DILATION THEORY AND SYSTEMS OF SIMULTANEOUS EQUATIONS IN THE PREDUAL OF AN OPERATOR ALGEBRA. I

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1. Introduction. Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$, let $\mathcal{A}(T)$ denote the smallest subalgebra of $\mathcal{L}(\mathcal{H})$ that contains T and $1_{\mathcal{H}}$ and is closed in the ultraweak operator topology. (For a discussion of this topology, cf. [8, Chapter I].) Moreover, let $Q(T)$ denote the quotient space $(\tau c)/{}^{\perp}\mathcal{A}(T)$, where (τc) is the trace-class ideal in $\mathcal{L}(\mathcal{H})$ under the trace norm, and ${}^{\perp}\mathcal{A}(T)$ denotes the preannihilator of $\mathcal{A}(T)$ in (τc). One knows (cf. [5]) that $\mathcal{A}(T)$ is the dual space of $Q(T)$ and that the duality is given by

$$(1) \qquad \langle A, [L]\rangle = \operatorname{tr}(AL), \quad A \in \mathcal{A}(T), \ [L] \in Q(T),$$

where $[L]$ is the image in $Q(T)$ of the operator L in (τc). Furthermore the weak* topology that accrues to $\mathcal{A}(T)$ by virtue of this duality coincides with the ultraweak operator topology on $\mathcal{A}(T)$. If x and y are vectors in $\mathcal{H}$ and we write, as usual, $x \otimes y$ for the rank-one operator in (τc) defined by $(x \otimes y)(u) = (u, y)x$, $u \in \mathcal{H}$, then $[x \otimes y] \in Q(T)$ and an easy calculation shows that for any A in $\mathcal{A}(T)$ we have

$$(2) \qquad \langle A, [x \otimes y]\rangle = \operatorname{tr}(A(x \otimes y)) = (Ax, y).$$

Suppose now that n is any cardinal number less than or equal to $\aleph_0$, and let N_n be an initial segment of the positive integers whose cardinality is n. The purpose of this paper is to study some classes of operators for which arbitrary systems of simultaneous equations in $Q(T)$ of the form

$$(3) \qquad [L_{ij}] = [x_i \otimes y_j], \quad i, j \in N_n,$$

can be solved for the unknown vectors x_i and y_j, $i, j \in N_n$, where the $[L_{ij}]$ are any given elements of $Q(T)$. (This project is of interest because it was shown in [3] that all operators in the classes $(\mathrm{BCP})_\theta$, $0 \leqslant \theta < 1$, to be defined in §2, have the property that systems of the form (3) can always be solved, even when $n = \aleph_0$.) In §3 we concentrate on operators for which finite systems (3) are solvable, and §4 is devoted to studying operators for which $\aleph_0 \times \aleph_0$ systems are solvable. Our results in this latter case constitute a rather extensive dilation theory that may have other applications.

2. Preliminaries. In this section we establish some preliminary notation and terminology, and we define precisely the classes of operators in which we will be interested. Let $\mathbf{D} = \{\lambda \in \mathbf{C} : |\lambda| < 1\}$ be the open unit disc in the complex plane,

Received March 11, 1983. Revision received August 10, 1983.
Michigan Math. J. 30 (1983).

and write $\mathbf{T}=\partial\mathbf{D}$. If we denote by $L^\infty = L^\infty(\mathbf{T})$ the Banach algebra of essentially bounded (Lebesgue) measurable functions on $\mathbf{T}$, then we recall that L^∞ is the dual space of $L^1 = L^1(\mathbf{T})$ under the usual pairing, and that $H^\infty = H^\infty(\mathbf{T})$ is a weak* closed subspace of L^∞ whose preannihilator in L^1 is the space H_0^1 of those functions f in $H^1 = H^1(\mathbf{T})$ whose analytic extension $\tilde{f}$ to $\mathbf{D}$ satisfies $\tilde{f}(0)=0$. Thus H^∞ is the dual space of L^1/H_0^1 under the pairing

$$\langle f, [g]\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})g(e^{it})\,dt. \tag{4}$$

We begin by noting that any study of classes of operators in $\mathcal{L}(\mathcal{H})$ for which systems of the form (3) can be solved loses no generality if the operators are all assumed to be contractions. Furthermore recall that any contraction T can be written as a direct sum $T = T_1 \oplus T_2$, where T_1 is a completely nonunitary contraction (i.e., T_1 has no nontrivial invariant subspace on which it acts as a unitary operator) and T_2 is a unitary operator. (Of course, either summand may act on the trivial space (0).) If T_2 is absolutely continuous (or acts on the space (0)), T will be called an *absolutely continuous contraction*. For such T (and certainly for completely nonunitary contractions, even if acting on a finite dimensional Hilbert space), one knows (cf. [16, Theorem III.2.1]) that there is a functional calculus $\Phi: H^\infty \to \mathcal{A}(T)$ defined by $\Phi(f) = f(T)$ for every f in H^∞. The mapping $\Phi = \Phi_T$ is a norm-decreasing, weak* continuous algebra homomorphism, and the range of Φ is weak* dense in $\mathcal{A}(T)$ (cf. [5, Theorem 3.2] for the completely nonunitary case; for absolutely continuous unitary operators, the weak*-continuity is an easy consequence of the spectral theorem). It therefore follows from general principles (cf. [5, Proposition 2.5]) that there exists a bounded, linear, one-to-one map $\phi = \phi_T$ of $Q(T)$ into L^1/H_0^1, such that $\Phi = \Phi^*$:

$$\begin{array}{ccc} H^\infty & \xrightarrow{\Phi=\phi^*} & \mathcal{A}(T) \\ L^1/H_0^1 & \xleftarrow{\ \phi\ } & Q(T). \end{array} \tag{5}$$

One could study arbitrary absolutely continuous contractions T for which systems of the form (3) are solvable, and, in fact, Olin and Thompson [13] proved that if T is subnormal, then every one-by-one system

$$[L] = [x \otimes y], \tag{6}$$

where $[L]$ is an arbitrary element of $Q(T)$, can be solved with vectors x, y in $\mathcal{H}$. But since, without making additional assumptions on T, the range of ϕ may not be all of L^1/H_0^1, there is not necessarily a good supply of interesting elements $[L]$ of $Q(T)$ for which to solve (3) or (6), as the following discussion shows.

For any λ in $\mathbf{D}$, let p_λ denote the Poisson kernel function

$$p_\lambda(e^{it}) = (1-|\lambda|^2)|1-\bar{\lambda}e^{it}|^{-2}, \quad e^{it} \in \mathbf{T},$$

in L^1. Then, from (4) and the well known properties of these functions, we obtain

$$\langle f, [p_\lambda]\rangle = \tilde{f}(\lambda), \quad f \in H^\infty. \tag{7}$$

If a completely nonunitary contraction T of norm one has the property that the range of the mapping ϕ in (5) contains some $[p_\lambda]$, let us write

$$\phi^{-1}([p_\lambda])=[C_\lambda]. \tag{8}$$

Then, from (1), (7), and (8), it follows that the element $[C_\lambda]$ of $Q(T)$ satisfies

$$\langle f(T),[C_\lambda]\rangle=\langle \Phi(f),[C_\lambda]\rangle=\langle f,[p_\lambda]\rangle=\hat{f}(\lambda),\quad f\in H^\infty. \tag{9}$$

Writing $\sigma(T)$, as usual, for the spectrum of an operator T, we have the following proposition.

PROPOSITION 2.1. *If a completely nonunitary contraction T in $\mathcal{L}(\mathcal{H})$ belongs to the class C_0 (in the terminology of* [16]) *and the minimal function m_T of T does not vanish on $\mathbf{D}$, then the range of the mapping $\phi: Q(T)\to L^1/H_0^1$ does not contain any of the $[p_\lambda]$, $\lambda\in\mathbf{D}$. Furthermore there exist such T satisfying $\sigma(T)=\mathbf{T}$.*

Proof. Suppose, for some such T and λ, the range of ϕ contains $[p_\lambda]$. Then

$$0=\langle m_T(T),[C_\lambda]\rangle=\hat{m}_T(\lambda)\neq 0$$

from the hypothesis and (9), a manifest contradiction. The fact that there exist such T in C_0 satisfying $\sigma(T)=\mathbf{T}$ is the content of [16, Corollary III.5.3]. □

In view of this proposition, we choose to limit our attention to those absolutely continuous contractions for which this unpleasantness does not occur.

DEFINITION 2.2. Let $\mathbf{A}$ denote the class of all absolutely continuous contractions T in $\mathcal{L}(\mathcal{H})$ for which the functional calculus Φ in (5) is an isometry of H^∞ onto $\mathcal{A}(T)$.

Some remarks concerning this definition are in order. First, it follows from [5, Theorem 2.7] and [6, Prop. 16.9 and Prob. 16K] that if $T\in\mathbf{A}$, then Φ is also a weak* homeomorphism and ϕ is an isometry of $Q(T)$ onto L^1/H_0^1. Thus, in this case, $Q(T)$ contains all of the elements $[C_\lambda]$, $\lambda\in\mathbf{D}$, defined by (8). Secondly, Scott Brown in [4] showed that (6) can always be solved for certain subnormal operators in $\mathbf{A}$, and thus originated the entire idea of considering systems of equations of the form (3). Thirdly, C. Apostol showed [1, Theorem 2.2] that if T is a completely nonunitary contraction in $\mathcal{L}(\mathcal{H})$ such that $\sigma(T)\supset\mathbf{T}$ and T has no nontrivial hyperinvariant subspace, then $T\in\mathbf{A}$. (It is elementary that if $T\in\mathbf{A}$, then $\sigma(T)\supset\mathbf{T}$.) Finally, the definition of the class $\mathbf{A}$ could be weakened somewhat by allowing Φ to be an invertible operator mapping H^∞ onto $\mathcal{A}(T)$; however this weaker hypothesis also implies that Φ is an isometry.

We say that a subset S of $\mathbf{D}$ is dominating for $\mathbf{T}$ if almost every point of $\mathbf{T}$ is a nontangential limit point of S. The following elementary proposition, whose proof is contained in that of [5, Theorem 3.2], gives a sufficient condition that an operator belong to $\mathbf{A}$.

PROPOSITION 2.3. *If T is an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$ and $\sigma(T)\cap\mathbf{D}$ is dominating for $\mathbf{T}$, then $T\in\mathbf{A}$.*

We turn now to the classes of operators to be studied in this paper.

DEFINITION 2.4. For any nonzero cardinal number n less than or equal to $\aleph_0$, we denote by $\mathbf{A}_n=\mathbf{A}_n(\mathcal{H})$ the set of all operators T in $\mathbf{A}$ such that if $\{[L_{ij}]\}_{i,j\in N_n}$ is an arbitrary $n\times n$ indexed family of elements from $Q(T)$, then the $n\times n$ system of simultaneous equations (3) in $Q(T)$ can be solved with vectors $\{x_i\}_{i\in N_n}$ and $\{y_j\}_{j\in N_n}$ from $\mathcal{H}$.

It is obvious from the definition that

$$\mathbf{A}\supset\mathbf{A}_1\supset\cdots\supset\mathbf{A}_n\supset\cdots\supset\mathbf{A}_{\aleph_0}. \tag{10}$$

Moreover, if $T\in\mathbf{A}_1$ and one solves the equation $[C_0]=[x\otimes y]$ for x and y in $\mathcal{H}$ (where $[C_0]$ is defined by (8)), then it is easy to see that x is nonzero and is not a cyclic vector for T (cf. [5]), so T has a nontrivial invariant subspace. Thus the question of whether $\mathbf{A}=\mathbf{A}_1$ is intimately related to the invariant subspace problem. (In this connection, see [10].) Furthermore, we show in §3 that if n is any positive integer, then certain subnormal operators in $\mathbf{A}_n$ do not belong to $\mathbf{A}_{n+1}$, so $\mathbf{A}_n\neq\mathbf{A}_{n+1}$. It is also important to know at the outset that we are not working in a vacuum, i.e., that $\mathbf{A}_{\aleph_0}\neq\varnothing$. Indeed, for any completely nonunitary contraction T in $\mathcal{L}(\mathcal{H})$ and for any μ in $\mathbf{D}$, let us write T_μ for the Möbius transform

$$T_\mu=(T-\mu I)(I-\bar{\mu}T)^{-1}. \tag{11}$$

Then, for each $0\leqslant\theta<1$, we define the class $(\mathrm{BCP})_\theta=(\mathrm{BCP})_\theta(\mathcal{H})$ to consist of all completely nonunitary contractions T in $\mathcal{L}(\mathcal{H})$ for which the set

$$\{\mu\in\mathbf{D}:\inf\sigma_e((T_\mu^*T_\mu)^{1/2})\leqslant\theta\quad\text{or}\quad\inf\sigma_e((T_\mu T_\mu^*)^{1/2})\leqslant\theta\}$$

is dominating for $\mathbf{T}$, where, as usual, we are writing $\sigma_e(A)$ for the essential spectrum of an operator A in $\mathcal{L}(\mathcal{H})$. It is easy to see that the nested family $\{(\mathrm{BCP})_\theta\}_{0\leqslant\theta<1}$ is increasing and that $(\mathrm{BCP})=(\mathrm{BCP})_0$ consists exactly of those completely nonunitary contractions T in $\mathcal{L}(\mathcal{H})$ for which $\sigma_e(T)\cap\mathbf{D}$ is dominating for $\mathbf{T}$. The following result is proved in [3] (cf. also [15]).

THEOREM 2.5. *For every θ, $0\leqslant\theta<1$, $(\mathrm{BCP})_\theta\subset\mathbf{A}_{\aleph_0}$.*

3. The classes $\mathbf{A}_n$ for n finite. In this section we will mostly be interested in obtaining results for operators T belonging to $\mathbf{A}_n$ for some positive integer n. We begin, however, with an equivalent formulation of the system (3). If $l\in L^1$, we will write the sequence of Fourier coefficients of l as $\{c_k(l)\}_{k=-\infty}^{\infty}$. Note that if $[l]\in L^1/H_0^1$ and $l_1,l_2\in[l]$, then $l_1-l_2\in H_0^1$, so $c_{-k}(l_1)=c_{-k}(l_2)$ for $k=0,1,2,\ldots$. We denote by $\{c_{-k}([l])\}_{k=0}^{\infty}$ this sequence of negative Fourier coefficients.

LEMMA 3.1. *Suppose T is an absolutely continuous contraction acting on a Hilbert space $\mathcal{H}$ of dimension less than or equal to $\aleph_0$, and n is some nonzero cardinal number less than or equal to $\aleph_0$. Then sequences $\{x_i\}_{i\in N_n}$ and $\{y_j\}_{j\in N_n}$ in $\mathcal{H}$ solve the system* (3) *if and only if*

$$c_{-k}(\phi_T([L_{ij}]))=(T^kx_i,y_j),\quad i,j\in N_n,\ k=0,1,2,\ldots, \tag{12}$$

where ϕ_T is the linear transformation of $Q(T)$ into L^1/H_0^1 in (5).

Proof. Since the polynomials $p(T)$ are weak* dense in $\mathcal{A}(T)$, vectors x and y in $\mathcal{H}$ solve $[L]=[x\otimes y]$ if and only if

$$\langle T^k, [L]\rangle = \langle T^k, [x\otimes y]\rangle, \quad k=0,1,2,\ldots. \tag{13}$$

But by (2), the right-hand side of (13) equals $(T^k x, y)$, and for the left-hand side, we have

$$\langle T^k, [L]\rangle = \langle \Phi_T(e^{ikt}), [L]\rangle = \langle e^{ikt}, \phi_T([L])\rangle = c_{-k}(\phi_T[L]). \qquad \square$$

In the remainder of the paper, the following notation will be quite useful. If T is an operator and $\mathcal{K}$ is a semi-invariant subspace for T, we shall write $T_{\mathcal{K}}$ for the compression of T to $\mathcal{K}$, i.e., for the operator $P_{\mathcal{K}}\, T \mid \mathcal{K}$, where $P_{\mathcal{K}}$ is the orthogonal projection whose range is $\mathcal{K}$. The next proposition shows that the property of belonging to $\mathbf{A}_n$ is inherited by certain dilations.

PROPOSITION 3.2. *If T is an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$, and n is a cardinal number such that $1 \leqslant n \leqslant \aleph_0$, then $T\in\mathbf{A}_n$ if and only if the compression of T to some infinite dimensional semi-invariant subspace $\mathcal{K}$ belongs to $\mathbf{A}_n(\mathcal{K})$.*

Proof. If $T\in\mathbf{A}_n$, then we may take $\mathcal{K}=\mathcal{H}\ominus(0)$. Thus it suffices to show that if $\mathcal{K}=\mathfrak{M}\ominus\mathfrak{N}$, where $\mathfrak{M}\supset\mathfrak{N}$ are invariant subspaces for T, and $\tilde{T}=T_{\mathcal{K}}\in\mathbf{A}_n(\mathcal{K})$, then $T\in\mathbf{A}_n(\mathcal{H})$. Since, for any function f in H^∞, we have

$$\|f\|_\infty \geqslant \|f(T)\| \geqslant \|P_{\mathcal{K}} f(T)\,|\,\mathcal{K}\| = \|f(\tilde{T})\| = \|f\|_\infty,$$

it follows easily that $T\in\mathbf{A}$. Now suppose that (3) is a given system of equations, where the $[L_{ij}]$ are arbitrary elements of $Q(T)$. For each pair $i,j\in N_n$, choose $[L'_{ij}]$ in $Q(\tilde{T})$ such that

$$\phi_T([L_{ij}]) = \phi_{\tilde{T}}([L'_{ij}]). \tag{14}$$

Since $\tilde{T}\in\mathbf{A}_n$, by Lemma 3.1 there exist sequences $\{\tilde{x}_i\}_{i\in N_n}$ and $\{\tilde{y}_j\}_{j\in N_n}$ in $\mathcal{K}$ that solve the system

$$c_{-k}(\phi_{\tilde{T}}([L'_{ij}])) = (\tilde{T}^k\tilde{x}_i, \tilde{y}_j), \quad i,j\in N_n, \quad k=0,1,2,\ldots. \tag{15}$$

We write $\mathcal{H}=\mathfrak{N}\oplus\mathcal{K}\oplus\mathfrak{M}^\perp$, and define the vectors x_i, y_j in $\mathcal{H}$ by

$$x_i = 0\oplus\tilde{x}_i\oplus 0, \qquad y_j = 0\oplus\tilde{y}_j\oplus 0, \quad i,j\in N_n.$$

Then, since $\mathcal{K}$ is semi-invariant for T, we deduce easily from (14) and (15) that

$$c_{-k}(\phi_T[L_{ij}])) = (T^k x_i, y_j), \quad i,j\in N_n, \quad k=0,1,2,\ldots, \tag{16}$$

and the result follows from Lemma 3.1. $\square$

Our next result relates the classes $\mathbf{A}_1$ and $\mathbf{A}_n$.

PROPOSITION 3.3. *Let n be a fixed cardinal number, $1\leqslant n<\aleph_0$, and for every T in $\mathbf{A}$, let $\tilde{T}$ denote the direct sum of n^2 copies of T, acting on the Hilbert space $\tilde{\mathcal{H}}$ which is the direct sum of n^2 copies of $\mathcal{H}$. Then $\tilde{T}\in\mathbf{A}_n(\tilde{\mathcal{H}})$ whenever $T\in\mathbf{A}_1(\mathcal{H})$.*

Proof. It is obvious that $\tilde{T}\in \mathbf{A}$, so let (3) be an arbitrary $n\times n$ system of equations in $Q(\tilde{T})$, and choose elements $[L'_{ij}]$ in $Q(\tilde{T})$ such that (14) holds for all $i,j\in N_n$. Then, in case $T\in \mathbf{A}_1$, for each fixed pair i,j in N_n, we may choose vectors x_{ij} and y_{ij} in $\mathcal{K}$ such that

$$c_{-k}(\phi_{\tilde{T}}[L'_{ij}])=(T^k x_{ij},y_{ij}),\quad k=0,1,2,\ldots. \tag{17}$$

Remembering that vectors in $\tilde{\mathcal{K}}$ may be regarded as column vectors of length n^2 with entries from $\mathcal{K}$, and that a column vector of length n^2 may be regarded as n vectors of length n laid end to end, we now define, for $i,j\in N_n$, the vectors $\tilde{x}_i$ and $\tilde{y}_j$ in $\tilde{\mathcal{K}}$ by

$$\tilde{x}_i=\left[\begin{array}{c} 0\\ \vdots\\ 0\\ \hline \vdots\\ \hline x_{i1}\\ x_{i2}\\ \vdots\\ x_{in}\\ \hline 0\\ \vdots\\ 0\\ \hline \vdots \end{array}\right],\qquad \tilde{y}_j=\left[\begin{array}{c} 0\\ \vdots\\ y_{1j}\\ \vdots\\ 0\\ \hline 0\\ \vdots\\ y_{2j}\\ \vdots\\ 0\\ \hline 0\\ \vdots\\ y_{3j}\\ \vdots\\ 0\\ \hline \vdots \end{array}\right],$$

where the nonzero entries in $\tilde{x}_i$ are in the $(in+1)$th through the $(in+n)$th positions, and the nonzero entries in $\tilde{y}_j$ occur at the jth position, $(n+j)$th position, $(2n+j)$th position, etc. An easy calculation using (14), (17), and the definition of the $\tilde{x}_i$ and $\tilde{y}_j$ shows that (15) is valid, so the proposition is proved. $\square$

Our next result shows that if $T\in \mathbf{A}_n$ then many different operators can be realized as compressions of T to finite-dimensional semi-invariant subspaces. Recall that a set $\{e_i\}_{i\in N_n}$ of vectors in a Hilbert space $\mathcal{K}$ is an n-cyclic set for an operator $A\in\mathcal{L}(\mathcal{K})$ if the smallest invariant subspace for A containing all of the e_i is $\mathcal{K}$ itself.

THEOREM 3.4. *Suppose $T\in \mathbf{A}_n$ for some positive integer n, and let A be any completely nonunitary contraction possessing an n-cyclic set of vectors and acting on a finite-dimensional Hilbert space $\mathcal{K}$. Then there exist invariant subspaces $\mathfrak{M}\supset\mathfrak{N}$ for T such that $T_{\mathfrak{M}\ominus\mathfrak{N}}$ is similar to A.*

Proof. Since $\mathcal{K}$ is finite-dimensional and A has an n-cyclic set of vectors $\{e_i\}_{i\in N_n}$, it is easy to see (via consideration of the Jordan canonical form of A)

that there is a set $\{f_j\}_{j\in N_n}$ of vectors in $\mathcal{K}$ that is an n-cyclic set for A^*. For each pair $1\leqslant i, j\leqslant n$, let $[L_{ij}]$ be the element in $Q(T)$ such that $\phi_A([e_i\otimes f_j])=\phi_T([L_{ij}])$. Choose sequences $\{x_i\}_{i\in N_n}$ and $\{y_j\}_{j\in N_n}$ from $\mathcal{H}$ that solve the $n\times n$ system (3). Then, by Lemma 3.1, we have

(18) $$(A^k e_i, f_j)=(T^k x_i, y_j), \quad i,j\in N_n,\ k=0,1,2,\ldots.$$

We define

(19) $$\begin{aligned}\mathfrak{M}&=\{p_1(T)x_1+\cdots+p_n(T)x_n: p_1,\ldots,p_n \text{ any polynomials}\}^-,\\ \mathfrak{M}_*&=\{q_1(T^*)y_1+\cdots+q_n(T^*)y_n: q_1,\ldots,q_n \text{ any polynomials}\}^-,\end{aligned}$$

and $\mathfrak{N}=\mathfrak{M}\cap\mathfrak{M}_*^\perp$. It is obvious from these definitions that $\mathfrak{M}$ and $\mathfrak{N}$ are invariant subspaces of T and that $\mathfrak{N}\subset\mathfrak{M}$. For each $1\leqslant i\leqslant n$, write $x_i=z_i+w_i$ where $z_i\in\mathfrak{M}\ominus\mathfrak{N}$ and $w_i\in\mathfrak{N}$. Since $T^{*k}y_j\in\mathfrak{M}_*$ for all j and k, we have $(T^k w_i,\ y_j)=(w_i, T^{*k}y_j)=0$ for all i, j, k, from which it results that

$$(A^k e_i, f_j)=(T^k z_i, y_j), \quad i,j\in N_n,\ k=0,1,2,\ldots,$$

or, equivalently, that $[x_i\otimes y_j]=[z_i\otimes y_j]$ for all i and j. Let us write $\tilde{T}$ for the compression $T_{\mathfrak{M}\ominus\mathfrak{N}}$. Then, of course, we may write, for any nonnegative integer k, $T^k z_i=\tilde{T}^k z_i+v_{ik}$ where $v_{ik}\in\mathfrak{N}$. Furthermore, since the y_j are orthogonal to $\mathfrak{N}$, we have

(20) $$(A^k e_i, f_j)=(T^k z_i, y_j)=(\tilde{T}^k z_i, y_j), \quad i,j\in N_n,\ k=0,1,2,\ldots.$$

We assert that the correspondence

(21) $$X: p_1(A)e_1+\cdots+p_n(A)e_n\to p_1(\tilde{T})z_1+\cdots+p_n(\tilde{T})z_n,$$

where the $p_i(\lambda)$ are any polynomials, is a one-to-one linear mapping from $\mathcal{K}$ onto $\mathfrak{M}\ominus\mathfrak{N}$. To prove that X is a mapping and that X is one-to-one, we observe that $\sum_{i\in N_n} p_i(A)e_i=0$ if and only if, for every family of polynomials $q_j(\lambda)$, $j\in N_n$, we have

$$\Big(\sum_{i\in N_n} p_i(A)e_i,\ \sum_{j\in N_n} q_j(A^*)f_j\Big)=0,$$

which is equivalent to

$$\Big(\sum_{i,j\in N_n} \tilde{q}_j(A)p_i(A)e_i,\ f_j\Big)=0, \quad \tilde{q}_j(\lambda)=\overline{q_j(\bar{\lambda})};$$

this, by virtue of (20), is equivalent to

(22) $$\Big(\sum_{i,j\in N_n} \tilde{q}_j(T)p_i(T)z_i,\ y_j\Big)=0.$$

On the other hand, (22) is equivalent to

$$\Big(\sum_{i\in N_n} p_i(T)z_i,\ \sum_{j\in N_n} q_j(T^*)y_j\Big)=0$$

for all families of polynomials $q_j(\lambda)$, $j\in N_n$, and this is clearly equivalent to saying that $\sum_i p_i(T)z_i$ belongs to $\mathfrak{N}$, or, what is the same thing, that $\sum_{i\in N_n} p_i(\tilde{T})z_i=0$. Thus X, given by (21), is a one-to-one mapping of $\mathcal{K}$ into $\mathfrak{M}\ominus\mathfrak{N}$. The linearity of X is clear from its definition (and the boundedness of X follows from the fact that $\mathcal{K}$ is finite dimensional). It follows immediately from the definition of $\mathfrak{M}$ and the equation

$$P_{\mathfrak{M}\ominus\mathfrak{N}}\big(\sum_i p_i(T)z_i\big)=\sum_i p_i(\tilde{T})z_i$$

that the range of X is dense in $\mathfrak{M}\ominus\mathfrak{N}$, and hence, since this range has the same dimension as $\mathcal{K}$, we have $\operatorname{range}(X)=\mathfrak{M}\ominus\mathfrak{N}$. Finally, it is clear from (21) that $XA=\tilde{T}X$, so $\tilde{T}$ is similar to A as desired. □

COROLLARY 3.5. *Suppose $T\in\mathbf{A}_n$ for some positive integer n, and let A be any completely nonunitary contraction acting on a Hilbert space $\mathcal{K}$ of dimension n. Then there exist invariant subspaces $\mathfrak{M}\supset\mathfrak{N}$ for T such that* $\dim(\mathfrak{M}\ominus\mathfrak{N})=n$ *and $T_{\mathfrak{M}\ominus\mathfrak{N}}$ is similar to A.*

Proof. Any basis for $\mathcal{K}$ is an n-cyclic set for a, and the result follows from Theorem 3.4. □

COROLLARY 3.6. *Suppose $T\in\mathbf{A}_n$ for some positive integer n, and $\lambda\in\mathbf{D}$. Then there exist invariant subspaces $\mathfrak{M}\supset\mathfrak{N}$ for T such that* $\dim(\mathfrak{M}\ominus\mathfrak{N})=n$ *and $T_{\mathfrak{M}\ominus\mathfrak{N}}=\lambda\cdot 1_{\mathfrak{M}\ominus\mathfrak{N}}$.*

Proof. This is immediate from the preceding corollary and the fact that an operator similar to a scalar λ must be equal to λ. □

It may be that Corollary 3.5 can be strengthened by replacing the relation of similarity by that of unitary equivalence. The following theorem lends evidence to this conjecture, and also shows that there exist subnormal operators in $\mathbf{A}_n$ that do not belong to $\mathbf{A}_{n+1}$. The last part of the proof is patterned after an argument of Hadwin and Nordgren [11].

THEOREM 3.7. *Let n be any fixed positive integer, and let U be a unilateral shift operator in $\mathcal{L}(\mathcal{K})$ of multiplicity n. Then $U\in\mathbf{A}_n$ and $U\notin\mathbf{A}_{n+1}$, so the nested sequence of sets $\{\mathbf{A}_n\}_{n=1}^{\infty}$ is strictly decreasing. Furthermore, if A is any completely nonunitary contraction acting on a Hilbert space of dimension n, then there exist invariant subspaces $\mathfrak{M}\supset\mathfrak{N}$ for U such that $U_{\mathfrak{M}\ominus\mathfrak{N}}$ is unitarily equivalent to A.*

Proof. Suppose first that $U\in\mathbf{A}_{n+1}$. Then, by Corollary 3.6 (with $\lambda=0$), there exist invariant subspaces $\mathcal{U}\supset\mathcal{V}$ of U such that $\dim(\mathcal{U}\ominus\mathcal{V})=n+1$ and the compression of U to $\mathcal{U}\ominus\mathcal{V}$ is 0. But it is well-known (cf. [16, Theorem VI.2.3]) that $U|\mathcal{U}$ must be a unilateral shift $V\in\mathcal{L}(\mathcal{U})$ of multiplicity $m\leqslant n$, and from above we have $V\mathcal{U}\subset\mathcal{V}$. But this implies that $\mathcal{U}\ominus\mathcal{V}\subset\operatorname{kernel} V^*$, which is impossible, since $m=\dim(\operatorname{kernel} V^*)\leqslant n$. This proves that $U\notin\mathbf{A}_{n+1}$.

Now suppose that A is any completely nonunitary contraction acting on a Hilbert space $\mathcal{K}$ of dimension n. Then $\sigma(A)$ lies in the open unit disc, and consequently $A\in C_{00}$ in the terminology of [16]. Hence, by [16, Theorem VI.2.3],

A is unitarily equivalent to the compression to a semi-invariant subspace of a unilateral shift operator of multiplicity $m=\dim\{(1_{\mathcal{K}}-AA^*)^{1/2}\mathcal{K}\}\leqslant n$, from which the stated result easily follows.

Finally, we show that $U\in\mathbf{A}_n$. For this purpose, consider the system (3), where the $[L_{ij}]$ are arbitrary elements of $Q(U)$. Let W be a unilateral shift of multiplicity $\aleph_0$ in $\mathcal{L}(\mathcal{H})$, and choose elements $[L'_{ij}]$ in $Q(W)$ that satisfy $\phi_U([L_{ij}])=\phi_W([L'_{ij}])$, $i,j\in N_n$. Since $\sigma_e(W)=\mathbf{D}^-$, $W\in(\mathrm{BCP})$, and, as mentioned earlier, we know from Theorem 2.5 that $(\mathrm{BCP})\subset\mathbf{A}_{\aleph_0}$. Thus there exist sequences $\{x_i\}_{i\in N_n}$ and $\{y_j\}_{j\in N_n}$ that satisfy the system

$$c_{-k}(\phi_U([L_{ij}]))=(W^kx_i,y_j),\quad i,j\in N_n,\ k=0,1,2,\ldots. \tag{23}$$

Let $\mathfrak{M}$ be the smallest invariant subspace for W containing the vectors $x_1,\ldots,x_n$, so $\mathfrak{M}$ is as in (19) with W replacing T. Then, as we have noted earlier, $W|\mathfrak{M}$ must be a unilateral shift Y in $\mathcal{L}(\mathfrak{M})$ of multiplicity $m\leqslant n$. For each $j\in N_n$, write $y_j=y'_j+y''_j$, where $y'_j\in\mathfrak{M}$ and $y''_j\in\mathfrak{M}^\perp$. It is trivial to verify that the sequences $\{x_i\}_{i\in N_n}$ and $\{y'_j\}_{j\in N_n}$ in $\mathfrak{M}$ satisfy

$$c_{-k}(\phi_U([L_{ij}]))=(Y^kx_i,y'_j),\quad i,j\in N_n,\ k=0,1,2,\ldots.$$

Since Y is unitarily equivalent to the restriction of U to some reducing subspace for U, it follows immediately from Lemma 3.1 and Proposition 3.2 that the system (3) is solvable, so the proof is complete. □

REMARK. The counterpart of Theorem 3.7 for $n=\aleph_0$ is also true. Indeed, if U is a unilateral shift of infinite multiplicity, then $\sigma_e(U)=\mathbf{D}^-$, so $U\in(\mathrm{BCP})\subset\mathbf{A}_{\aleph_0}$. If W is a bilateral shift of infinite multiplicity, then clearly $W\in\mathbf{A}$, and since W restricted to an invariant subspace is U, it follows from Proposition 3.2 that W also belongs to $\mathbf{A}_{\aleph_0}$.

If $\mathcal{H}_n$ is an n-dimensional Hilbert space, we write $\mathrm{Lat}(\mathcal{H}_n)$ for the lattice of all subspaces of $\mathcal{H}_n$. The next result relates the invariant subspace lattice $\mathrm{Lat}(T)$ of an operator T in $\mathbf{A}_n$ to $\mathrm{Lat}(\mathcal{H}_n)$.

THEOREM 3.8. *Suppose $T\in\mathbf{A}_n$ for some positive integer $n\geqslant 2$. Then there is a one-to-one mapping $\eta:\mathrm{Lat}(\mathcal{H}_n)\to\mathrm{Lat}(T)$ that is increasing, preserves closed spans, and has the property that if $\{\mathcal{K}_i\}_{i\in I}$ is any family of nontrivial subspaces of $\mathcal{H}_n$ such that $\bigcap_{i\in I}\mathcal{K}_i=(0)$, then $\bigcap_{i\in I}\eta(\mathcal{K}_i)=\bigcap_{i\in I}(T\eta(\mathcal{K}_i))^-$.*

Proof. As before, it follows from Corollary 3.6 that there exist $\mathfrak{M}\supset\mathfrak{N}$ in $\mathrm{Lat}(T)$ such that $\mathcal{H}_n=\mathfrak{M}\ominus\mathfrak{N}$ has dimension n and such that $T\mathfrak{M}\subset\mathfrak{N}$ (or, equivalently, $T_{\mathcal{H}_n}=0$). For any subspace $\mathcal{K}$ of $\mathcal{H}_n$, define $\eta(\mathcal{K})$ to be the smallest invariant subspace for T that contains $\mathcal{K}$ (so, in particular, $\eta(\mathcal{K})\subset\mathfrak{M}$). It is obvious from this definition that η is an increasing mapping that preserves closed spans, and the fact that $T\mathcal{H}_n\subset\mathfrak{N}\subset\mathcal{H}_n^\perp$ implies that $\mathcal{K}=\eta(\mathcal{K})\cap\mathcal{H}_n$, so η is one-to-one. We prove the last statement of the theorem in the case in which $\operatorname{card} I=2$; the proof in the general case is almost exactly the same. Thus let $\mathcal{K}_1$ and $\mathcal{K}_2$ be nonzero subspaces of $\mathcal{H}_n$ with $\mathcal{K}_1\cap\mathcal{K}_2=(0)$, and note that $T\mathcal{K}_i$ is orthogonal to $\mathcal{K}_j$ for $1\leqslant i,j\leqslant 2$. Since obviously $(T\eta(\mathcal{K}_1))^-\cap(T\eta(\mathcal{K}_2))^-\subset\eta(\mathcal{K}_1)\cap\eta(\mathcal{K}_2)$, it suffices to prove the reverse inclusion. Thus, let x be a nonzero vector in $\eta(\mathcal{K}_1)\cap\eta(\mathcal{K}_2)$, and let $\{e_1,\ldots,e_{k_1}\}$ and $\{f_1,\ldots,f_{k_2}\}$ be orthonormal bases for $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively. Then there exist sequences $\{p_{n,i}(\lambda)\}_{n=1}^{\infty}$, $1\leqslant i\leqslant k_1$, and $\{q_{n,j}(\lambda)\}_{n=1}^{\infty}$, $1\leqslant j\leqslant k_2$, of polynomials such that

$$\begin{aligned}&\lim_{n\to\infty}(p_{n,1}(T)e_1+\cdots+p_{n,k_1}(T)e_{k_1})=x, \quad \text{and}\\ &\lim_{n\to\infty}(q_{n,1}(T)f_1+\cdots+q_{n,k_2}(T)f_{k_2})=x.\end{aligned}\tag{24}$$

From the first equation in (24), we deduce easily that x is orthogonal to $\mathcal{H}_n\ominus\mathcal{K}_1$, and from the second that x is orthogonal to $\mathcal{H}_n\ominus\mathcal{K}_2$. But then, since

$$(\mathcal{H}_n\ominus\mathcal{K}_1)\vee(\mathcal{H}_n\ominus\mathcal{K}_2)=\mathcal{H}_n\ominus(\mathcal{K}_1\cap\mathcal{K}_2)=\mathcal{H}_n,$$

x is orthogonal to $\mathcal{H}_n$ and hence to all of the e_i and f_j. Thus, again from (24) we see that $p_{n,i}(0)\to 0$ and $q_{n,j}(0)\to 0$ for all i and j. Thus if we define sequences of polynomials $\{r_{n,i}(\lambda)\}$ and $\{s_{n,j}(\lambda)\}$ by setting $p_{n,i}(\lambda)-p_{n,i}(0)=\lambda r_{n,i}(\lambda)$ and $q_{n,j}(\lambda)-q_{n,j}(0)=\lambda s_{n,j}(\lambda)$, then

$$\begin{aligned}&\lim_{n\to\infty}(T[r_{n,1}(T)e_1+\cdots+r_{n,k_1}(T)e_{k_1}])=x, \quad \text{and}\\ &\lim_{n\to\infty}(T[s_{n,1}(T)f_1+\cdots+s_{n,k_2}(T)f_{k_2}])=x,\end{aligned}$$

from which the result follows. □

COROLLARY 3.9. *Suppose T is an invertible operator in $\mathbf{A}_n$ for some positive integer $n\geqslant 2$, and, in the notation of Theorem* 3.8, *suppose $\{\mathcal{K}_i\}$ is any family of nontrivial subspaces of $\mathcal{H}_n$ such that $\bigcap\mathcal{K}_i=(0)$. Then we have $T(\bigcap_i\eta(\mathcal{K}_i))=\bigcap_i\eta(\mathcal{K}_i)$, so that either T^{-1} has a nontrivial invariant subspace, or $\bigcap_i\eta(\mathcal{K}_i)=(0)$.*

Proof. This follows immediately from Theorem 3.8 and the fact that if T is invertible, then $T\eta(\mathcal{K}_i)$ is closed and $\bigcap_i T\eta(\mathcal{K}_i)=T(\bigcap_i\eta(\mathcal{K}_i))$. □

The question of whether the inverse of every operator in $\mathbf{A}_2$ has a nontrivial invariant subspace is important. For example, we have the following.

COROLLARY 3.10. *If the inverse of every invertible operator in $\mathbf{A}_{\aleph_0}$ has a nontrivial invariant subspace, then every operator in $\mathcal{L}(\mathcal{H})$ whose norm is equal to its spectral radius has a nontrivial invariant subspace.*

Proof. As noted above, $(\mathrm{BCP})\subset\mathbf{A}_{\aleph_0}$, and it follows from [10, Corollary 5.2] that if the inverse of every invertible operator in (BCP) has a nontrivial invariant subspace, then so does every operator in $\mathcal{L}(\mathcal{H})$ whose norm is equal to its spectral radius. □

4. The space $\mathbf{A}_{\aleph_0}$ of universal dilations. In this section we establish some dilation theorems for operators in the class $\mathbf{A}_{\aleph_0}$. These theorems are sufficiently strong that operators in $\mathbf{A}_{\aleph_0}$ have the right to be called "universal dilations". Recall from Sections 2 and 3 that the class $\mathbf{A}_{\aleph_0}$ contains many operators. In particular, for $0\leqslant\theta<1$, we have $(\mathrm{BCP})_\theta\subset\mathbf{A}_{\aleph_0}$; moreover, as we shall see shortly

(Prop. 4.5), if $T \in \mathbf{A}_1$, then $T \otimes 1_{\mathcal{H}} \in \mathbf{A}_{\aleph_0}$. If X is a closed, densely defined, linear transformation, we write $\mathfrak{D}(X)$ for the domain of X.

THEOREM 4.1. *Suppose $T \in \mathbf{A}_{\aleph_0}$, and let $A \in \mathcal{L}(\mathcal{H})$ be any absolutely continuous contraction. If L is any countable subset of $\mathcal{H}$, then there exist invariant subspaces $\mathfrak{M} \supset \mathfrak{N}$ for T and a closed one-to-one linear transformation $X: \mathfrak{D}(X) \to \mathfrak{M} \ominus \mathfrak{N}$ such that*

(a) *the linear manifold $\mathfrak{D}(X)$ is dense in $\mathcal{H}$ and contains L,*
(b) *the range of X is dense in $\mathfrak{M} \ominus \mathfrak{N}$, and*
(c) *$T_{\mathfrak{M} \ominus \mathfrak{N}} Xz = XAz$ for all z in $\mathfrak{D}(X)$.*

Proof. The argument is similar to that given to prove Theorem 3.4, and we content ourselves with a sketch. Let $\{e_i\}_{i=1}^{\infty}$ be a sequence that is dense in $\mathcal{H}$ and contains all of the elements of L, and choose elements L_{ij} in $Q(T)$, $1 \leqslant i, j < \infty$, such that

$$\phi_A([e_i \otimes e_j]) = \phi_T([L_{ij}]), \quad 1 \leqslant i, j < \infty.$$

Then, by hypothesis, there exist sequences $\{x_i\}_{i=1}^{\infty}$ and $\{y_j\}_{j=1}^{\infty}$ in $\mathcal{H}$ that solve the system (3) (with $n = \aleph_0$). Thus, by Lemma 3.1, we have

$$(A^k e_i, e_j) = (T^k x_i, y_j), \quad 1 \leqslant i, j < \infty, \quad k = 0, 1, 2, \ldots.$$

We define $\mathfrak{M} = \bigvee \{T^k x_i : i \geqslant 1, \ k \geqslant 0\}$, $\mathfrak{M}_* = \bigvee \{T^{*k} y_j : j \geqslant 1, \ k \geqslant 0\}$, and $\mathfrak{N} = \mathfrak{M} \cap \mathfrak{M}_*^{\perp}$. Clearly $\mathfrak{M}$ and $\mathfrak{N}$ are invariant subspaces for T with $\mathfrak{M} \supset \mathfrak{N}$, and if we write $x_i = z_i + w_i$ where $z_i \in \mathfrak{M} \ominus \mathfrak{N}$ and $w_i \in \mathfrak{N}$, then, as before, we deduce easily that

$$(A^k e_i, e_j) = (T^k z_i, y_j), \quad 1 \leqslant i, j < \infty, \quad k = 0, 1, 2, \ldots. \tag{25}$$

Furthermore, writing $\tilde{T}$ for the compression of T to $\mathfrak{M} \ominus \mathfrak{N}$, we have $T^k z_i = \tilde{T}^k z_i + v_{ik}$, where $v_{ik} \in \mathfrak{N}$, so we conclude, as before, that

$$(A^k e_i, e_j) = (\tilde{T}^k z_i, y_j), \quad 1 \leqslant i, j < \infty, \quad k = 0, 1, 2, \ldots.$$

Now, just as in the proof of Theorem 3.4, the correspondence

$$X_0 : p_1(A) e_1 + \cdots + p_m(A) e_m \to p_1(\tilde{T}) z_1 + \cdots + p_m(\tilde{T}) z_m, \tag{26}$$

where m is any positive integer and the $p_i(\lambda)$, $1 \leqslant i \leqslant m$, are any polynomials, is a one-to-one linear mapping of a dense linear manifold in $\mathcal{H}$ containing L onto a dense linear manifold in $\mathfrak{M} \ominus \mathfrak{N}$. Furthermore, it turns out that X_0 is closable, and that its closure X is one-to-one. To prove these assertions, it suffices to show that if $p_i^{(n)}(\lambda)$ are polynomials such that

$$p_1^{(n)}(A) e_1 + \cdots + p_{m_n}^{(n)}(A) e_{m_n} \to e', \quad \text{and}$$
$$p_1^{(n)}(\tilde{T}) z_1 + \cdots + p_{m_n}^{(n)}(\tilde{T}) z_{m_n} \to z',$$

then $e' = 0$ if and only if $z' = 0$. But $e' = 0$ if and only if, for every positive integer p and every sequence of polynomials $q_1(\lambda), \ldots, q_p(\lambda)$, we have

$$(e', q_1(A^*) e_1 + \cdots + q_p(A^*) e_p) = 0, \tag{27}$$

since the sequence $\{e_i\}$ is dense in $\mathcal{H}$; furthermore, (27) is equivalent to

$$\lim_n (p_1^{(n)}(A)e_1+\cdots+p_{m_n}^{(n)}(A)e_{m_n}, q_1(A^*)e_1+\cdots+q_p(A^*)e_p)=0,$$

which, by virtue of (25), is equivalent to

$$\lim_n (p_1^{(n)}(T)z_1+\cdots+p_{m_n}^{(n)}(T)z_{m_n}, q_1(T^*)y_1+\cdots+q_p(T^*)y_p)=0.$$

This is equivalent to

$$\lim_n (p_1^{(n)}(\tilde{T})z_1+\cdots+p_{m_n}^{(n)}(\tilde{T})z_{m_n}, q_1(T^*)y_1+\cdots+q_p(T^*)y_p)=0,$$

which is equivalent to

$$(z', q_1(T^*)y_1+\cdots+q_p(T^*)y_p)=0,$$

which is equivalent to $z'=0$. Thus X_0 is closable, and its closure X is one-to-one. Since the range of X_0 is dense in $\mathfrak{M}\ominus\mathfrak{N}$, just as in the proof of Theorem 3.4, the same is true of X, and since $X_0A=\tilde{T}X_0$ on the domain of X_0 by virtue of (26), conclusion (c) in the statement of theorem follows easily from the fact that X is the closure of X_0. □

Of course the relation between A and the compression $T_{\mathfrak{M}\ominus\mathfrak{N}}$ of T to $\mathfrak{M}\ominus\mathfrak{N}$ is very weak in the above theorem. There are, however, classes of operators A for which this relation can be dramatically improved.

PROPOSITION 4.2. *Suppose $T\in\mathbf{A}_{\aleph_0}$ and $\{\lambda_k\}_{k=1}^{\infty}$ is any sequence of (not necessarily distinct) points from* $\mathbf{D}$. *Then there exists a semi-invariant subspace $\mathcal{K}$ for T such that $T_{\mathcal{K}}$ is unitarily equivalent to a normal operator N in $\mathcal{L}(\mathcal{H})$ whose matrix relative to some orthonormal basis $\{e_k\}_{k=1}^{\infty}$ for $\mathcal{H}$ is the diagonal matrix* $\mathrm{Diag}(\{\lambda_k\}_{k=1}^{\infty})$.

Proof. Set $A=1_{\mathcal{H}}\otimes N$, and note that each λ_k is an eigenvalue of A of infinite multiplicity. Define the countable set L so that it contains an infinite number of linearly independent eigenvectors corresponding to each λ_k, $1\leqslant k<\infty$, and let $\mathfrak{M}$, $\mathfrak{N}$ and X be as provided in Theorem 4.1. Then, if we write $\tilde{T}=T_{\mathfrak{M}\ominus\mathfrak{N}}$, it is easy to see from (c) that each λ_k, $1\leqslant k<\infty$, is an eigenvalue of $\tilde{T}$ whose corresponding eigenspace is infinite dimensional. Therefore we may choose by induction an orthonormal sequence $\{f_k\}_{k=1}^{\infty}$ in $\mathfrak{M}\ominus\mathfrak{N}$ such that $\tilde{T}f_k=\lambda_k f_k$, $1\leqslant k<\infty$. Setting $\mathcal{K}=\bigvee_{k=1}^{\infty}\{f_k\}$, we see that $\mathcal{K}$ (regarded as a subspace of $\mathfrak{M}\ominus\mathfrak{N}$) is an invariant subspace for $\tilde{T}$, and hence that $\mathcal{K}$ (regarded as a subspace of $\mathcal{H}$) is a semi-invariant subspace for T. Clearly $T_{\mathcal{K}}=\tilde{T}|\mathcal{K}$ is unitarily equivalent to N, so the proof is complete. □

COROLLARY 4.3. *If $T\in\mathbf{A}_{\aleph_0}$, then there exists an invariant subspace $\mathfrak{M}$ for T such that $\mathfrak{M}\ominus(T\mathfrak{M})^-$ is infinite dimensional.*

Proof. Define $\lambda_k=0$ for $1\leqslant k<\infty$, and apply Proposition 4.2 to obtain an infinite dimensional semi-invariant subspace $\mathcal{K}=\mathfrak{M}\ominus\mathfrak{N}$ for T such that $T_{\mathfrak{M}\ominus\mathfrak{N}}=0$. Then $(T\mathfrak{M})^-\subset\mathfrak{N}$ and the result follows. □

Another easy corollary of Proposition 4.2 is the following useful characterization of the class $\mathbf{A}_{\aleph_0}$.

COROLLARY 4.4. *An absolutely continuous contraction T in $\mathcal{L}(\mathcal{H})$ belongs to $\mathbf{A}_{\aleph_0}$ if and only if there exists a semi-invariant subspace $\mathcal{K}$ for T such that the compression $T_{\mathcal{K}}$ belongs to* (BCP). *Thus $\mathbf{A}_{\aleph_0}$ is self-adjoint.*

Proof. If $T \in \mathbf{A}_{\aleph_0}$, then it follows from Proposition 4.2 that T has a compression to a semi-invariant subspace that is a (BCP)-operator. Indeed, the operator $\operatorname{Diag}(\{\lambda_k\}) \in$ (BCP) if the sequence $\{\lambda_k\}$ is taken to be dense in $\mathbf{D}$. On the other hand, if T has such a compression, then it follows immediately from the fact that $(\mathrm{BCP}) \subset \mathbf{A}_{\aleph_0}$ and Proposition 3.2 that $T \in \mathbf{A}_{\aleph_0}$. That $\mathbf{A}_{\aleph_0}$ is self-adjoint now follows from the fact that if A is the compression to a semi-invariant subspace of T, then A^* is the compression to a semi-invariant subspace of T^*, together with the fact that (BCP) is self-adjoint. (That $\mathbf{A}_{\aleph_0}$ is self-adjoint could also have been deduced easily from Lemma 3.1.) □

The following consequence of Corollary 4.4 is the counterpart of Proposition 3.3 for $n = \aleph_0$.

PROPOSITION 4.5. *If $T \in \mathbf{A}_1(\mathcal{H})$, $\tilde{\mathcal{H}}$ is the direct sum of $\aleph_0$ copies of $\mathcal{H}$, and $\tilde{T}$ is the direct sum of $\aleph_0$ copies of T acting on $\tilde{\mathcal{H}}$, then $\tilde{T} \in \mathbf{A}_{\aleph_0}(\tilde{\mathcal{H}})$.*

Proof. Write $\tilde{\mathcal{H}} = \sum_1^\infty \oplus \mathcal{H}_n$, where each $\mathcal{H}_n$ equals $\mathcal{H}$, and write $\tilde{T} = \sum_1^\infty \oplus T_n$, where each T_n equals T. If $\{\lambda_n\}_{n=1}^\infty$ is any sequence dense in $\mathbf{D}$, then, since $T_n \in \mathbf{A}_1(\mathcal{H}_n)$, there exists a one-dimensional semi-invariant subspace $\mathcal{K}_n \subset \mathcal{H}_n$ for T_n such that $(T_n)_{\mathcal{K}_n} = \lambda_n 1_{\mathcal{K}_n}$. Let $\mathcal{K} \subset \tilde{\mathcal{H}}$ be the span of the orthogonal spaces $\mathcal{K}_n$, $1 \leqslant n < \infty$. It is easy to see that $\mathcal{K}$ is semi-invariant for $\tilde{T}$ and that $\tilde{T}_{\mathcal{K}}$ is unitarily equivalent to the normal operator $\operatorname{Diag}(\{\lambda_n\})$, which is a (BCP)-operator. Thus by Corollary 4.4, $\tilde{T} \in \mathbf{A}_{\aleph_0}(\tilde{\mathcal{H}})$. □

REMARK. The preceding argument can be used to give a different proof of the following stronger result due to Exner [9]: If $T_n \in \mathbf{A}_1(\mathcal{H})$, $1 \leqslant n < \infty$, and $\tilde{T} = \sum_{n=1}^\infty \oplus T_n$, then $\tilde{T} \in \mathbf{A}_{\aleph_0}(\tilde{\mathcal{H}})$.

The following theorem, which is of independent interest in its own right (and may be known), is a stepping-stone to be used to improve Proposition 4.2.

PROPOSITION 4.6. *Suppose A is an operator in $\mathcal{L}(\mathcal{H})$ with range $\mathcal{H}$ and infinite dimensional kernel such that*

$$\inf\{\|Ax\| : \|x\| = 1,\ x \in (\ker A)^\perp\} \geqslant \alpha > 0.$$

Then there exists an invariant subspace $\mathcal{M}$ for A such that $A|\mathcal{M}$ is unitarily equivalent to αU^*, where U is a (*forward, unweighted*) *unilateral shift of infinite multiplicity.*

Proof. If we write the polar decomposition of A as $A = VP$, and define $B = [(P|(\ker A)^\perp)^{-1} \oplus 0_{\ker A}]V^*$, then clearly $AB = 1_{\mathcal{H}}$ and $\|B\| \leqslant 1/\alpha$. Furthermore, since $\ker A$ is infinite dimensional, we can choose an isometry W in $\mathcal{L}(\mathcal{H})$ such that $W\mathcal{H}\subset\ker A$ and $\ker A\ominus W\mathcal{H}$ is infinite dimensional. Define now

$$Z=W(1_{\mathcal{H}}-\alpha^2B^*B)^{1/2}+\alpha B.$$

Since $AW=0$ and the ranges of B and W are orthogonal, we have

$$Z^*Z=(1_{\mathcal{H}}-\alpha^2B^*B)^{1/2}W^*W(1_{\mathcal{H}}-\alpha^2B^*B)^{1/2}+\alpha^2B^*B=1_{\mathcal{H}}$$
$$\text{and}\quad AZ=A(W(1_{\mathcal{H}}-\alpha^2B^*B)^{1/2}+\alpha B)=\alpha AB=\alpha 1_{\mathcal{H}},$$

so, in particular, Z is also an isometry. Let $\{e_{n,0}\}_{n=0}^{\infty}$ be an orthonormal sequence in the space $\ker A\ominus W\mathcal{H}$. Then all the terms of this sequence are orthogonal to $W\mathcal{H}\vee B\mathcal{H}$, from which it follows that each $e_{m,0}$ is orthogonal to the range of Z. Thus if we define $e_{n,j}=Z^je_{n,0}$ for all $0\leqslant n,j<\infty$, it follows that $e_{n,0}$ is orthogonal to $e_{m,j+1}$ for $0\leqslant j,n,m<\infty$. Consequently, since Z is an isometry, the entire family $\{e_{n,j}\}_{0\leqslant n,j<\infty}$ is orthonormal. Since

$$Ae_{n,0}=0\quad\text{and}\quad Ae_{n,j+1}=AZe_{n,j}=\alpha e_{n,j},\quad 0\leqslant n,j<\infty,$$

it is obvious that if we define $\mathfrak{M}$ to be $\bigvee_{n,j}\{e_{n,j}\}$, then $A\mathfrak{M}\subset\mathfrak{M}$ and $A|\mathfrak{M}$ is unitarily equivalent to αU^*, so the theorem is proved. □

COROLLARY 4.7. *Suppose $T\in\mathbf{A}_{\aleph_0}$ and $0<\alpha<1$. Then there exists a semi-invariant subspace $\mathcal{K}$ for T such that $T_{\mathcal{K}}$ is unitarily equivalent to αU^*, where U is a (forward, unweighted) unilateral shift operator of infinite multiplicity.*

Proof. Let $\{\lambda_k\}$ be a sequence of distinct points that is dense in the annulus $\{\lambda\in\mathbf{C}:\alpha<|\lambda|<1\}$, and let N be a diagonal normal operator whose matrix relative to some orthonormal basis for $\mathcal{H}$ is $\mathrm{Diag}(\{\lambda_k\})$. Then, according to Proposition 4.2, there exists a semi-invariant subspace $\mathcal{K}_1$ for T such that $\tilde{T}=T_{\mathcal{K}_1}$ is unitarily equivalent to N. Since a semi-invariant subspace $\mathcal{K}\subset\mathcal{K}_1$ for $\tilde{T}$, regarded as a subspace of $\mathcal{H}$ (the Hilbert space of T), is also a semi-invariant subspace for T, it suffices to show that N has a semi-invariant subspace $\mathcal{P}$ such that $N_{\mathcal{P}}$ is unitarily equivalent to αU^*. Note that $\sigma_e(N)=\{\lambda\in\mathbf{C}:\alpha\leqslant|\lambda|\leqslant 1\}$, so $N\in(\mathrm{BCP})\subset\mathbf{A}_{\aleph_0}$, and thus we may apply Corollary 4.3 to N to deduce the existence of an invariant subspace $\mathfrak{M}\subset\mathcal{H}$ for N such that $\mathfrak{M}\ominus N\mathfrak{M}$ is infinite dimensional. If we write $\mathcal{H}=N\mathfrak{M}\oplus(\mathfrak{M}\ominus N\mathfrak{M})\oplus\mathfrak{M}^\perp$, then, relative to this decomposition of $\mathcal{H}$, N has a matrix of the form

$$\begin{pmatrix} A & B & C\\ 0 & 0 & D\\ 0 & 0 & E\end{pmatrix}.$$

We define R to be the compression of N to the semi-invariant subspace $\mathcal{R}=\mathcal{H}\ominus N\mathfrak{M}$, so that the corresponding matrix for R is

$$\begin{pmatrix} 0 & D\\ 0 & E\end{pmatrix}.$$

It is obvious that the kernel of R is infinite dimensional. Furthermore, since N (together with N^*) is invertible and has lower bound α, and since $\|N^*x\|=\|R^*x\|$

for all x in $\mathcal{R}$, it follows easily that the range of R is $\mathcal{R}$ and that R^* is bounded below by α. Consideration of the polar decomposition of R then easily gives the fact that

$$\inf\{\|Rx\|: \|x\|=1,\ x\in \mathcal{R}\ominus \ker R\} \geq \alpha.$$

Thus we may apply Proposition 4.6 to R to conclude the existence of an invariant subspace $\mathcal{P}$ for R such that $R|\mathcal{P}$ is unitarily equivalent to αU^*. Since $\mathcal{P}$, regarded as a subspace of $\mathcal{H}$, is semi-invariant for N and $N_{\mathcal{P}}=R|\mathcal{P}$, the result follows. □

The following theorem supports the view that operators in $\mathbf{A}_{\aleph_0}$ deserve to be called universal dilations. Recall that an operator A is called a strict contraction if $\|A\|<1$.

THEOREM 4.8. *Suppose $T\in \mathbf{A}_{\aleph_0}$, and let $\{A_j\}_{j=1}^{\infty}$ be any sequence of strict contractions acting on Hilbert spaces of dimension less than or equal to $\aleph_0$. Then there exists a semi-invariant subspace $\mathcal{K}$ for T such that $T_{\mathcal{K}}$ is unitarily equivalent to the direct sum $\sum_j \oplus A_j$.*

Proof. Choose a sequence of positive numbers $\{\alpha_j\}$ such that $\|A_j\|<\alpha_j<1$ for $1\leq j<\infty$. Then each of the operators $\alpha_j^{-1}A_j$ is a strict contraction, and it is well-known (cf. [16, Theorem VI.2.3*]) that every strict contraction is unitarily equivalent to the restriction of a backward unilateral shift operator U^* in $\mathcal{L}(\mathcal{H})$ of infinite multiplicity to some invariant subspace. Thus A_j is unitarily equivalent to the restriction of $\alpha_j U^*$ to some invariant subspace $\mathfrak{M}_j$, and $\sum_j\oplus A_j$ is unitarily equivalent to the restriction of $\sum_j\oplus\alpha_j U^*$ to the invariant subspace $\sum_j\oplus\mathfrak{M}_j$. Thus it suffices to prove that T has a compression to a semi-invariant subspace that is unitarily equivalent to $\sum_j\oplus\alpha_j U^*$. For this purpose, we now choose a sequence $\{\lambda_j\}$ of points that is dense in $\mathbf{D}$ and that has the property that every term is repeated infinitely often. Then, according to Proposition 4.2, there is a semi-invariant subspace $\mathcal{K}_1$ for T such that the compression $T_{\mathcal{K}_1}$ is unitarily equivalent to a normal operator N whose matrix relative to some orthonormal basis is $\mathrm{Diag}(\{\lambda_j\})$. Thus, it suffices to prove that the compression of N to some semi-invariant subspace is unitarily equivalent to $\sum\oplus\alpha_j U^*$. By the way the sequence $\{\lambda_j\}$ was chosen, we may write N as the direct sum $N=\sum_{j=1}^{\infty}\oplus N_j$ of countably many normal operators N_j each of which satisfies $\sigma_e(N_j)=\mathbf{D}^-$. (In fact, all of the N_j may be taken to be the same operator.) Since each $N_j\in(\mathrm{BCP})\subset\mathbf{A}_{\aleph_0}$, we may apply Corollary 4.7 to conclude that, for $1\leq j<\infty$, there exists a semi-invariant subspace $\mathfrak{N}_j$ for N_j such that the compression of N_j to $\mathfrak{N}_j$ is unitarily equivalent to $\alpha_j U^*$. It follows easily that the subspace $\sum\oplus\mathfrak{N}_j$ is a semi-invariant subspace for $N=\sum\oplus N_j$, and, of course, the compression of N to $\sum\oplus N_j$ is unitarily equivalent to $\sum\oplus\alpha_j U^*$. Thus the proof is complete. □

Theorem 4.8 makes it worthwhile to make the following definition.

DEFINITION 4.9. If A is an operator on a Hilbert space of dimension less than or equal to $\aleph_0$, and *every* operator T in $\mathbf{A}_{\aleph_0}$ has the property that some compression of T to a semi-invariant subspace is unitarily equivalent to A, then we call

A a *universal* $\mathbf{A}_{\aleph_0}$*-compression*, and we denote the set of all universal $\mathbf{A}_{\aleph_0}$-compressions by $\mathcal{C}(\mathbf{A}_{\aleph_0})$.

It is obvious that every A in $\mathcal{C}(\mathbf{A}_{\aleph_0})$ is a completely nonunitary contraction. The following proposition sets forth some additional properties of the class $\mathcal{C}(\mathbf{A}_{\aleph_0})$.

PROPOSITION 4.10. *The set* $\mathcal{C}(\mathbf{A}_{\aleph_0})$ *is self-adjoint, contains every (separably acting) strict contraction, is contained in* C_{00} *(in the terminology of* [16]*), and is closed under the formation of countable direct sums. Furthermore, if* $T\in\mathcal{C}(\mathbf{A}_{\aleph_0})$ *and* x *is any nonzero vector in the Hilbert space of* T*, then* $\|Tx\|<\|x\|$.

Proof. That $\mathcal{C}(\mathbf{A}_{\aleph_0})$ contains every (separably acting) strict contraction is immediate from Theorem 4.8. That $\mathcal{C}(\mathbf{A}_{\aleph_0})$ is self-adjoint follows easily from the fact that $\mathbf{A}_{\aleph_0}$ is self-adjoint and the obvious equations

$$\mathfrak{M}\ominus\mathfrak{N}=\mathfrak{M}\cap\mathfrak{N}^{\perp}=\mathfrak{N}^{\perp}\ominus\mathfrak{M}^{\perp},$$

$$(T_{\mathfrak{M}\ominus\mathfrak{N}})^{*}=(T^{*})_{\mathfrak{N}^{\perp}\ominus\mathfrak{M}^{\perp}},$$

where $\mathfrak{M}\supset\mathfrak{N}$ are invariant subspaces of T. To see that $\mathcal{C}(\mathbf{A}_{\aleph_0})\subset C_{00}$, it suffices to note that (BCP) contains operators in C_{00} (for example, the normal operator N that appeared in the proof of Corollary 4.7) and to observe that the compression to a semi-invariant subspace of an operator belonging to C_{00} also belongs to C_{00}. A similar argument proves the last statement of the proposition. Finally, that $\mathcal{C}(\mathbf{A}_{\aleph_0})$ is closed under countable direct sums is proved via a construction that is almost identical to that in the proof of Theorem 4.8, and thus no more need be said about it. □

The question of exactly which completely nonunitary contractions belong to $\mathcal{C}(\mathbf{A}_{\aleph_0})$ is interesting and important. The following lemma leads to the construction of some such operators that are not direct sums of strict contractions.

LEMMA 4.11. *If* $\{\alpha_j\}_{j=0}^{\infty}$ *is any sequence of numbers such that* $0<\alpha_j<1$ *for every* j *and* $\prod_{j=0}^{\infty}\alpha_j=0$, *then there exists a square summable sequence* $\{\xi_j\}_{j=0}^{\infty}$ *of positive numbers such that*

$$\left(\sum_{j=n+1}^{\infty}\xi_j^2\right)\Big/\left(\sum_{j=n}^{\infty}\xi_j^2\right)=\alpha_n,\quad n=0,1,2,\ldots. \tag{28}$$

Proof. It suffices to set

$$\xi_0^2=1-\alpha_0,\qquad \xi_j^2=\alpha_0\alpha_1\ldots\alpha_{j-1}(1-\alpha_j),\quad j\geqslant 1,$$

and to observe that $\sum_{j=n}^{\infty}\xi_j^2=\alpha_0\alpha_1\ldots\alpha_{n-1}$ for every $n=1,2,\ldots$, while $\sum_{j=0}^{\infty}\xi_j^2=1$. □

THEOREM 4.12. *Let* $\alpha=\{\alpha_j\}_{j=0}^{\infty}$ *be any sequence of positive numbers as in Lemma* 4.11. *If* S_α *is a forward weighted shift operator of multiplicity one with weight sequence* α, *then* $S_\alpha\in\mathcal{C}(\mathbf{A}_{\aleph_0})$. *Conversely, if* S_β *is a forward weighted shift operator with positive weight sequence* β *and* $S_\beta\in\mathcal{C}(\mathbf{A}_{\aleph_0})$, *then* β *must satisfy the hypotheses of Lemma* 4.11.

Proof. We define first a sequence $\{A_n\}_{n=0}^{\infty}$ of strict contractions. Let A_0 be the zero operator acting on a 1-dimensional space with orthonormal basis $\{e_{0,0}\}$, and for each $n>0$, let A_n be a truncated forward weighted shift acting on an $(n+1)$-dimensional Hilbert space whose definition relative to an orthonormal basis $\{e_{n,0}, e_{n,1}, \ldots, e_{n,n}\}$ for the space is given by

$$A_n e_{n,j} = \alpha_j^{1/2} e_{n,j+1}, \quad 0 \leqslant j \leqslant n-1,$$
$$A_n e_{n,n} = 0.$$

By Proposition 4.10 the operator $A = \sum_{n=0}^{\infty} \oplus A_n$ belongs to $\mathcal{C}(\mathbf{A}_{\aleph_0})$, and therefore it will suffice to show that S_α is unitarily equivalent to the restriction of A to some invariant subspace for A. Observe that the space on which A acts has an orthonormal basis $\{e_{n,j} : 0 \leqslant n < \infty, 0 \leqslant j \leqslant n\}$, and the action of A on this basis is given by

$$Ae_{n,j} = \alpha_j^{1/2} e_{n,j+1}, \quad Ae_{n,n} = 0, \quad 0 \leqslant n < \infty, \ 0 \leqslant j \leqslant n-1.$$

Let $\{\xi_j\}_{j=0}^{\infty}$ be the square summable sequence provided by Lemma 4.11, and for $n \geqslant 0$ define $x_n = \sum_{j=n}^{\infty} \xi_j e_{j,n}$. Then

$$Ax_n = \sum_{j=n}^{\infty} \xi_j A e_{j,n} = \sum_{j=n+1}^{\infty} \xi_j \alpha_n^{1/2} e_{j,n+1} = \alpha_n^{1/2} x_{n+1}$$

or, equivalently,

$$\begin{aligned} A(x_n/\|x_n\|) &= (\alpha_n^{1/2}/\|x_n\|) x_{n+1} = \alpha_n^{1/2} (\|x_{n+1}\|/\|x_n\|)(x_{n+1}/\|x_{n+1}\|) \\ &= \alpha_n^{1/2} \left(\sum_{j=n+1}^{\infty} \xi_j^2 \Big/ \sum_{j=n}^{\infty} \xi_j^2 \right)^{1/2} (x_{n+1}/\|x_{n+1}\|) \qquad (29) \\ &= \alpha_n (x_{n+1}/\|x_{n+1}\|). \end{aligned}$$

Thus the orthonormal family $\{x_n/\|x_n\|\}_{n=0}^{\infty}$ generates an invariant subspace $\mathfrak{M}$ for A such that $A|\mathfrak{M}$ is unitarily equivalent to S_α, which proves the first part of the theorem. Conversely, if S_β is a forward weighted shift in $\mathcal{C}(\mathbf{A}_{\aleph_0})$ with positive weight sequence β, then $\beta_n < 1$ for all n because of the last statement of Proposition 4.10, and $\prod_{n=0}^{\infty} \beta_n = 0$ because $S_\beta \in C_{00}$. □

Of course, it is immediate from Proposition 4.10 that every backward unilateral weighted shift S_α^* with a weight sequence α as in Lemma 4.11 also belongs to $\mathcal{C}(\mathbf{A}_{\aleph_0})$. As for bilateral shifts, we have the following.

THEOREM 4.13. *If $\alpha = \{\alpha_j\}_{j=-\infty}^{\infty}$ is any sequence of numbers such that $0 < \alpha_j < 1$ for every j and $\prod_{j=0}^{\infty} \alpha_j = \prod_{j=1}^{\infty} \alpha_{-j} = 0$, then the bilateral weighted shift operator W_α of multiplicity one with weight sequence α belongs to $\mathcal{C}(\mathbf{A}_{\aleph_0})$.*

Proof. For each integer n, let $\{e_{n,j} : 0 \leqslant j < \infty\}$ be an orthonormal basis for a Hilbert space $\mathcal{H}_n$, and let A_n be a backward weighted shift operator on $\mathcal{H}_n$ defined by

$$A_n e_{n,0} = 0, \quad A_n e_{n,j} = \alpha_{n-j}^{1/2} e_{n,j-1}, \quad 1 \leqslant j < \infty.$$

It is clear that the weight sequence for each A_n satisfies the condition of Lemma 4.11, so, by the above remark, we have $A_n \in \mathcal{C}(\mathbf{A}_{\aleph_0})$ for all n. Furthermore, by Proposition 4.10 the direct sum $A=\sum_{n=-\infty}^{\infty}\oplus A_n$ acting on $\sum_{n=-\infty}^{\infty}\oplus\mathcal{H}_n$ belongs to $\mathcal{C}(\mathbf{A}_{\aleph_0})$, so as before it suffices to construct an invariant subspace $\mathfrak{M}$ for A such that $A|\mathfrak{M}$ is unitarily equivalent to W_α. For this purpose, let $\{\xi_j\}_{j=0}^{\infty}$ be the sequence provided by Lemma 4.11 corresponding to the given sequence $\{\alpha_j\}_{j=0}^{\infty}$. We can define ξ_n inductively for $n<0$ such that the relation (28) remains valid for all integers n. Indeed, we set $\xi_n^2=(\alpha_n^{-1}-1)\sum_{j=n+1}^{\infty}\xi_j^2$, and a one-line computation shows that (28) is valid. We define next, for $-\infty<n<\infty$, $x_n=\sum_{j=n}^{+\infty}\xi_j e_{j,j-n}$, and, as before, the family $\{x_n/\|x_n\|\}_{n=-\infty}^{\infty}$ is orthonormal and generates an invariant subspace $\mathfrak{M}$. To see that $A|\mathfrak{M}$ is unitarily equivalent to W_α, we compute

$$Ax_n=\sum_{j=n}^{+\infty}\xi_j A e_{j,j-n}=\sum_{j=n+1}^{+\infty}\xi_j\alpha_n^{1/2}e_{j,j-n-1}=\alpha_n^{1/2}x_{n+1},$$

and the result now follows from (29), just as before. □

From these last two theorems, we obtain this interesting corollary.

COROLLARY 4.14. *Suppose* $T\in\mathbf{A}_{\aleph_0}$, *and let* V *be an operator in* $\mathcal{L}(\mathcal{H})$ *that is either a unitary operator or a forward or backward* (*unweighted*) *shift of finite multiplicity. Then* T *is unitarily equivalent to an operator* T' *acting on* $\mathcal{H}\oplus\mathcal{H}\oplus\mathcal{H}$ *of the form*

$$T'=\begin{pmatrix} A & K_1 & B\\ 0 & V+K_3 & K_2\\ 0 & 0 & D\end{pmatrix}$$

where the K_i, $1\leqslant i\leqslant 3$, *are compact. Consequently* T *is unitarily equivalent to* $(V\oplus T_1)+K$, *where* $T_1\in\mathcal{L}(\mathcal{H})$ *and* K *is compact.*

Proof. We first show that there exists an operator C in $\mathcal{C}(\mathbf{A}_{\aleph_0})$ such that $K_3=C-V$ is compact. That this is possible when V is unitary follows from Proposition 4.2 and the well-known theorem that V may be written as a diagonal unitary operator plus a compact operator. That this is possible when V is a forward or backward shift of finite multiplicity follows easily from Theorem 4.12 and Proposition 4.10. Next, using the fact that $C\in\mathcal{C}(\mathbf{A}_{\aleph_0})$, we deduce the existence of invariant subspaces $\mathfrak{M}\supset\mathfrak{N}$ for T such that $T_{\mathfrak{M}\ominus\mathfrak{N}}$ is unitarily equivalent to C, and hence T is unitarily equivalent to some operator of the form

$$T'=\begin{pmatrix} A & K_1 & B\\ 0 & V+K_3 & K_2\\ 0 & 0 & D\end{pmatrix}$$

acting on the space $\mathcal{H}\oplus\mathcal{H}\oplus\mathcal{H}$.

The compactness of K_1 and K_2 will now follow from that of K_3 and the properties of V. Indeed, in any case $V^*V=1-E$ where E is a projection onto some finite-dimensional space (perhaps (0)). Since T' is a contraction, we have, for every unit vector x in $\mathcal{H}$,

$$\|K_1x\|^2+\|(V+K_3)x\|^2=\|K_1x\|^2+(V^*Vx,x)+2\operatorname{Re}(Vx,K_3x)+\|K_3x\|^2\leqslant 1,$$

which implies that

$$\|K_1x\|^2\leqslant\|Ex\|^2+2\|K_3x\|-\|K_3x\|^2,$$

and the compactness of K_1 is now immediate from that of E and K_3. The argument that K_2 is compact proceeds similarly by taking adjoints, so the proof is complete. □

In the terminology of [14], the last conclusion of Corollary 4.14 is that T is compalent to $V\oplus T_1$. Another version of this result could, alternatively, be deduced from the fact that $\sigma_e(T)\supset\mathbf{T}$ and [1].

Added in proof. Since this paper was written the authors (in collaboration with C. Apostol and B. Chevreau) have proved the following theorems, pertinent to the above results, that will appear in subsequent papers: I. *Every T in $\mathbf{A}_{\aleph_0}$ is reflexive.* II. $\mathbf{A}_1\cap C_{00}=\mathbf{A}_{\aleph_0}\cap C_{00}$.

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