DENSE SUBSPACES OF ENTIRE FUNCTIONS

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1. Introduction and terminology. Suppose that $g(z)$ is an entire function and that T is a "suitable" linear operator. We indicate below which operators are suitable, but note for now that some suitable operators are $T(f(z))=f'(z)$ or $f(z+a)$ or $f'(z+a)$. Let M be the linear span of $\{T^k g\}_{k\geq 0}$. When $g(z)$ has no more than *minimal type* (that is, $|g(z)|=O(e^{A|z|})$ for all $A>0$) and $g(z)$ is not a polynomial, we show in §2 that M is dense in the natural topology on the space of entire functions of no more than minimal type. This implies that M is also dense in the space of all entire functions in the weaker topology of uniform convergence on compact sets. When $g(z)$ has exponential order less than 1, we show that M is dense in a stronger topology and that more operators are suitable. If $g(z)$ has no more than exponential type τ, we show in §3 by more elementary means that M is dense in the space of functions of exponential type no more than τ, unless $g(z)$ is a finite linear combination of functions of the form $z^i e^{a_j z}$ with $|a_j|\leq\tau$.

We say that a Frechét space topology on a vector space E of entire functions, or of formal power series, is *natural* if the coefficient projections $\sum_0^\infty c_n z^n \to c_n$ are all continuous. There may be many different countable collections of norms, each of which defines a natural Frechét topology on E, but the resulting topology is uniquely determined. In fact it follows easily from the closed graph theorem that if $E\subseteq F$ are Frechét spaces of entire functions or of formal power series and if F has a natural topology, then the topology on E is natural if and only if E is continuously imbedded in F. Thus, in particular, convergence of a sequence in a natural Frechét topology on a space of entire functions will always imply that the sequence also converges uniformly on compact sets. For us the most important natural Frechét spaces will be the Banach spaces $c_0(n!/w_n)$ of all functions $g(z)=\sum_0^\infty c_n z^n$ with $\lim(c_n n!/w_n)=0$ and norm $\|g(z)\|=\sup(|c_n|n!/w_n)$ (where $\{w_n\}$ is some sequence of positive numbers), and the Frechét space

$$E_\tau=\bigcap_{r>\tau} c_0\left(\frac{n!}{r^n}\right), \tag{1.1}$$

which is the space of entire functions of type no more than τ [6, Th. 4.13.1, p. 78].

We will prove the assertions given above about the space M by considering its annihilator $M^\perp$ in the dual space of the space of entire functions containing M, and showing that either $M^\perp$ is $\{0\}$ or has finite codimension. To do this we will represent the dual space as an algebra of formal or convergent power series in which $M^\perp$ is an ideal. This is most easily done by using the version of the Heaviside operational calculus given by Roman [11, pp. 6–17] as a preliminary to his

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study of the umbral calculus. In a subsequent paper we will show that the umbral operators [11, pp. 37–42] and related operators studied by Roman are bounded in appropriate norms, and use this fact to prove norm convergence in the expansion and polynomial expansion theorems [11, p. 18].

Following Roman [11, p. 12], we define the action of the formal power series $f(t)=\sum_0^\infty c_n t^n$ on polynomials by

$$f(t)(p(z))=\left(\sum_0^\infty c_n t^n\right)p(z)=\sum_0^\infty c_n p^{(n)}(z). \tag{1.2}$$

We use the same formula when $p(z)$ is not a polynomial, provided the series converges in the natural Frechét topology on $C[[z]]$; that is, the topology of convergence for each coefficient. The duality between formal power series in t and polynomials in z is defined so that formal power series multiplication by $f(t)$ is the adjoint of the action of $f(t)$ on polynomials [11, Th. 2.2.5, p. 13]. The explicit definition of this duality is [11, p. 6]

$$\left\langle \sum_{n=0}^\infty \frac{a_n t^n}{n!} \,\middle|\, \sum_{n=0}^\infty b_n z^n \right\rangle=\sum_0^\infty a_n b_n. \tag{1.3}$$

We use this formula whenever the series converges. This will occur precisely when the sequence $\{a_n\}_0^\infty$ is in the β dual [13, Def. 4.3.5, p. 62] of the set of $\{b_n\}$ which are the coefficients of the functions in the vector space under consideration. For the spaces we consider, it will be clear that this β dual can be identified with the usual dual space (cf. [13, Th. 7.2.9, p. 107]). In particular, whenever $\{w_n\}$ is a sequence of positive numbers we have

$$\ell^1(w_n)=c_0(n!/w_n)^*, \tag{1.4}$$

where $\ell^1(w_n)$ is the space of formal power series $f(t)=\sum_0^\infty c_n t^n$ for which the norm $\|f(t)\|=\sum |c_n| w_n$ is finite.

In §2, where we study functions of exponential order less than 1, the dual spaces will contain series with zero radius of convergence and will be studied by the methods of radical Banach algebras of power series (in particular from [2] and [3]). In §3, where we study functions of exponential type, the dual spaces are algebras of analytic functions. The functions of minimal type are a boundary case, and the same result appears as Theorem 2.1 and as the case $\tau=0$ of Corollary 3.3, though the proof in §2 actually shows density in a stronger topology.

The "suitable" operators mentioned at the beginning of this section are the operators $f(t)$ of formula (1.2) for which $c_1\neq 0$ and for which $f(t)$ satisfies a growth condition appropriate to the space under consideration. In particular $tg(z)=g'(z)$ is always suitable; $e^{at}g(z)=g(z+a)$ is suitable for all $a\neq 0$ for functions of no more than minimal type, and for sufficiently small a for functions of exponential type. For other examples of operators $f(t)$ see [11, pp. 14–15]. The operators $f(t)$ are precisely the linear operators that commute with translations [11, Cor. 2.2.9, p. 17].

2. Functions of order less than one or of minimal type. Using the terminology described in the previous section, we can now prove the promised result about functions of minimal type.

THEOREM 2.1. *Suppose that $g(z)$ is an entire function of at most minimal type but is not a polynomial. If $f(t)$ is a power series with positive radius of convergence and $f'(0) \neq 0$, then $M = \operatorname{span}\{f(t)^k g(z)\}_{k \geq 0}$ is dense in the natural Frechét topology on the space E_0 of functions of at most minimal type.*

Proof. Since M is unchanged by replacing $f(t)$ by $f(t) - f(0)$, we assume without loss of generality that $f(0) = 0$. Let $g(z) = \sum_0^\infty (c_n/n!) z^n$. Since $g(z)$ is of minimal type, it follows, from formula (1.1) with $\tau = 0$, that $\lim |c_n|^{1/n} = 0$. Let $d_n = n|c_n|$ so that $(c_n/d_n) \to 0$ and $\lim d_n^{1/n} = 0$.

Fix an integer $k > 0$. We show that there is a sequence $\{w_n\}$ of positive numbers for which $\lim w_n^{1/n} = 0$ and M is a subspace of $c_0(n!/w_n)$ with P_k, the space of polynomials of degree k or less, in the closure of M. It is clear from formula (1.1) that $c_0(n!/w_n) \subseteq E_0$, so that the closure of M in E_0 would then contain P_k. Since k is arbitrary this will show that the closure of M in E_0 contains all polynomials and hence that M is dense in E_0.

We now proceed to define the sequence $\{w_n\}$ by $(w_{n+k})^{1/n} = \sup_{j \geq n} (d_{j+k})^{1/j}$ for $n \geq 0$ and $w_n = d_n$ for $n < k$. Then $\lim_{n \to \infty} (w_{n+k})^{1/n} = \lim_{n \to \infty} (d_{n+k})^{1/n} = 0$ and $\{(w_{n+k})^{1/n}\}_{n \leq 0}^\infty$ is non-increasing, so $\{w_{n+k}\}_{n=0}^\infty$ is submultiplicative [5, Th. 7.2.4, p. 239] and thus $\ell^1(w_{n+k})$ is a radical Banach algebra with identity adjoined [2, Lemma (2.4), p. 643]. We also have $\ell^1(w_{n+k}) = \{f(t) \in C[[t]] : f(t)t^k \in \ell(w_n)\}$ (see [2, p. 645]), so that $\ell^1(w_n)$ is just the sum of $\ell^1(w_n)t^k$ plus the polynomials. Thus $\ell^1(w_n)$ is also a radical Banach algebra with identity adjoined (cf. [2, pp. 655–656]).

Since $d_n \leq w_n$, it follows that $g(z)$ and hence M is contained in $c_0(n!/w_n)$. Now let $M^\perp$ be the annihilator of M in $\ell^1(w_n)$. Since multiplication by $f(t)$ on $\ell(w_n)$ is the adjoint of the operator $f(t)$ on $c_0(n!/w_n)$, it follows that $f(t)M^\perp \subseteq M^\perp$. But $f(t)$ has positive radius of convergence, so that its compositional inverse $\hat{f}(t)$ also has positive radius of convergence. Hence the series obtained by substituting $f(t)$ for t in the series for $\hat{f}(t)$ converges in the norm of $\ell^1(w_n)$ to $\hat{f} \circ f = t$ (cf. [2, p. 643]). Thus $tM^\perp \subseteq M^\perp$ and hence $M^\perp$ is a closed ideal in $\ell^1(w_n)$.

We now show that $M^\perp \subseteq P_k^\perp$, which is the collection of series in $\ell^1(w_n)$ of the form $\sum_{n=k+1}^\infty a_n t^n$. If this were not so, $M^\perp$ would contain a series of the form $h(t) = t^k(1 + r(t))$, where $r(t)$ is a formal power series with 0 constant term. Then $1 + r(t)$ belongs to $\ell^1(w_{n+k})$, which is a radical algebra with identity adjoined, so that $(1 + r(t))^{-1}$ belongs to $\ell^1(w_{n+k})$ and hence $t^k(1 + r(t))^{-1}$ belongs to $\ell^1(w_n)$. Thus $t^{2k} = h(t)(t^k(1 + r(t))^{-1})$ belongs to the ideal $M^\perp$ which must then contain $P_{2k}^\perp$. Then $g(z) \in M^{\perp\perp} \subseteq P_{2k}$, contradicting the hypothesis that $g(z)$ is not a polynomial.

Now we have $M^\perp \subseteq P_k^\perp$ so that the closure of M in $c_0(n!/w_n)$, which is just $M^{\perp\perp}$, contains $P_k^{\perp\perp} = P_k$. This completes the proof. □

For functions of exponential order less than one, it is convenient to work with the spaces $c_0(n!^{1+\epsilon})$ for $\epsilon > 0$. The space $c_0(n!^{1+\epsilon})$ contains all functions of order less than $1/(1+\epsilon)$ and contains only functions of order less than or equal to $1/(1+\epsilon)$ [6, Th. 4.12.1, p. 74]. After the next theorem we will indicate other possible spaces we could use.

THEOREM 2.2. *Suppose that $g(z)$ belongs to $c_0(n!^{1+\epsilon})$ but is not a polynomial. If $f(t)$ is a series in $\ell^1(1/n!^{\epsilon})$ with non-zero coefficient of t, then $M = \operatorname{Span}\{f(t)^k g(z)\}_{k\geq 0}$ is dense in the Banach space $c_0(n!^{1+\epsilon})$.*

Proof. We let $M^{\perp}$ be the annihilator of M in the dual space $\ell^1(1/n!^{\epsilon})$ and prove that $M^{\perp}=\{0\}$, so that M is dense. As in the previous theorem, we may assume that $f(t)$ has zero constant term. This implies that $f(t)$ is an algebra generator of $\ell^1(w_n)$ [1; 3, pp. 41–42], so that $M^{\perp}$ is a closed ideal in $\ell^1(1/n!^{\epsilon})$. If $M^{\perp}$ were not $\{0\}$ it would contain some t^k [2, pp. 644–645] and hence $M^{\perp}\supseteq P_k^{\perp}$. Thus $g(z)\in M^{\perp\perp}\subseteq P_k^{\perp\perp}=P_k$, contradicting the hypothesis that $g(z)$ is not a polynomial. This completes the proof. □

Instead of considering $c_0(n!^{1+\epsilon})$, we could use $c_0(n!/w_n)$ for any $\{w_n\}$ for which nw_{n+k}/w_n is bounded for some k and $\ell^1(w_n)$ is an algebra with only the standard closed non-zero ideals $P_k^{\perp}$ ([3, pp. 41–42; 2; 4; 12]). In fact, under these hypotheses we could replace $c_0(n!/w_n)$ by $\ell^p(n!/w_n)$ for $1<p<\infty$, since the results in [2], [3], and [4] apply to $\ell^p(n!/w_n)^*=\ell^q(w_n)$ and even to more general sequence spaces.

3. Functions of exponential type. In this section we determine the closure of $f(t)$-invariant subspaces M of Frechét spaces of functions of exponential type. We cannot use the obvious space $c_0(n!/\tau^n)$ because, although much is known about the ideal structure of the dual space $\ell^1(\tau^n)$ [7, Chapter 11], the total picture is still unclear. If we consider $\ell^2(n!/\tau^n)$, its dual space $\ell^2(\tau^n)$ is just the Hardy space H^2 of the disc $D(\tau)$ of radius τ. Then $M^{\perp}$ is a closed $f(t)$-invariant subspace of $H^2(D(\tau))$. When $f(t)$ is an algebra generator of $\ell^1(\tau^n)$ (see [9] for sufficient conditions for this), then $M^{\perp}$ is t-invariant. Hence $M^{\perp}$ is all multiples of some inner function $\phi(t)$ and M is the null-space of $\phi(t)$, acting according to formula (1.4). Though this is a complete description of the closed $f(t)$-invariant subspaces of $\ell^2(n!/\tau^n)$, the description seems too abstract to be useful. For instance, for a fixed $g(z)$ it is hard to see how one could determine whether $\{t^k g(z)\}_{k\geq 0}$ has dense span. Instead we need to concentrate on Frechét spaces of entire functions of exponential type whose dual spaces are algebras of analytic functions with simple ideal structure. The best choice seems to be the spaces E_τ of entire functions of type no more than $\tau\geq 0$ (see formula (1.1) above). The dual space of E_τ is easily shown to be the space $H(\bar{D}(\tau))$ of functions analytic on a neighborhood of the closed disc $\bar{D}(\tau)$ of radius τ (just combine formula (1.3) above with [8, Prop. (6.5), p. 46]). For completeness we sketch a proof of the following simple lemma describing the ideals of $H(\bar{D}(\tau))$ (cf. [8, pp. 109–110]).

LEMMA 3.1. *The ideals in $H(\bar{D}(\tau))$ are all principal ideals generated by polynomials all of whose zeroes lie in the closed disc $\bar{D}(\tau)$.*

Proof. Let I be a non-zero ideal in $H(\bar{D}(\tau))$. Since a non-zero analytic function can have only a finite number of zeroes on the closed disc $\bar{D}(\tau)$, the set of common zeroes of I, counting multiplicity, is a finite set. In other words, there is a polynomial $p(t)$, with all of its zeroes in the closed disc, which is the greatest common divisor of the functions in I. To complete the proof, we must show that

$p(t)$ belongs to I. The functions $f(t)/p(t)$ have no common zeroes for $f(t)$ in I, so a simple compactness argument (cf. [8, p. 110]) shows that there is a finite set of functions $f_1(t), f_2(t), \ldots, f_n(t)$ in I for which the functions $f_i(t)/p(t)$ have no common zeroes. Thus $p(t)$ is the greatest common divisor of the $f_i(t)$, and an obvious variant of Helmer's Theorem [8, Th. (13.6), p. 109], with essentially the same proof, shows that there are functions $h_1(t), h_2(t), \ldots, h_n(t)$ in $H(\bar{D}(\tau))$ for which $p(t)=f_1(t)h_1(t)+f_2(t)h_2(t)+\cdots+f_n(t)h_n(t)$. Hence $p(t)$ belongs to I, and the proof is complete. □

We can now describe the $f(t)$-invariant subspaces of E_τ.

THEOREM 3.2. *Suppose that E_τ is the space of all entire functions of at most exponential order $\tau \geq 0$ and that $f(t)$ is analytic and univalent on a neighborhood of the closed disc $\bar{D}(\tau)$. If M is a non-zero linear subspace of E_τ with $f(t)M \subseteq M$, then either M is dense in the natural Frechét topology on E_τ, or there is a finite set of points $a_1, a_2, \ldots, a_k$ in $\bar{D}(\tau)$ and a finite set of non-negative integers $n(1), n(2), \ldots, n(k)$ for which M is the linear span of the functions $z^i e^{a_j z}$ for $j = 1, 2, \ldots, k$ and $0 \leq i \leq n(j)$.*

Proof. Suppose we knew that $M^\perp$ was an ideal in $H(\bar{D}(\tau))$. If $M^\perp=\{0\}$, then M would be dense. If $M^\perp \neq \{0\}$ but $M^\perp$ were an ideal, there would be (by Lemma 3.1) a polynomial $p(t)$ with its zeroes in $\bar{D}(\tau)$ for which $M^\perp$ was the image of $H(\bar{D}(\tau))$ under multiplication by $p(t)$. In that case M would be the solution space of the differential equation $p(t)y(z)=0$, and hence M would have the form indicated in the theorem.

To show that $M^\perp$ is an ideal we choose $g(t)$ in $H(\bar{D}(\tau))$ and $h(t)$ in $M^\perp$ and show that $g(t)h(t)$ belongs to $M^\perp$. Choose a number $r>\tau$ for which $f(t)$, $g(t)$, and $h(t)$ are all analytic on a neighborhood of $\bar{D}(r)$ and for which $f(t)$ is univalent on this neighborhood. Then $f(t)$, $g(t)$, and $h(t)$ all belong to $\ell^1(r^n)$, and $f(t)$ is a generator of this algebra [9]. Since $M^\perp \cap \ell(r^n)$ is a closed $f(t)$-invariant subspace of $\ell^1(r^n)$, it must then be an ideal in $\ell^1(r^n)$. Hence

$$g(t)h(t) \in M^\perp \cap \ell^1(r^n) \subseteq M^\perp.$$

This completes the proof. □

The following corollary essentially restates the above theorem in the terms used in §1 and §2.

COROLLARY 3.3. *Suppose that $g(z)$ is an entire function of at most type $\tau \geq 0$ and that $f(t)$ is analytic and univalent on a neighborhood of the closed disc $\bar{D}(\tau)$. Then either $g(z)$ is a finite linear combination of functions of the form $z^i e^{a_j z}$ with $|a_j| \leq \tau$, or $M = \mathrm{Span}\{f(t)^k g(z)\}_{k \geq 0}$ is dense in the natural Frechét topology on the space of functions of exponential type at most τ.*

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