THE RANGE INCLUSION PROBLEM FOR ELEMENTARY OPERATORS

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Dedicated to the memory of Constantin Apostol

1. Introduction. Let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on a separable infinite-dimensional complex Hilbert space $\mathcal{H}$. Let $A=(A_1,\dots,A_n)$ and $B=(B_1,\dots,B_n)$ denote n-tuples of operators and let $R=R_{AB}$ denote the *elementary operator* on $\mathcal{L}(\mathcal{H})$ defined by

$$R(X)=\sum_{i=1}^{n} A_iXB_i.$$

Let $\mathcal{J}$ denote a 2-sided ideal of $\mathcal{L}(\mathcal{H})$ ($\mathcal{J}\neq\mathcal{L}(\mathcal{H})$). The purpose of this note is to draw attention to the range inclusion problem for elementary operators, which asks for a characterization of the structure of an elementary operator R_{AB} whose range is contained in $\mathcal{J}$,

$$\operatorname{Ran} R_{AB}\subset\mathcal{J}. \tag{1.1}$$

(We note that if $\mathcal{J}\neq\{0\}$, then it is impossible to achieve the identity $\operatorname{Ran} R_{AB}=\mathcal{J}$; this is because each nonzero ideal contains $\mathcal{F}$, the ideal of finite rank operators, and if $\mathcal{F}\subset\operatorname{Ran} R_{AB}$, then $\operatorname{Ran} R_{AB}=\mathcal{L}(\mathcal{H})$ [2, Thm. 2.3].)

It is easy to illustrate sufficient conditions for the range inclusion (1.1). If for each i, $A_i\in\mathcal{J}$ or $B_i\in\mathcal{J}$, then clearly (1.1) holds. This condition is not, however, necessary for range inclusion, as shown by the following.

EXAMPLE 1.1. For $1\le p\le\infty$, let $\mathcal{C}_p$ denote the Schatten p-ideal [5, p. 91]. Suppose $1<p,q<\infty$ with $1/p+1/q=1$, and let $A\in\mathcal{C}_p\backslash\mathcal{C}_1$ and $B\in\mathcal{C}_q\backslash\mathcal{C}_1$; then for every $X\in\mathcal{L}(\mathcal{H})$, $AXB\in\mathcal{C}_1$ [5, p. 92].

For $T\in\mathcal{L}(\mathcal{H})$, let $s(T)$ denote the sequence of *s-numbers* of T [5, p. 59]; in the case when T is compact, the s-numbers are the eigenvalues of $(T^*T)^{1/2}$ arranged in decreasing order and repeated according to multiplicity. For an ideal $\mathcal{J}$, let J denote the *ideal set* of $\mathcal{J}$ (see, e.g., [3; 6; 7; 8]); thus $T\in\mathcal{J}$ if and only if $s(T)\in J$ [3; 8]. For example, if $\mathcal{J}=\mathcal{C}_p$, then $J=l_p$ ($1\le p\le\infty$). The range inclusion (1.1) for $n=1$ has been characterized by Loebl and the author [3] as follows.

THEOREM 1.2 [3, Thm. 5.6]. *Let $A,B\in\mathcal{L}(\mathcal{H})$ and let $\mathcal{J}$ be an ideal of $\mathcal{L}(\mathcal{H})$. The* elementary multiplication *operator $S=S_{AB}$, defined by $S(X)=AXB$ ($X\in\mathcal{L}(\mathcal{H})$), satisfies* $\operatorname{Ran} S\subset\mathcal{J}$ *if and only if the product sequence $s(A)s(B)$ belongs to J.*

(Note that Example 1.1 follows from Theorem 1.2 and the fact that $l_p\cdot l_q\subset l_1$.)

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In the general case, let $S_i(X)=A_iXB_i$ $(1\le i\le n)$. Clearly, if $\operatorname{Ran} S_i\subset \mathfrak{J}$ for each i (as explained by Theorem 1.2), then $\operatorname{Ran} R\subset \mathfrak{J}$. Once again, this sufficient condition is not necessary, as the following example from [3] shows.

EXAMPLE 1.3. Let $\{e_n\}_{n=1}^{\infty}$ denote an orthonormal basis for $\mathcal{H}$ and let M and N denote the compact normal operators defined by $Me_n=(1/n^{1/2})e_n$ and $Ne_n=(1/n)e_n$. Let $A_1=M\oplus N$, $B_1=N\oplus M$, $A_2=-M\oplus 0$, $B_2=-N\oplus M$; let $R(X)=A_1XB_1+A_2XB_2$. Using a 2×2 operator matrix calculation and Theorem 1.2, it is not difficult to check that $\operatorname{Ran} R\subset \mathcal{C}_1$ while $\operatorname{Ran} S_i\not\subset \mathcal{C}_1$ $(i=1,2)$.

In this example, it is interesting to observe that R admits an alternate representation, $R=S_{A_1'B_1'}+S_{A_2'B_2'}$, where $\operatorname{Ran} S_{A_i'B_i'}\subset\mathcal{C}_1$ $(i=1,2)$; indeed, let $A_1'=2M\oplus 0$, $B_1'=N\oplus 0$, $A_2'=0\oplus N$, $B_2'=N\oplus M$, and $S_i'(X)=A_i'XB_i'$ $(i=1,2)$.

This example suggests consideration of the following possible properties of a given elementary operator $R=R_{AB}=\sum_{i=1}^{n}S_{A_iB_i}$:

$$R=\sum_{i=1}^{p}S_{A_i'B_i'}, \tag{1.2}$$

where for each i, $A_i'\in\mathfrak{J}$ or $B_i'\in\mathfrak{J}$;

$$R=\sum_{i=1}^{k}S_{A_i''B_i''}, \tag{1.3}$$

where for each i, $\operatorname{Ran} S_{A_i''B_i''}\subset\mathfrak{J}$;

$$\operatorname{Ran} R\subset\mathfrak{J}. \tag{1.4}$$

Clearly, (1.2) $\Rightarrow$ (1.3) $\Rightarrow$ (1.4), and we are interested in the extent to which reverse implications are possible.

QUESTION A. Does (1.4) $\Rightarrow$ (1.2)?

QUESTION B [3]. Does (1.4) $\Rightarrow$ (1.3)?

QUESTION C. Does (1.3) $\Rightarrow$ (1.2)?

The main result of this note implies a strong negative answer to Question C (and so also to Question A): in Theorem 2.1 we prove that if A and B are each linearly independent modulo the ideal $\mathfrak{J}$, then R_{AB} admits no representation of the form (1.2). As Example 1.1 shows, elementary operators R_{AB} exist with A,B each independent modulo $\mathfrak{J}$ and $\operatorname{Ran} R_{AB}\subset\mathfrak{J}$, so the negative answer to Question C follows. Thus the focus of the range inclusion problem shifts to Question B, which remains open; an affirmative answer to Question B, together with Theorem 1.2, would effectively solve the range inclusion problem.

Although (1.2) fails even for elementary multiplications S_{AB}, it does hold for S_{AB} in a strong sense if we restrict the ideal $\mathfrak{J}$: Loebl [7] calls an ideal $\mathfrak{J}$ *multiplicatively prime* if $A\in\mathfrak{J}$ or $B\in\mathfrak{J}$ whenever $\operatorname{Ran} S_{AB}\subset\mathfrak{J}$ (so that S_{AB} clearly satisfies (1.2)). In [7] Loebl showed that among the *norm ideals* of $\mathcal{L}(\mathcal{H})$ (in the sense of [5, Chap. 3] and [10]), the only multiplicatively prime ideals are $\{0\}$ and

$\mathcal{K}(\mathcal{H})$, the ideal of compact operators on $\mathcal{H}$. The ideal $\mathfrak{F}$ is also multiplicatively prime [7], and Loebl conjectured that among non-norm ideals it is the only multiplicatively prime ideal. In [6], Lin showed that there exist other non-norm multiplicatively prime ideals, for example, $\mathfrak{J} = \bigcup_{k=0}^{\infty} \mathcal{C}_{2^k}$. In Section 3 we include a result communicated to us by Salinas [9] which shows that an ideal is multiplicatively prime if and only if it is prime in the sense of [8]; thus multiplicatively prime ideals exist in abundance.

Note that an ideal $\mathfrak{J}$ is (multiplicatively) prime if and only if $A \in \mathfrak{J}$ whenever $AXB \in \mathfrak{J}$ for all $X \in \mathcal{L}(\mathcal{H})$ and $\{B\}$ is independent modulo $\mathfrak{J}$ (i.e., $B \notin \mathfrak{J}$!). We say that a (necessarily prime) ideal $\mathfrak{J}$ is *strongly prime* if it satisfies the following condition:

(1.5) If R_{AB} is an elementary operator with $\operatorname{Ran} R \subset \mathfrak{J}$, and if B is independent modulo $\mathfrak{J}$, then $A \subset \mathfrak{J}$.

We will show in Proposition 3.2 that an ideal $\mathfrak{J}$ is strongly prime if and only if R has some representation as in (1.2) whenever $\operatorname{Ran} R_{AB} \subset \mathfrak{J}$. We also say that an ideal $\mathfrak{J}$ is *strong* if each elementary operator R with $\operatorname{Ran} R \subset \mathfrak{J}$ has the structure of (1.3); thus from Proposition 3.2 it follows that an ideal is strongly prime if and only if it is both strong and prime.

Since (1.5) is apparently a much stronger condition than that defining a (multiplicatively) prime ideal, it is perhaps unclear that strongly prime ideals actually exist; however, the following results of Fong and Sourour [4] provide important examples of such ideals. The first result, which shows that $\{0\}$ is strongly prime, is the basic ingredient in the proofs of our results.

THEOREM 1.4 [4, Thm. I]. *If $\{B_1, \ldots, B_n\}$ is linearly independent and $R_{AB} = 0$, then each $A_i = 0$.*

THEOREM 1.5 [4, Thm. III]. *If $\{B_1, \ldots, B_n\}$ is independent modulo $\mathcal{K}(\mathcal{H})$ and* $\operatorname{Ran} R_{AB} \subset \mathcal{K}(\mathcal{H})$, *then each $A_i \in \mathcal{K}(\mathcal{H})$ (i.e., $\mathcal{K}(\mathcal{H})$ is strongly prime).*

For another example, a result of Apostol and the author [1, Prop. 5.2], together with Proposition 3.2, shows that $\mathfrak{F}$ is strongly prime (Corollary 3.3 below). The preceding results motivate the following question.

QUESTION D. Which prime ideals are strongly prime? Is *every* prime ideal strongly prime?

Although non-prime ideals cannot be strongly prime, there is a "mixed" analogue of primeness valid for arbitrary ideals.

THEOREM 1.6 [1, Cor. 3.5]. *Let $\mathfrak{J}$ be an ideal of $\mathcal{L}(\mathcal{H})$. If B is independent modulo $\mathcal{K}(\mathcal{H})$ and* $\operatorname{Ran} R_{AB} \subset \mathfrak{J}$, *then $A \subset \mathfrak{J}$.*

Returning to the range inclusion problem, note that Question B has an affirmative answer if and only if every prime ideal is strongly prime and every non-prime ideal is strong. For a prime ideal $\mathfrak{J}$, Question A has an affirmative answer if and only if $\mathcal{J}$ is strongly prime. The proofs of strong primeness for $\{0\}$, $\mathcal{K}(\mathcal{H})$, and $\mathcal{F}$ differ from one another considerably: the proof for $\{0\}$ uses only standard functional analysis, but is nontrivial; the proof for $\mathcal{K}(\mathcal{H})$ entails Voiculescu's theorem [11] (as does the proof of Theorem 1.6); the proof for $\mathcal{F}$ employs an intricate geometric construction. Thus it may be difficult to answer Question D, and it would be interesting merely to find additional examples of strongly prime ideals.

In a different direction, we note that Question A has an affirmative answer if we restrict the type of elementary operator under consideration. We say that an elementary operator R is *strongly representable* if, whenever $\mathcal{J}$ is an ideal with $\operatorname{Ran} R \subset \mathcal{J}$, then R admits a representation as in (1.2); R is *weakly representable* if, whenever $\operatorname{Ran} R \subset \mathcal{J}$, then R can be expressed as in (1.3). For $A, B \in \mathcal{L}(\mathcal{H})$, let T_{AB} denote the *generalized derivation* defined by $T_{AB}(X) = AX - XB$. A result of [3] shows that if $\operatorname{Ran} T_{AB} \subset \mathcal{J}$ then there exists $\lambda \in \mathbf{C}$ such that $A - \lambda \in \mathcal{J}$ and $B - \lambda \in \mathcal{J}$, and clearly $T_{AB} = T_{A-\lambda, B-\lambda}$; thus generalized derivations are strongly representable. In Section 4 we show that, if A and B each consist of mutually commuting positive compact operators, then R is weakly representable.

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2. On representations of elementary operators. Our main result, which provides negative answers to Questions A and C, is as follows.

THEOREM 2.1. *Let $\mathcal{J}$ be an ideal of $\mathcal{L}(\mathcal{H})$. Suppose A and B are each independent modulo $\mathcal{J}$. Then R_{AB} has no representation of the form $\sum_{j=1}^{m} S_{C_j D_j}$ where for each j, either $C_j \in \mathcal{J}$ or $D_j \in \mathcal{J}$.*

We require three preliminary lemmas. For $\mathcal{S} \subset \mathcal{L}(\mathcal{H})$, we denote the linear span of $\mathcal{S}$ by $\langle \mathcal{S} \rangle$.

LEMMA 2.2. *Given an elementary operator R_{AB}, where $A = \{A_1, \ldots, A_n\}$ and $B = \{B_1, \ldots, B_n\}$, assume that $B' = \{B_{n_1}, \ldots, B_{n_k}\} \subset B$ is independent. Then there exists an integer $p \geq k$ and an independent subset of B, $B'' = \{B_{n_1}, \ldots, B_{n_p}\}$ ($\supseteq B'$), such that for every X in $\mathcal{L}(\mathcal{H})$,*

$$R_{AB}(X) = \sum_{j=1}^{p} A_j' X B_{n_j},$$

where $A_j' \in \langle A_{n_j}, \{A_m\}_{m \neq n_i} \ (i = 1, \ldots, p) \rangle$.

Proof. Let $\{B_{n_1}, \ldots, B_{n_p}\}$ be a maximal independent subset of B containing $\{B_{n_1}, \ldots, B_{n_k}\}$. For $1 \leq j \leq n$, if $j \neq n_1, \ldots, n_p$ then there exist scalars $b_{j_1}, \ldots, b_{j_p}$ such that

$$B_j = b_{j1} B_{n_1} + \cdots + b_{jp} B_{n_p}.$$

Then, for $X \in \mathcal{L}(\mathcal{H})$,

$$\begin{aligned} R_{AB}(X) &= A_{n_1}XB_{n_1} + \cdots + A_{n_p}XB_{n_p} + \sum_{j \neq n_i} A_j XB_j \\ &= A_{n_1}XB_{n_1} + \cdots + A_{n_p}XB_{n_p} + \sum_{j \neq n_i} A_j X(b_{j1}B_{n_1} + \cdots + b_{jp}B_{n_p}) \\ &= \Big(A_{n_1} + \sum_{j \neq n_i} b_{j1}A_j\Big)XB_{n_1} + \cdots + \Big(A_{n_p} + \sum_{j \neq n_i} b_{jp}A_j\Big)XB_{n_p}. \end{aligned}$$

To complete the proof, let

$$A'_m = A_{n_m} + \sum_{j \neq n_i} b_{jm}A_j, \quad m = 1, \dots, p.$$

LEMMA 2.3. *Let $\mathcal{J}$ be an ideal of $\mathcal{L}(\mathcal{H})$. Suppose that for every $X \in \mathcal{L}(\mathcal{H})$,*

$$\sum_{i=1}^{n} A_i X B_i = \sum_{j=1}^{m} C_j X D_j,$$

where $\{B_1, \dots, B_n\}$ is independent modulo $\mathcal{J}$ and each $D_j \in \mathcal{J}$. Then $A_i = 0$, $i = 1, \dots, n$.

Proof. If each $D_j = 0$, the result follows from Theorem 1.4. We may thus assume some $D_j \neq 0$, so Lemma 2.2 implies that there exist an independent set $\{D'_1, \dots, D'_p\} \subset D$ and operators $\{C'_1, \dots, C'_p\} \subset \langle C \rangle$ such that

$$\sum_{j=1}^{m} C_j X D_j = \sum_{k=1}^{p} C'_k X D'_k, \quad X \in \mathcal{L}(\mathcal{H}).$$

Thus,

$$\sum_{i=1}^{n} A_i X B_i = \sum_{k=1}^{p} C'_k X D'_k, \quad X \in \mathcal{L}(\mathcal{H}).$$

Note that $B' = \{B_1, \dots, B_n, D'_1, \dots, D'_p\}$ is independent: for suppose $b_1, \dots, b_n$, $d_1, \dots, d_p$ are scalars with $b_1B_1 + \cdots + b_nB_n + d_1D'_1 + \cdots + d_pD'_p = 0$. Since each $D'_i \in \mathcal{J}$ and B is independent modulo $\mathcal{J}$, then each $b_i = 0$; since $\{D'_1, \dots, D'_p\}$ is independent, it follows that each $d_j = 0$. Since B' is independent, Theorem 1.4 implies that each $A_i = 0$; the proof is complete. □

LEMMA 2.4. *Given an elementary operator $R = R_{AB}$ and an ideal $\mathcal{J}$, assume that $\{B_1, \dots, B_k\}$ is a maximal independent subset of B modulo $\mathcal{J}$. Then R_{AB} admits a representation of the form*

$$R = \sum_{j=1}^{k} S_{C_jB_j} + \sum_i S_{D_iJ_i},$$

where each $J_i \in \mathcal{J}$. If, furthermore, for each $j > k$, either $B_j \in \mathcal{J}$ or $A_j \in \mathcal{J}$, then in the above representation each C_j can be chosen so that $C_j - A_j \in \mathcal{J}$.

Proof. Assume that $k < n$ and that $B_l \notin \mathcal{J}$ for some $l > k$. We can write

$$B_l = \sum_{j=1}^{k} c_j B_j + J$$

for some $c_1, \ldots, c_k \in \mathbf{C}$ and some $J \in \mathcal{J}$. Hence

$$R_{AB}(X) = \sum_{j=1}^{k} (A_j + c_j A_l) X B_j + A_l X J + \sum_{\substack{j>k \\ j \neq l}} A_j X B_j.$$

Thus, among B_j with $j > k$, the number of operators not in $\mathcal{J}$ has been reduced by one; moreover, under the additional hypothesis, $A_l \in \mathcal{J}$, so each $C_j \equiv A_j + c_j A$ satisfies $C_j - A_j \in \mathcal{J}$. □

Proof of Theorem 2.1. Assume to the contrary that

$$R_{AB} = \sum_{i=1}^{n} S_{A_i B_i} = \sum_{j=1}^{m} S_{C_j D_j},$$

where A and B are each independent modulo $\mathcal{J}$, while for each j, either C_j or D_j is in $\mathcal{J}$. By relabeling if necessary, we may assume $\{B_1, \ldots, B_n, D_1, \ldots, D_k\}$ is a maximal independent subset of $B \cup D$ modulo $\mathcal{J}$. By Lemma 2.4,

$$0 = \sum_{i=1}^{n} S_{A_i B_i} - \sum_{j=1}^{m} S_{C_j D_j}$$

admits a representation of the form

$$0 = \sum_{i=1}^{n} S_{A_i' B_i} - \sum_{j=1}^{k} S_{C_j' D_j} + \sum_{p} S_{L_p J_p},$$

where $A_i' - A_i \in \mathcal{J}$ for each i and $J_p \in \mathcal{J}$ for each p. Thus, by Lemma 2.3, each $A_i' = 0$, whence each $A_i \in \mathcal{J}$. This contradicts the assumption that $\{A_1, \ldots, A_n\}$ is independent modulo $\mathcal{J}$. □

3. Multiplicatively prime ideals. In this section we examine the range inclusion problem for multiplicatively prime ideals. If $\mathcal{J}$ and $\mathcal{L}$ are ideals, let $\mathcal{J}\mathcal{L} = \{JL : J \in \mathcal{J}, L \in \mathcal{L}\}$; for $\mathcal{S} \subset \mathcal{L}(\mathcal{H})$, let $(\mathcal{S})$ denote the 2-sided ideal generated by $\mathcal{S}$,

$$(\mathcal{S}) = \left\{ \sum_{i=1}^{n} A_i X_i B_i : n \geq 1,\ A_i, B_i \in \mathcal{L}(\mathcal{H}),\ X_i \in \mathcal{S} \right\}.$$

Note that if $\mathcal{M}$ is an ideal, then $\mathcal{J}\mathcal{L} \subset \mathcal{M}$ if and only if $(\mathcal{J}\mathcal{L}) \subset \mathcal{M}$. Recall from [8] that an ideal $\mathcal{J}$ is *semiprime* if, for every ideal $\mathcal{I}$ such that $\mathcal{I}\mathcal{I} \subset \mathcal{J}$, then $\mathcal{I} \subset \mathcal{J}$; $\mathcal{J}$ is *irreducible* if, given ideals $\mathcal{I}$ and $\mathcal{K}$ with $\mathcal{I} \cap \mathcal{K} \subset \mathcal{J}$, then $\mathcal{I} \subset \mathcal{J}$ or $\mathcal{K} \subset \mathcal{J}$. An ideal $\mathcal{J}$ is *prime* if, given ideals $\mathcal{I}$ and $\mathcal{K}$ with $\mathcal{I}\mathcal{K} \subset \mathcal{J}$, then $\mathcal{I} \subset \mathcal{J}$ or $\mathcal{K} \subset \mathcal{J}$. It is not difficult to check that $\mathcal{J}$ is prime if and only if it is semiprime and irreducible [8]; a characterization of prime ideals in terms of characteristic sequences is given in [8, Thm. 3.7].

The following result is due to N. Salinas, who has kindly allowed us to include it here; we have simplified part of the original proof somewhat. For $T \in \mathcal{L}(\mathcal{H})$, we denote $(\{T\})$ by (T).

THEOREM 3.1. *An ideal $\mathcal{J}$ is multiplicatively prime if and only if it is prime.*

Proof. Suppose $\mathcal{J}$ is prime and assume $AXB \in \mathcal{J}$ for every $X \in \mathcal{L}(\mathcal{H})$. Then $(YAX)(WBZ) = Y(AXWB)Z \in \mathcal{J}$ $(X, Y, Z, W \in \mathcal{L}(\mathcal{H}))$, whence $(A)(B) \subset \mathcal{J}$.

Since $\mathcal{J}$ is prime, $(A) \subset \mathcal{J}$ or $(B) \subset \mathcal{J}$; thus $A \in \mathcal{J}$ or $B \in \mathcal{J}$, so $\mathcal{J}$ is multiplicatively prime.

Conversely, suppose $\mathcal{J}$ is multiplicatively prime; we will show that $\mathcal{J}$ is semiprime and irreducible. Indeed, suppose $\mathcal{I}$ is an ideal and $\mathcal{I}\mathcal{I} \subset \mathcal{J}$; then for $A \in \mathcal{I}$ and $X \in \mathcal{L}(\mathcal{H})$, $AXA = A(XA) \in \mathcal{J}$. Since $\mathcal{J}$ is multiplicatively prime, then $A \in \mathcal{J}$; thus $\mathcal{I} \subset \mathcal{J}$, so $\mathcal{J}$ is semiprime. Next, suppose $\mathcal{I}$ and $\mathcal{K}$ are ideals such that $\mathcal{I} \cap \mathcal{K} \subset \mathcal{J}$. We seek to show that $\mathcal{I} \subset \mathcal{J}$ or $\mathcal{K} \subset \mathcal{J}$. Suppose to the contrary that there exist $A \in \mathcal{I} \backslash \mathcal{J}$ and $B \in \mathcal{K} \backslash \mathcal{J}$. For every $X \in \mathcal{L}(\mathcal{H})$, $AXB \in \mathcal{I} \cap \mathcal{K}$, whence $AXB \in \mathcal{J}$; since $\mathcal{J}$ is multiplicatively prime, $A \in \mathcal{J}$ or $B \in \mathcal{J}$, and this contradiction completes the proof. □

The next result provides several characterizations of strongly prime ideals and is thus helpful in studying Question D and the range inclusion problem for prime ideals.

PROPOSITION 3.2. *Let $\mathcal{J}$ be a prime ideal. Then the following are equivalent:*

(1) *Whenever R is an elementary operator with* $\operatorname{Ran} R \subset \mathcal{J}$, *then R has a representation as in* (1.2);

(2) *Whenever R is an elementary operator with* $\operatorname{Ran} R \subset \mathcal{J}$, *then R has a representation as in* (1.3);

(3) *Whenever $R = R_{AB}$ satisfies* $\operatorname{Ran} R \subset \mathcal{J}$, *then A or B is dependent modulo $\mathcal{J}$;*

(4) *$\mathcal{J}$ is strongly prime.*

REMARK. To see the content of Proposition 3.2, consider $\mathcal{J} = \{0\}$. It is obvious that $\mathcal{J}$ satisfies property (1); the fact that $\mathcal{J}$ satisfies (4) is Theorem 1.4.

Note also that if an ideal $\mathcal{J}$ satisfies (1), (3), or (4), then it is necessarily prime, so the hypothesis on $\mathcal{J}$ is reasonable.

Proof of Proposition 3.2. Observe that (1) $\Rightarrow$ (3) follows immediately from Theorem 2.1. (We do not know of a simpler proof in the present case when $\mathcal{J}$ is prime.) Also, since $\mathcal{J}$ is prime, the equivalence of (1) and (2) is trivial, as is the implication (4) $\Rightarrow$ (3). To complete the proof we will prove (4) $\Rightarrow$ (1) and (3) $\Rightarrow$ (4).

(4) $\Rightarrow$ (1). Assume R is an elementary operator with $\operatorname{Ran} R \subset \mathcal{J}$, where $\mathcal{J}$ is strongly prime. We seek a representation for R as in (1.2). By Lemma 2.4, R can be written as R_{CD} where $D \subset \mathcal{J}$, or as $R_{AB} + R_{CD}$, where B is independent modulo $\mathcal{J}$ and $D \subset \mathcal{J}$. In the latter case, clearly $\operatorname{Ran} R_{AB} \subset \mathcal{J}$, and since $\mathcal{J}$ is strongly prime, it follows that $A \subset \mathcal{J}$.

(3) $\Rightarrow$ (4). Suppose $A_1XB_1 + \cdots + A_nXB_n \in \mathcal{J}$ $(X \in \mathcal{L}(\mathcal{H}))$ and $\{B_1, \ldots, B_n\}$ is independent modulo $\mathcal{J}$. We seek to prove that each $A_i \in \mathcal{J}$. The proof is by induction on $n \geq 1$. Since $\mathcal{J}$ is multiplicatively prime, the result is clear for $n = 1$. In general, (3) implies that $\{A_1, \ldots, A_n\}$ is dependent modulo $\mathcal{J}$; we may thus assume there exist scalars $a_2, \ldots, a_n$ and $J \in \mathcal{J}$ such that

$$A_1 = a_2A_2 + \cdots + a_nA_n + J. \tag{3.2}$$

Thus, for each X,

$$A_2X(B_2+a_2B_1)+\cdots+A_nX(B_n+a_nB_1)\in\mathcal{J}.$$

Since $\{B_i+a_iB_1\}_{i=2}^n$ is independent modulo $\mathcal{J}$, then by induction $A_2\in\mathcal{J},\ldots,A_n\in\mathcal{J}$, whence (3.2) implies $A_1\in\mathcal{J}$. Thus $\mathcal{J}$ satisfies (4). □

The following result provides an affirmative answer to Question A for $\mathcal{J}=\{0\}$, $\mathcal{K}(\mathcal{H})$, or $\mathcal{F}$.

COROLLARY 3.3. *In each of the following cases, if R is an elementary operator and* Ran $R\subset\mathcal{J}$, *then $R=\sum S_{A_i'B_i'}$, where for each i, $A_i'\in\mathcal{J}$ or $B_i'\in\mathcal{J}$:*
(i) $\mathcal{J}=\{0\}$;
(ii) $\mathcal{J}=\mathcal{K}(\mathcal{H})$;
(iii) $\mathcal{J}=\mathcal{F}$.

Proof. The proof of (i) is trivial.
(ii) Theorem 1.5 shows that $\mathcal{K}(\mathcal{H})$ is strongly prime, so the result follows from Proposition 3.2 ((4) ⇒ (1)).
(iii) [1, Prop. 5.2] shows that $\mathcal{F}$ satisfies property (3) of Proposition 3.2. □

4. Positive coefficients. In this section we give an example of a weakly representable elementary operator.

PROPOSITION 4.1. *Let $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^n$ be sequences of commuting positive operators in $\mathcal{K}(\mathcal{H})$. Then R_{AB} is weakly representable; moreover, if* Ran R_{AB} *is contained in an ideal $\mathcal{J}$, then* Ran $S_{A_iB_i}\subset\mathcal{J}$ *for each i.*

Proof. It suffices to prove that Ran $S_{A_1B_1}\subset\mathcal{J}$. Since the A_i's are commuting and compact, there exists an orthonormal basis $\{e_m\}$ relative to which $A_ie_m=\alpha_{im}e_m$ with $\alpha_{im}\geq 0$, $i=1,\ldots,n$. Similarly, there exists an orthonormal basis $\{f_p\}$ such that $B_if_p=\beta_{ip}f_p$ with $\beta_{ip}\geq 0$, $i=1,\ldots,n$.

Let $\{S_k(A_1)\}_{k=1}^\infty$ denote the sequence of s-numbers of A_1; thus there is a sequence $\{m_k\}_{k=1}^\infty$ of distinct positive integers such that $s_k(A_1)=\alpha_{1,m_k}$ $(k\geq 1)$. Similarly, there is a sequence $\{p_k\}_{k=1}^\infty$ of distinct positive integers such that $s_k(B_1)=\beta_{1,p_k}$. Let $\mathfrak{N}=\langle\{f_{p_k}\}_{k=1}^\infty\rangle$. Define a partial isometry X such that $Xf_{p_k}=e_{m_k}$, $k\geq 1$, and $X=0$ on $\mathfrak{N}^\perp$. Then

$$\begin{aligned}R_{AB}(X)f_{p_k}&=\sum A_iXB_if_{p_k}\\&=\left(\sum_{i=1}^n\alpha_{i,m_k}\beta_{i,p_k}\right)e_{m_k},\quad k\geq 1,\end{aligned}$$

and $R_{AB}(X)=0$ on $\mathfrak{N}^\perp$.

Since $R_{AB}(X)\in\mathcal{J}$, then $\{\sum_{i=1}^n\alpha_{i,m_k}\beta_{i,p_k}\}_{k=1}^\infty\in J$ (the ideal set of $\mathcal{J}$ [3; 8]). Since

$$\alpha_{1,m_k}\beta_{1,p_k}\leq\sum_{i=1}^n\alpha_{i,m_k}\beta_{i,p_k},$$

then $s(A_1)s(B_1)=\{\alpha_{1,m_k}\beta_{1,p_k}\}\in J$, so Theorem 1.2 implies Ran $S_{A_1B_1}\subset\mathcal{J}$. The proof is complete. □

QUESTION 4.2. If A and B each consist of commuting compact normal operators, is R_{AB} weakly representable?

Example 1.3 offers some positive evidence concerning this question, but it also shows that we cannot expect Ran $S_{A_i B_i} \subset \mathcal{J}$ as in the positive case.

REFERENCES

1. C. Apostol and L. Fialkow, *Structural properties of elementary operators,* Canad. J. Math. 38 (1986), 1485–1524.
2. L. Fialkow, *Spectral properties of elementary operators,* Acta Sci. Math. (Szeged) 46 (1983), 269–283.
3. L. Fialkow and R. Loebl, *Elementary mappings into ideals of operators,* Illinois J. Math. 28 (1984), 555–578.
4. C. K. Fong and A. R. Sourour, *On the operator identity* $\sum A_k X B_k = 0$, Canad. J. Math. 31 (1979), 845–857.
5. I. C. Gohberg and M. G. Krein, *Introduction to the theory of linear nonselfadjoint operators,* Transl. Math. Monographs, 18, Amer. Math. Soc., Providence, R.I., 1969.
6. P. Lin, *A note on multiplicatively prime ideals of operators,* preprint.
7. R. Loebl, *A note on ideals of operators,* Proc. Amer. Math. Soc. 96 (1986), 62–64.
8. D. Morris and N. Salinas, *Semiprime ideals and irreducible ideals of the ring of bounded operators on Hilbert spaces,* Indiana Univ. Math. J. 23 (1974), 575–589.
9. N. Salinas, private communication.
10. D. Voiculescu, *A non-commutative Weyl–von Neumann theorem,* Rev. Roumaine Math. Pures Appl. 21 (1976), 97–113.

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