H_*(X, A, G) \to H_*(Y, B, G)$ is an isomorphism for every $G \in \mathbf{G}$.

LEMMA 3.

(1) *On the category* $\mathbf{A}_p$ *the bifunctor homology theory* $\bar{H}_*$ *(resp., on the categories* $\mathbf{A}_C$ *and* $\mathbf{A}_{CM}$ *the Steenrod bifunctor homology theory) with coefficients in the category of abelian groups has the Vietoris property.*

(2) *On the categories* $\mathbf{A}_C$ *and* $\mathbf{A}_{CM}$ *the Čech bifunctor homology theory with coefficients in the category of abelian groups has the Vietoris property.*

Proof. (1) Let $f: (X, A) \to (Y, B)$ be a Vietoris map for $\bar{H}_*$. For any $y \in Y$ and every abelian group G we have the exact sequence

$$0 \to \operatorname{Ext}(\tilde{\check{H}}^{i+1}(f^{-1}(y), Z), G) \to \tilde{\bar{H}}_i(f^{-1}(y), G)$$
$$\to \operatorname{Hom}(\tilde{\check{H}}^i(f^{-1}(y), Z), G) \to 0$$

for every $i \geq -1$. Thus $\operatorname{Hom}(\check{H}^i(f^{-1}(y), Z), I) = 0$ for every infinitely divisible group I and all $i \geq 0$. Therefore

$$\tilde{\check{H}}^i(f^{-1}(y), Z) = 0 \quad \text{for every } i \geq 0.$$

Since the Vietoris–Begle theorem holds for paracompact pairs and for the Čech cohomology theory, the map f induces an isomorphism $\check{H}^i(Y, B, Z) \xrightarrow{\approx} \check{H}^i(X, A, Z)$ for every $i \geq 0$. Consider now the commutative diagram with exact rows

$$0 \to \operatorname{Ext}(\check{H}^{i+1}(X, A, Z), G) \to \bar{H}_i(X, A, G) \to \operatorname{Hom}(\check{H}^i(X, A, Z), G) \to 0$$
$$\downarrow \approx \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow \approx$$
$$0 \to \operatorname{Ext}(\check{H}^{i+1}(Y, B, Z), G) \to \bar{H}_i(Y, B, G) \to \operatorname{Hom}(\check{H}^i(Y, B, Z), G) \to 0$$

induced by the map f, where the left and right vertical arrows are isomorphisms. It follows that the homomorphism $\bar{H}_i(X, A, G) \to \bar{H}_i(Y, B, G)$ is an isomorphism for all $i \geq -1$ and every abelian group G. For the categories $\mathbf{A}_C$ and $\mathbf{A}_{CM}$ the proof is similar. This ends the proof of (1). Note that if $f: (X, A) \to (Y, B)$ is a surjective map such that $f^{-1}(B) = A$ and the reduced homology $\tilde{\bar{H}}_*(f^{-1}(y), Z)$ is trivial for any $y \in Y$, then the induced homomorphism $\bar{H}_*(f, G): \bar{H}_*(X, A, G) \to \bar{H}_*(Y, B, G)$ is an isomorphism for every abelian group.

(2) Consider now the Čech bifunctor homology theory $\check{H}_*$ on the category $\mathbf{A}_C$ with coefficients in the category of abelian groups. Let $f: (X, A) \to (Y, B)$ be a Vietoris map for $\check{H}_*$. It follows from [22] that for a compact pair (X, A) and for every abelian group G there is a functorial exact sequence

$$0 \to \operatorname{Pext}(\check{H}^{i+1}(X, A, Z), G) \to \bar{H}_i(X, A, G) \to \check{H}_i(X, A, G) \to 0$$

for every $i \geq 0$, where $\operatorname{Pext}(X, Y)$ is the subgroup of pure extensions of the group $\operatorname{Ext}(X, Y)$ (on Pext see also [11]). Therefore, for every abelian group G and all $i \geq 0$, the map f induces the following commutative diagram with exact rows:

(7)
$$0 \to \operatorname{Pext}(\check{H}^{i+1}(X, A, Z), G) \to \bar{H}_i(X, A, G) \to \check{H}_i(X, A, G) \to 0$$
$$\downarrow f_{i+1}^{(1)} \qquad\qquad\qquad \downarrow f_i \qquad\qquad\qquad \downarrow \check{f}_i$$
$$0 \to \operatorname{Pext}(\check{H}^{i+1}(Y, B, Z), G) \to \bar{H}_i(Y, B, G) \to \check{H}_i(Y, B, G) \to 0.$$

Since the map f is also a Vietoris map for the Steenrod bifunctor homology theory $\bar{H}_*$, by (1) of Lemma 3 the homomorphism f_i is an isomorphism

for all $i \geq 0$ and every abelian group G. As proved in (1), the homomorphism $\check{H}^i(Y, B, Z) \to \check{H}^i(X, A, Z)$ is an isomorphism for all $i \geq 0$, and therefore $f_{i+1}^{(1)}$ is an isomorphism for all $i \geq 0$ and every abelian group G. It follows from diagram (7) that $\check{f}_i$ is also an isomorphism for all $i \geq 0$ and every abelian group G. This completes the proof of Lemma 3. $\qquad \square$

On the category $\mathbf{A}_{CM}$ of compact metric pairs, Kaul [15] characterized the Čech homology theory with the Eilenberg–Steenrod axioms, the property of partial continuity, and the Vietoris property, when the coefficient group is either an elementary compact topological group or a field. Lemma 3 gives the possibility of obtaining a similar characterization of the Steenrod and Čech bifunctor homology theories with coefficients in the category of abelian groups.

THEOREM 4.

(1) *On the categories* $\mathbf{A}_C$ *and* $\mathbf{A}_{CM}$, *there exists one and only one exact bifunctor homology theory with coefficients in the category of abelian groups* (*up to natural equivalence*) *which has the Vietoris property and satisfies the axioms of homotopy, excision, dimension, and partial continuity for every infinitely divisible group.*

(2) *On the categories* $\mathbf{A}_C$ *and* $\mathbf{A}_{CM}$, *there exists one and only one partially exact bifunctor homology theory with coefficients in the category of abelian groups* (*up to natural equivalence*) *which has the Vietoris property and satisfies the axioms of homotopy, excision, dimension, and partial continuity.*

Proof. By Lemma 3 the existence is clear. To prove the uniqueness, consider the canonical map $\vartheta : (K, L) \to (X, A)$ (see the proof of Theorem 1). In case (1), from the partial continuity it follows that

$$\tilde{H}_*(\vartheta^{-1}(x), I) \approx \varprojlim_{\alpha} \tilde{H}_*(|\bar{\sigma}_\alpha(x)|, I) = 0$$

for any $x \in X$ and every infinitely divisible group I. From the exactness of H_* under exact coefficient sequences we deduce that $\tilde{H}_*(\vartheta^{-1}(x), G) = 0$ for every abelian group G, and therefore the map ϑ is a Vietoris map for H_*. In case (2), the map ϑ is also a Vietoris map for H_*. Therefore, in both cases, from the Vietoris property it follows that the map ϑ induces an isomorphism $H_*(K, L, G) \xrightarrow{\approx} H_*(X, A, G)$ for every abelian group G. Since $(K, L) = \varprojlim_{\alpha}\{(K_\alpha, L_\alpha), |p_\beta^\alpha|\}$ is the inverse limit of pairs of finite polyhedra and simplicial maps, from the partial continuity property it follows that we have the isomorphism $H_*(K, L, G) \xrightarrow{\approx} \varprojlim_{\alpha} H_*(K_\alpha, L_\alpha, G)$ for every infinitely divisible group G in case (1) and for every abelian group G in case (2). For case (1) the rest of the proof is similar to the proof of Theorem 1, and it is more simple for case (2) (see [15]). $\qquad \square$

.REMARK. All results of this paper are true when the category $\mathbf{G}$ of coefficients is the category of modules over a principal ideal domain.

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