INCOMPRESSIBILITY OF SURFACES AFTER DEHN SURGERY

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Introduction. This paper was, at first, motivated by Hatcher and Thurston's question [4] of whether each Dehn surgery on a two-bridge knot along the boundary of an incompressible and ∂-incompressible surface leads to a Haken manifold. However, a deeper motivation was that our investigation was intended as a first step to approaching the Waldhausen Conjecture that each P^2-irreducible manifold with infinite fundamental group has a finite sheeted covering which is a Haken manifold [13].

First we prove Theorem 1.4, which says that incompressibility of a surface is preserved after Dehn surgery for an unknotted surface with one boundary component and with some extra conditions with two boundary components. Then we show that the condition that a surface is unknotted cannot be dropped, and the assumption about the number of the boundary components of the surface is essential too (Example 1.14). We use Theorem 1.4 to answer the Hatcher–Thurston question. Later we solve the similar problem for punctured torus bundles over S^1. This allows us to construct a large class of non-Haken, non-Seifert but almost Haken manifolds (Propositions 3.1–3.3). We consider also branched coverings of some non-Haken manifolds (Proposition 3.4).

Finally we show that the assumptions of Theorem 1.4 are often satisfied and Theorem 1.4 has several applications. Using Jaco's theorem about hierarchies [6], we find a condition for 3-manifolds which is sufficient to get unknotted surfaces (Proposition 4.3 and Corollary 4.5), and then we give examples of manifolds satisfying this condition (some closed 3-braids).

We end the paper by the remark that Theorem 1.4 can be used to prove property R for a huge class of knots.

1. Main theorem. We work in the PL-category.

DEFINITION 1.1. (a) Let M be a 3-manifold and F a surface which is either properly embedded in M or contained in ∂M. We say that F is *compressible in M* if one of the following conditions is satisfied:

(i) F is a 2-sphere which bounds a 3-cell in M, or

(ii) F is a 2-cell and either $F \subset \partial M$ or there is a 3-cell $X \subset M$ with $\partial X \subset F \cup \partial M$, or

(iii) there is a 2-cell $D \subset M$ with $D \cap F = \partial D$ and with ∂D not contractible in F.

We say that F is *incompressible* if it is not compressible.

(b) Let F be a submanifold of a manifold M. We say that F is *π_1-injective in M* if the inclusion-induced homomorphism from $\pi_1(F)$ to $\pi_1(M)$ is an injection.

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(c) Let F be a surface properly embedded in a compact 3-manifold M. We say that F is *∂-incompressible in M* if there is no 2-disk $D \subset M$ such that $D \cap F = \alpha$ is an arc in ∂D, $D \cap \partial M = \beta$ is an arc in ∂D, with $\alpha \cap \beta = \partial\alpha = \partial\beta$ and $\alpha \cup \beta = \partial D$, and α is not parallel to ∂F in F.

DEFINITION 1.2. Let M be a 3-manifold and λ a simple closed, 2-sided curve on ∂M. We define a new 3-manifold M_λ to be M with a 2-handle glued along λ. That is: Let A_λ be a regular neighborhood of λ in ∂M. Let (D^3, A) be a 3-disk with an annulus on the boundary and ϕ a homeomorphism $A_\lambda \to A$; then $M_\lambda = (M, A_\lambda) \cup_\phi (D^3, A)$. If $\{\lambda_i\}_{i=1}^n$ is a finite collection of pairwise disjoint, simple closed, 2-sided curves on ∂M then $M_{\{\lambda_i\}} \overset{\text{def.}}{=} (\dots((M_{\lambda_1})_{\lambda_2})\dots)_{\lambda_n}$. The definition does not depend on the order of the λ_i. If $\partial_1 M = T^2$ is a boundary component of M, and λ is a nontrivial, simple closed curve on $\partial_1 M$, then by the *Dehn surgery on M along* λ we mean the manifold, M^λ, obtained from M_λ by capping off the new boundary component of M_λ, which equals S^2. If $\{\lambda_i\}_{i=1}^n$ is a finite collection of nontrivial, disjoint, parallel, simple closed curves on $\partial_1 M = T^2$, then by $M^{\{\lambda_i\}}$ we mean M^{λ_1}. The manifold M^λ is obtained from M by the operation which is in fact only the second part of the original Dehn surgery (which consists of drilling and filling) and, perhaps, should be called Dehn filling.

DEFINITION 1.3. Let $(F, \partial F) \to (M, \partial M)$ be a surface properly embedded in a 3-manifold M. We say that F is *unknotted in M* if and only if $M - \operatorname{int} V_F$ is a collection of handlebodies, where V_F is a regular neighborhood of F in M.

THEOREM 1.4. *Let $(M, \partial M)$ be a compact 3-manifold with $\partial M = T^2$, and $(F, \partial F)$ a properly embedded, unknotted surface in $(M, \partial M)$. Suppose that the following two conditions are satisfied:*

1. *Each component of ∂F is not trivial in ∂M.*
2. *Either* (i) *F has one boundary component, or* (ii) *F is two-sided, non-parallel to the boundary, has two boundary components and disconnects M.*

Let $F\hat{}$ be a natural extension of F to $M^{\partial F}$. Then: F is incompressible and π_1-injective if and only if $F\hat{}$ is incompressible and π_1-injective.

The assumptions of Theorem 1.4 allow many classes of 3-manifolds and surfaces. For example:

PROPOSITION 1.5. *Let $(M, \partial M)$ be a compact 3-manifold with $\partial M = T^2$, and $(F, \partial F)$ a properly embedded surface in $(M, \partial M)$. Then:*

(a) *If M does not contain any closed, 2-sided, non-boundary parallel, incompressible surface, and F is incompressible, π_1-injective and not parallel to the boundary, then F is unknotted.*

(b) *If the image of the inclusion-induced homomorphism*

$$i_*: H_1(\partial M, \mathbf{Z}) \to H_1(M, \mathbf{Z})/\operatorname{Tor}(H_1(M, \mathbf{Z}))$$

is of finite, odd index in $H_1(M, \mathbf{Z})/\operatorname{Tor}(H_1(M, \mathbf{Z}))$ (in particular if M is orientable and $H_1(M, \mathbf{Z}) = \mathbf{Z}$), then each two-sided surface F with an even number of nontrivial boundary components disconnects M.

REMARK 1.6. The assumptions of (b) are satisfied, for example, by complements of knots in S^3 or once-punctured surface-bundles over S^1, with $H_1(M, \mathbf{Q}) = \mathbf{Q}$. However, this last condition alone is not sufficient as shown by a manifold obtained from the torus-bundle over S^1 with monodromy map $\left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right]$, by cutting out the interior of a regular neighborhood of a curve which cuts each fiber twice.

Proof of Proposition 1.5. (a) Will be seen to follow from Corollary 4.5, which will be proved in the fourth part of the paper.

(b) Assume that F does not disconnect M. Then there exists a curve $\lambda \subset M$ transverse to F such that $\lambda \cap F =$ one point. Therefore, by the assumption of (b), there exists an odd number, $2k+1$, such that $(2k+1)\lambda$ is homologous (mod $\operatorname{Tor}(H_1(M, \mathbf{Z}))$) to a curve in ∂M, or its power. However each curve in ∂M cuts ∂F an even number of times algebraically, and each cycle representing a torsion element in $H_1(M, \mathbf{Z})$ has crossing index with F equal to 0, so we have a contradiction.

If $H_1(M, \mathbf{Z}) = \mathbf{Z}$, consider $\ker(i_*: H_1(\partial M, \mathbf{Z}) \to H_1(M, \mathbf{Z}))$. Some element of $\ker i_*$ is realized by a simple, closed curve, λ_0, in ∂M. λ_0 bounds a 2-chain C in M. Let λ_1 be a simple closed curve in ∂M which cuts λ_0 once. Obviously λ_1 generates an element of infinite order in $H_1(M, \mathbf{Z})$, since M is orientable. In fact, λ_1 generates $H_1(M, \mathbf{Z})$, as it cuts C once algebraically. So i_* is onto. □

We start the proof of Theorem 1.4 with the following lemma, in fact the main lemma of the proof.

LEMMA 1.7. *Let H_n $(n>0)$ be a handlebody (orientable or not) of genus n, and $\lambda \subset \partial H_n$ be a 2-sided, simple closed curve which has a regular neighborhood, A_λ, in ∂H_n. Then $\partial((H_n)_\lambda)$ is π_1-injective in $(H_n)_\lambda$ if and only if $\partial H_n - \operatorname{int} A_\lambda$ is π_1-injective in H_n, with the exception of the case $n=1$ and λ is a meridian of ∂H_1.*

To prove Lemma 1.7, we have to describe some algebraic properties of incompressible surfaces. The results of Lyon [7] and Shenitzer [11] are crucial in the proof. First we use the technique of Lyon [7].

Let $W \subset F$ be a set of words (or cyclic words) in the basis X of a free group F. The incidence graph $J(W)$ is the graph whose vertices are in 1-1 correspondence with the non-trivial words in W with an edge joining vertices w_1 and w_2 if there exists $x \in X$ such that x or x^{-1} lies in w_1 and x or x^{-1} lies in w_2. W is connected with respect to the basis X if $J(W)$ is connected, and is *connected* if it is connected with respect to each basis of F. If the set W of elements (or cyclic elements) is not contained in any proper free factor of F and if W is connected, we say *W binds F*. Now, we can formulate Lyon's result:

LEMMA 1.8. [7] *Let $\{\gamma_i\}_{i=1}^s$ be a collection of pairwise disjoint, 2-sided, simple closed curves in ∂H_n. Then $\partial H_n - \{\gamma_i\}_{i=1}^s$ is incompressible if and only if $\{[\gamma_i]\}_{i=1}^s$ binds $F_n = \pi_1(H_n)$ and no γ_i is contractible in ∂H_n.*

Lemma 1.8 is proved in [7] for an orientable handlebody, but the proof can be extended to a nonorientable handlebody as well (using the fact that each automorphism of a free group $\pi_1(H_n)$ is induced by a homeomorphism of H_n). □

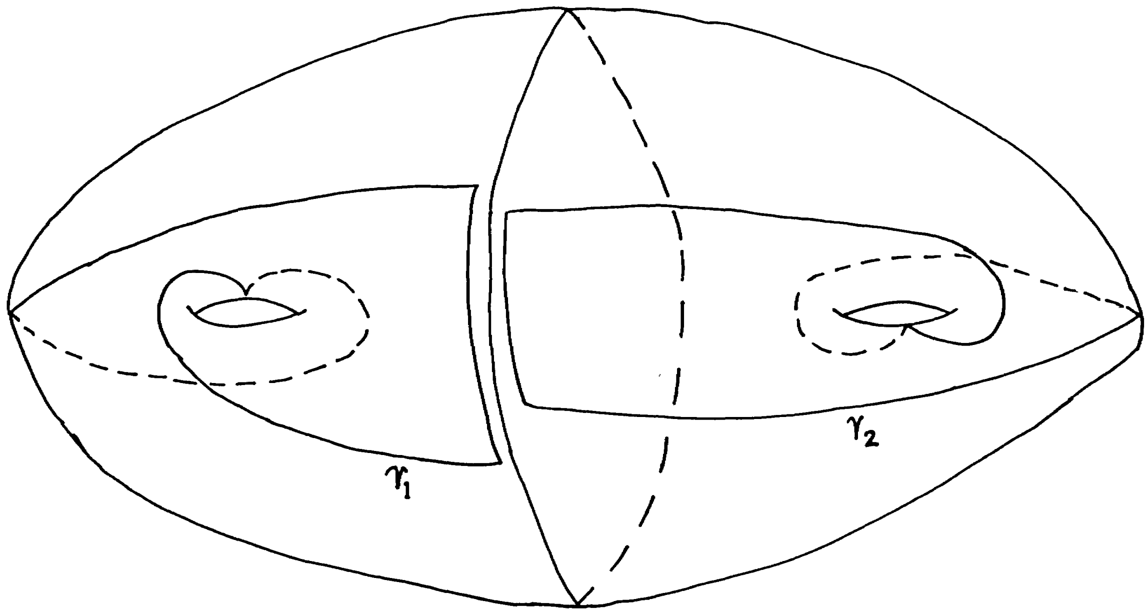

Figure 1.1.

Shenitzer [11] noticed that the following is an immediate consequence of the Grushko Theorem and the Freiheitssatz.

PROPOSITION 1.9. [11] *If* $G=\{F_n: r\}$ *and* $n>1$, *then the following conditions are equivalent:*

1. *G can be decomposed into a free product or* $n=2$ *and r is a free generator of* F_2 *(i.e.,* $G=\mathbf{Z}$*).*
2. *r does not bind* F_n.

Now, we can prove Lemma 1.7. If $n=1$, the proof consists in verification of several easy cases, and we omit it. Assume $n>1$.

($\Rightarrow$) Assume that $\partial H_n - \operatorname{int} A_\gamma$ is not π_1-injective. Let D^2 be a compressing disk of $\partial H_n - \operatorname{int} A_\gamma$ in H_n (existing by the Loop Theorem). Then ∂D^2 is not contractible in $\partial H_n - \operatorname{int} A_\gamma$ but it is contractible in $\partial(H_n)_\gamma$. Therefore either (i) ∂D^2 is parallel to γ in ∂H_n, or (ii) ∂D^2 and ∂A_γ bound a pair of pants in ∂H_n, so D^2 cuts off from H_n a genus one handlebody which contains γ. In both cases we use the assumption $n>1$ in order to find a compressing disk of $\partial H_n - \operatorname{int} A_\gamma$ in H_n which does not satisfy (i) and (ii). This disk will also be a compressing disk of $\partial((H_n)_\gamma)$ in $(H_n)_\gamma$, which contradicts the π_1-injectivity of $\partial((H_n)_\gamma)$.

($\Leftarrow$) $\partial H_n - \operatorname{int} A_\gamma$ is π_1-injective so, by Lemma 1.8, $[\gamma]$ binds $F_n = \pi_1(H_n)$. Then, by Proposition 1.9, the group $\{F_n: [\gamma]\}$ cannot be decomposed into a free product and, for $n=2$, $[\gamma]$ is not a free generator of F_2 (i.e., $\{F_2: [\gamma]\} \neq Z$). Therefore we conclude, using the Loop Theorem, that $\partial((H_n)_\gamma)$ is π_1-injective in $(H_n)_\gamma$. $\square$

REMARK 1.10. Proposition 1.9 (resp. Lemma 1.7) is not true for more than one relator (resp. curve). The simplest example is: $G=\{x_1, x_2: x_1^2x_2^2, x_2^2\} = \mathbf{Z}_2 * \mathbf{Z}_2$ but $\{x_1^2x_2^2, x_2^2\}$ binds F_2 (see [7]).

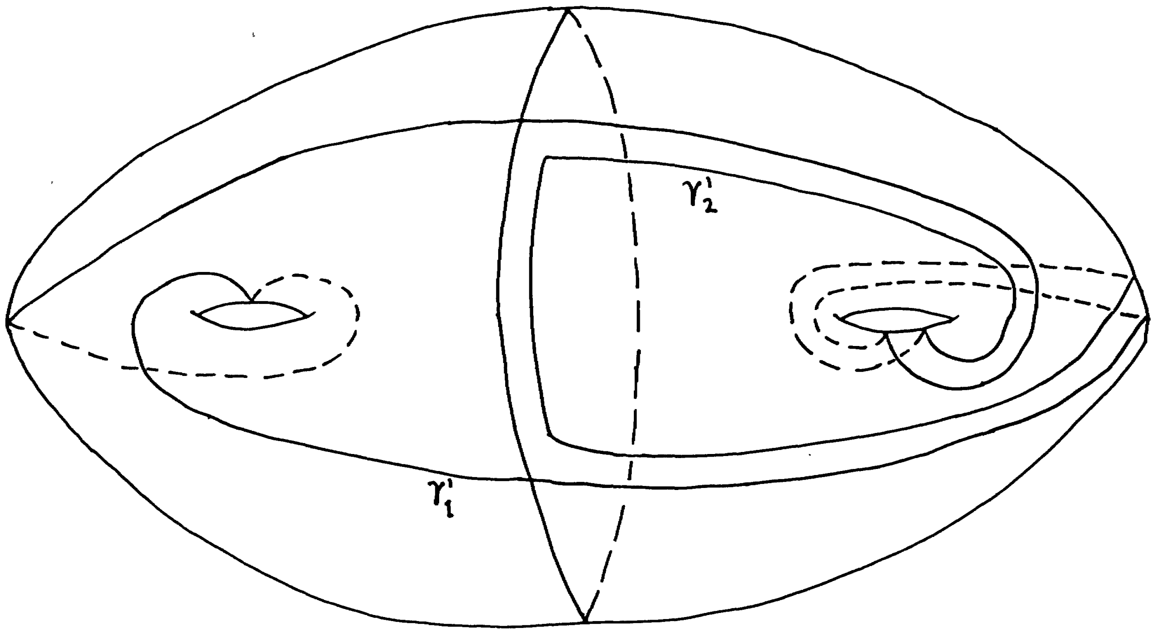

Figure 1.2.

Now we give some geometric examples which illustrate Remark 1.10.

EXAMPLE 1.11. Consider a genus 2, orientable handlebody, H_n, with two systems of curves: γ_1, γ_2 (Figure 1.1) and γ_1', γ_2' (Figure 1.2).

In the natural presentation of $\pi_1(H_2)$, when H_2 is embedded in $\mathbf{R}^3$ as on the picture we have: $[\gamma_1]=x_1^2$, $[\gamma_2]=x_2^2$ and $\partial H_2 - \operatorname{int} A_{\gamma_1} - \operatorname{int} A_{\gamma_2}$ is compressible in H_2, where A_{γ_i} is a tubular neighborhood of γ_i in ∂H_2 $(i=1,2)$.

In $\pi_1(H_2)$, curves γ_1' and γ_2' can be written: $[\gamma_1'] = x_1^2 x_2^2$, $[\gamma_2'] = x_2^2$ and $\partial H_2 - \operatorname{int} A_{\gamma_1'} - \operatorname{int} A_{\gamma_2'}$ is incompressible (compare [7]; $x_1^2x_2^2$ and x_2^2 are minimal cyclic words which bind F_2). But $(H_2)_{\{\gamma_1,\gamma_2\}} = (H_2)_{\{\gamma_1',\gamma_2'\}} = L(2;1)\#L(2;1)\#D^3$ with $\pi_1((H_2)_{\{\gamma_1,\gamma_2\}}) = \mathbf{Z}_2 * \mathbf{Z}_2$.

EXAMPLE 1.12. Consider a genus 4, orientable handlebody with two curves γ_1, γ_2 (Figure 1.3).

In the natural presentation of $\pi_1(H_4)$, we have: $[\gamma_1] = x_2^2x_1^2x_4^2x_3^2$, $[\gamma_2] = x_4^2x_3^2$. We have in this example that $\partial H_4 - \operatorname{int} A_{\gamma_1} - \operatorname{int} A_{\gamma_2}$ is incompressible (see [7]) but $\partial((H_4)_{\{\gamma_1,\gamma_2\}})$ is a compressible genus 2 surface in $(H_4)_{\{\gamma_1,\gamma_2\}}$.

EXAMPLE 1.13. Let $F=P^2\#P^2\#\cdots\#P^2$ be a nonorientable surface of genus n $(n \geqslant 3)$. Let $F'=F\#D^2$ (surface with one hole). Consider the I-bundle over F' such that the bundle space is orientable. In fact it is a genus n orientable handlebody, H_n, because its fundamental group is free of rank n. Now $(H_n)_{\gamma_1}$, where $\gamma_1 = \partial F'$, is a natural extension of the I-bundle over F' to an I-bundle over F. Of course, $\partial(H_n)_{\gamma_1}$ is incompressible in $(H_n)_{\gamma_1}$. Now, let γ_2'' be any nontrivial (in F) simple, closed curve in F' which does not change orientation on F', so the restriction of the bundle to γ_2'' is a product bundle. Assume additionally that the boundary components, γ_2 and γ_2', of this bundle are not parallel in $\partial(H_n)_{\gamma_1}$. Then

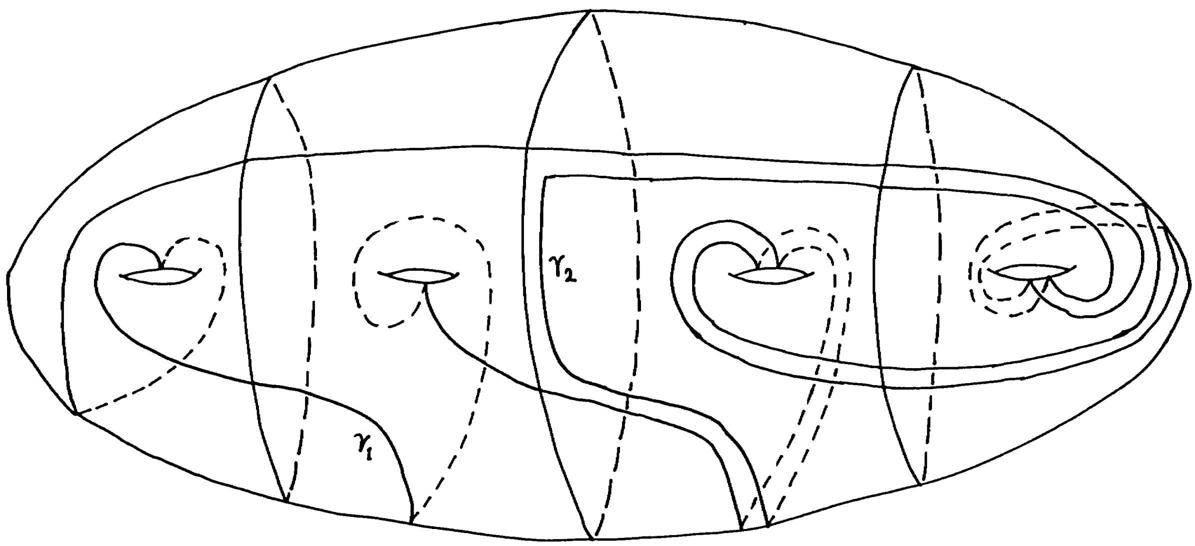

Figure 1.3.

$\partial(H_n)_{\gamma_1} - \operatorname{int} A_{\gamma_2}$ is incompressible in $(H_n)_{\gamma_1}$ but $\partial((H_n)_{\gamma_1})_{\gamma_2}$ is compressible in $((H_n)_{\gamma_1})_{\gamma_2}$. In order to show the compressibility, let the annulus $A_{\gamma_2''}$ be the restriction of our I-bundle to the curve γ_2''. γ_2, which is one of the boundary components of $A_{\gamma_2''}$, bounds a disk, D_{γ_2}, in $((H_n)_{\gamma_1})_{\gamma_2}$. So $A_{\gamma_2''} \cup D_{\gamma_2}$ forms a properly embedded disk in $(H_n)_{\gamma_1})_{\gamma_2}$ with boundary γ_2' and it follows from our assumptions that γ_2' is not trivial in $\partial((H_n)_{\gamma_1})_{\gamma_2}$. So $A_{\gamma''} \cup D_{\gamma_2}$ is the compressing disk of $\partial((H_n)_{\gamma_1})_{\gamma_2}$ in $((H_n)_{\gamma_1})_{\gamma_2}$.

Proof of Theorem 1.4.

($\Leftarrow$) If $F = D^2$ it is trivial. If F is a Möbius band and F is not π_1-injective, we conclude that $M = M_1 \#$ (genus one handlebody) and we can choose a meridian disk of the handlebody disjoint from F (F is one- or two-sided). If we cut M open along this meridian disk we obtain a contradiction with the assumption that F is unknotted (compare (b) below). Let $F \neq D^2$ and Möbius band. Assume F is not π_1-injective (it includes the case of F compressible). First consider the case of two-sided F. So F is compressible. Let $\gamma \subset F$ be the boundary of a compressing disk. We have three possibilities for γ:

(a) γ is not parallel to ∂F, and γ does not cut out from F a pair of pants. In this case γ is not trivial in $F\hat{}$ so $F\hat{}$ is not π_1-injective.

(b) γ is parallel to ∂F. Then ∂F bounds a disk in M, so $M = M_1 \#$ (genus one handlebody) and ∂F is parallel to the meridian of the handlebody, but this contradicts the assumption that F is unknotted and $F\hat{}$ incompressible (we cut M along a meridian disk of the handlebody which is disjoint from F. F should still be unknotted but this contradicts the incompressibility of $F\hat{}$).

(c) γ cuts out from F a pair of pants. This contradicts the assumption that F is unknotted and $F\hat{}$ incompressible.

If F is not two-sided, let $\tilde{F}$ denote the boundary of the tubular neighborhood of F in M. Since $F \neq$ Möbius band, $\tilde{F}\hat{}$ is incompressible. In fact $\tilde{F}$ satisfies all assumptions of Theorem 1.4, so $\tilde{F}$ is incompressible, π_1-injective, so F is π_1-injective (in particular incompressible).

($\Rightarrow$) We assume F is incompressible and π_1-injective. If F is an incompressible disk then $F\hat{}$ is a nonseparating sphere, so incompressible. If F=Möbius band, $F\hat{}$ is a projective space, thus incompressible, π_1-injective. If F is an annulus, $F\hat{}$ is a 2-sphere which cuts $M^{\partial F}$ into a nontrivial connected sum (each factor has a genus one Heegaard splitting and is different than S^3 because F is not parallel to ∂M), so $F\hat{}$ is incompressible. Let $F \neq D^2$, Möbius band or annulus. Now it is enough to prove the theorem for two-sided F (if F is one-sided, consider the boundary of a tubular neighborhood of F). Let V_F be a tubular neighborhood of F in M. $M - \operatorname{int} V_F$ is a handlebody, H_n (or a pair of handlebodies, H_n and H_n'), $n>1$. Let $A = H_n \cap \partial M$ (resp. $A = H_n \cap \partial M$ and $A' = H_n' \cap \partial M$). $\partial H_n - \operatorname{int} A$ is π_1-injective in H_n. Now we use Lemma 1.7 to get $\partial((H_n)_{\text{core of } A})$ π_1-injective, so incompressible (resp. $\partial((H_n)_{\text{core of } A})$ and $\partial((H_n')_{\text{core of } A'})$ π_1-injective). $M^{\partial F}$ is obtained by gluing together either two components of the boundary of $(H_n)_{\text{core of } A}$ or the boundaries of $(H_n)_{\text{core of } A}$ and $(H_n')_{\text{core of } A'}$, so $F\hat{}$ is π_1-injective, incompressible in $M^{\partial F}$. □

Theorem 1.4 is not valid if we drop the assumption that the surface is unknotted or the assumption about the number of the boundary components of the surface.

EXAMPLE 1.14. We use the notation of Example 1.13.

(a) Consider two copies of the manifold $(H_n)_{\gamma_1}$, say M_1 and M_2. Let F_1 and F_2 denote the incompressible surfaces $\partial M_1 - \operatorname{int} A_{\gamma_2}$ and $\partial M_2 - \operatorname{int} A_{\gamma_2}$, respectively. Now glue together M_1 and M_2 along F_1 and F_2. As the result we have a 3-manifold, M, with ∂M=torus and F_1 is π_1-injective, incompressible in M. F_1 is *not unknotted* in M and has two boundary components. Since $\partial(M_1)_{\gamma_2}$ is compressible, $F_1\hat{}$ is compressible in $M^{\partial F_1}$. F_1 (and $F_1\hat{}$) is connected if γ_2 does not disconnect ∂M_1 and has two components if γ_2 disconnects ∂M_1.

(b) Let us assume that γ_2 does not disconnect $\partial(H_n)_{\gamma_1}$, so $\gamma_1 \cup \gamma_2$ does not disconnect ∂H_n. Consider two copies of H_n, H_n' and H_n'', with the appropriate curves γ_1', γ_2' and γ_1'', γ_2''. Let the boundary of $A_{\gamma_1'}$ be denoted by $\partial^+(A_{\gamma_1'})$ and $\partial^-(A_{\gamma_1'})$, similarly the boundary components of $A_{\gamma_2'}$, $A_{\gamma_1''}$ and $A_{\gamma_2''}$. Let $F' = \partial H_n' - \operatorname{int} A_{\gamma_1'} - \operatorname{int} A_{\gamma_2'}$ and $F'' = \partial H_n'' - \operatorname{int} A_{\gamma_1''} - \operatorname{int} A_{\gamma_2''}$. F' and F'' are homeomorphic to an orientable surface of genus $n-2$, with four holes. We can find a homeomorphism $f: F' \to F''$ such that

$$f(\partial^+(A_{\gamma_1'})) = \partial^+(A_{\gamma_1''}), \quad f(\partial^-(A_{\gamma_1'})) = \partial^+(A_{\gamma_2''}),$$
$$f(\partial^+(A_{\gamma_2'})) = \partial^-(A_{\gamma_1''}) \quad \text{and} \quad f(\partial^-(A_{\gamma_2'})) = \partial^-(A_{\gamma_2''}).$$

If we glue together H_n' and H_n'' along F' and F'', using f, we get a manifold, say N, with F' incompressible, π_1-injective, unknotted, two-sided in N. Furthermore $\partial N = T^2$. However, $F'\hat{}$ is compressible in $N^{\partial F}$. F' has four boundary components.

2. Corollaries of Theorem 1.4 and related topics.

COROLLARY 2.1. (Answer to a question of Hatcher and Thurston [4]). *Let $K_{p/q}$ be the 2-bridge knot of type p/q. Let $M_{m/l}(K_{p/q})$ denote the manifold* *obtained from Dehn surgery on $K_{p/q}$, where a tubular neighborhood of $K_{p/q}$ is cut out and reglued so as to make a meridian disk kill a curve in $S^3 - K_{p/q}$, wrapping l times around $K_{p/q}$ longitudinally and m times meridianally. Let F be an incompressible, ∂-incompressible surface in $S^3 - \operatorname{int} V_{K_{p/q}}$, where $V_{K_{p/q}}$ is a tubular neighborhood of $K_{p/q}$ in S^3. Let the boundary components of F have slope m/l. Then $M_{m/l}(K_{p/q})$ is a Haken manifold or a connected sum of lens spaces.*

This gives an affirmative answer to the question posed in [4; after Theorem 2].

Proof. It follows from [4] that if m/l is the slope of the boundary of some incompressible, ∂-incompressible surface in $S^3 - \operatorname{int} V_{K_{p/q}}$, then there exists an incompressible, ∂-incompressible surface F (orientable or not) with one boundary component and the slope of ∂F equal to m/l. Furthermore, each incompressible, ∂-incompressible surface in $S^3 - \operatorname{int} V_{K_{p/q}}$ is π_1-injective and unknotted. So $F\hat{}$ in $M_{m/l}(K_{p/q})$ is either projective space (this case leads us to a connected sum of lens spaces and we deal with a torus knot), or $\chi(F\hat{}) \leqslant 0$ and it follows from Theorem 1.4 that $F\hat{}$ is incompressible and π_1-injective. So $F\hat{}$ or the boundary of a regular neighborhood of $F\hat{}$ (if $F\hat{}$ is not orientable) gives us a 2-sided, incompressible surface. Therefore $M_{m/l}(K_{p/q})$ is a Haken manifold. It is irreducible when we are not in a case of a connected sum of lens spaces. □

COROLLARY 2.2. *Let M be a punctured-torus bundle over S^1 with a hyperbolic monodromy map. If F is an incompressible, ∂-incompressible and π_1-injective surface in M, then $F\hat{}$ is incompressible, π_1-injective in $M^{\partial F}$.*

Corollary 2.2 was proved independently by Culler et al. [1, Remark 2.5.2].

Proof. It follows from Theorem 1.4 in the case of one boundary component of F. Generally we use Proposition 1.9. We use the terminology of [2]. Each incompressible, ∂-incompressible, π_1-injective surface, F, with boundary on ∂M has one, two or four boundary components. If F=Möbius band, $F\hat{}$ is a projective space and we are done. Now we can assume that F is orientable, because for non-orientable F it is enough to deal with the boundary of a tubular neighborhood of F in M. Consider the sequence of groups

$$\pi_1(\partial_1 F) \xrightarrow{(i_0)_*} \pi_1(F) \xrightarrow{(i_1)_*} \pi_1(M) \xrightarrow{p_*} \pi_1(S^1)$$

where i_0 and i_1 are embeddings, p is the projection of the fibration, and $\partial_1 F$ is a single component of ∂F. With the exception of F=fiber of the fibration (in which case Corollary 2.2 is obvious), we have $(pi_1 i_0)_*(\pi_1(\partial F)) \neq 0$. Let $\tilde{M}$ be a covering of M associated with the subgroup $p_*^{-1}((pi_1 i_0)_*(\pi_1(\partial F))$ of $\pi_1(M)$. $\tilde{M}$ is a one-, two- or four-sheeted covering. $\tilde{F}$, the (connected) lift of F associated with the covering $\tilde{M} \to M$, has four boundary components.

FACT 2.3. Consider the diagram

$$\begin{array}{ccc} \tilde{S} & \hookrightarrow & \tilde{N} \\ \downarrow & & \downarrow \\ S & \hookrightarrow & N \end{array}$$

where $\tilde{N}$ is a finite sheeting covering of a 3-manifold N and $\tilde{S}$ is a lifting of a

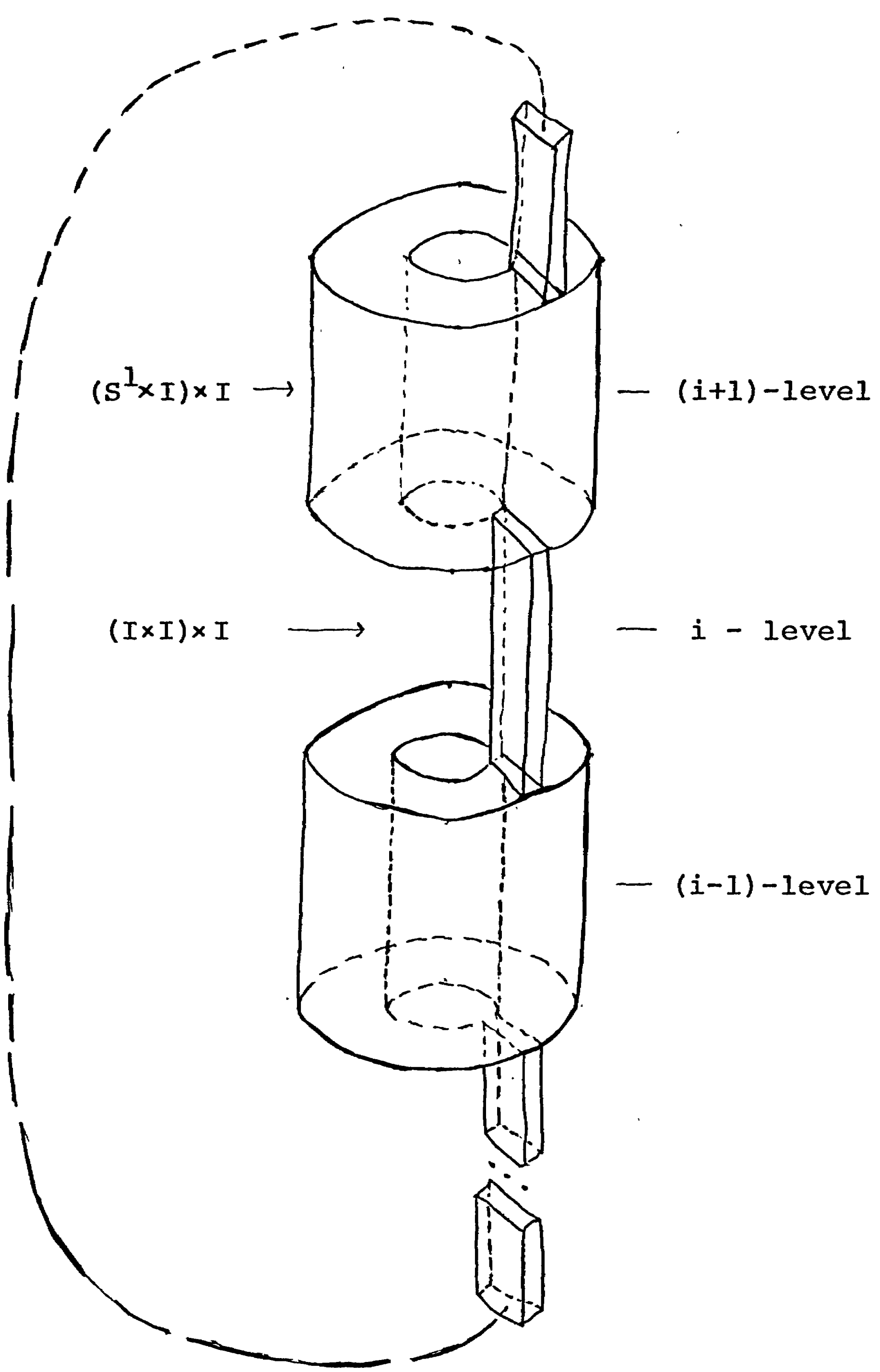

Figure 2.1

surface $S\subset N$. S and $\tilde{S}$ satisfy: $\tilde{S}$ is π_1-injective if and only if S is π_1-injective. Furthermore, if S is compressible, then $\tilde{S}$ is compressible. Therefore, in order to prove Corollary 2.2, it is enough to study the case of F with exactly four boundary components.

F is unknotted in M, and in the case of four boundary components, M cut open along F is a disjoint union of two handle-bodies. If F is associated with a minimal edge-path of k edges (in our case k is even), then each handlebody is of genus $(k/2)+1$ and topologically is depicted in Figure 2.1.

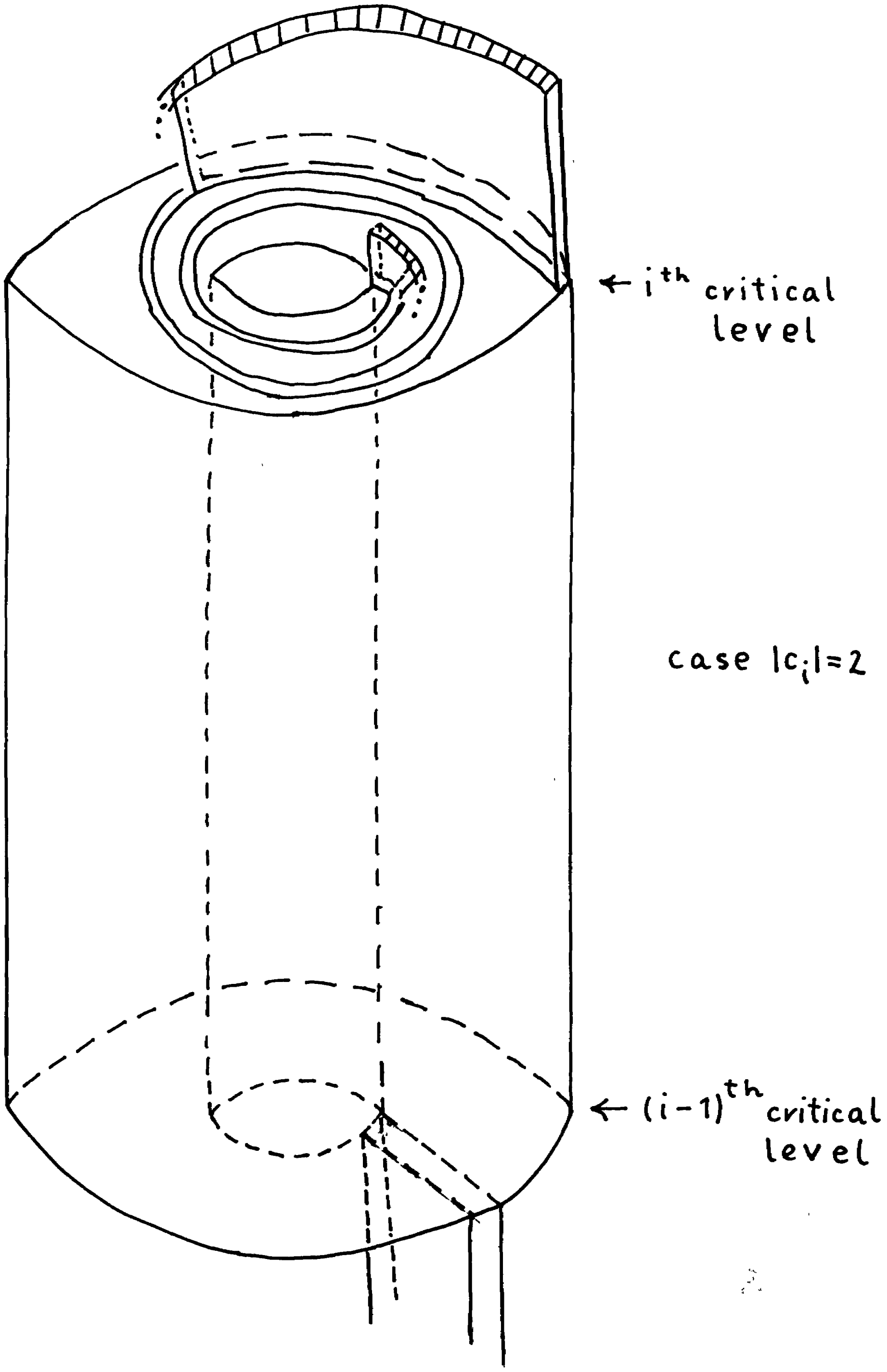

Figure 2.2

Now, it remains to find how the "trace" of ∂F (i.e., ∂(a handlebody $\cap\, \partial M$)) looks on each handlebody. Recall (cf. [2]) that if an invariant edge-path is defined by vertices $\ldots a_{-1}/b_{-1}, a_0/b_0, a_1/b_1, \ldots$ and c_i are defined to satisfy $a_i/b_i = (a_{i-2}+c_i a_{i-1})/(b_{i-2}+c_i b_{i-1})$, then the edge-path is minimal and the associated surface is π_1-injective if and only if, for each i, $|c_i| \geqslant 2$.

"Geometrically", one piece of our handlebody is depicted in Figure 2.2. We can see that the "trace" of ∂F on the boundaries of handlebodies $H_{(k/2)+1}$ may

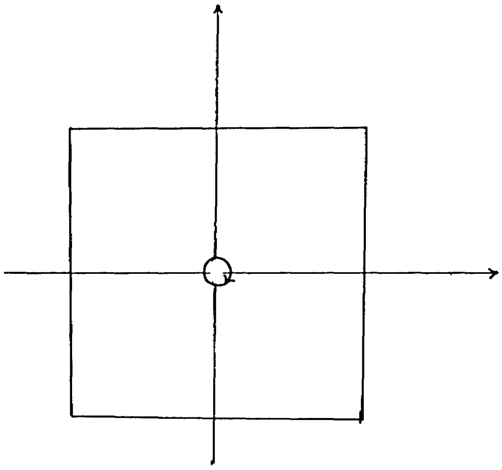

Figure 3.0

be written, in an appropriately chosen base of $F_{(k/2)+1}=\pi_1(H_{(k/2)+1})$, as follows:

(a) First handlebody. The trace of ∂F is represented by words:

$$w_1=w_2=x_0;\quad w_3=w_4=x_0x_1^{c_1}x_3^{c_3}x_5^{c_5}\dots x_{k-1}^{c_{k-1}}.$$

(b) Second handlebody. The trace of ∂F is represented by words:

$$w_1=w_2=x_0;\quad w_3=w_4=x_0x_2^{c_2}x_4^{c_4}\dots x_k^{c_k}.$$

In both cases the group $G=\{F_{(k/2)+1}: w_1,w_2,w_3,w_4\}=\{F_{k/2}: w_1^{-1}w_3\}$ is indecomposable under free product and $G\neq \mathbf{Z}$ ($w_1^{-1}w_3$ binds $F_{k/2}$) (see [11] or [7]). Therefore, we can end the proof of Corollary 2.2 similarly as the proof of Lemma 1.7. In the case of $k=2$, one gets that $H_{(k/2)+1}=H_2$, $\pi_1(M^{\partial F})=\mathbf{Z}_{|c_1|}*\mathbf{Z}_{|c_2|}$ and $M^{\partial F}$ is a connected sum of lens spaces with $F^{\wedge}$ the sphere which defines the connected sum. □

The method of the proof of Corollary 2.2 may also be used to show that a surface F is π_1-injective if and only if, for all i, $|c_i|\geqslant 2$. This method of the proof seems to be shorter than that of [2] or [1].

3. Almost Haken manifolds. A 3-manifold is said to be almost Haken if it allows a finite sheeted covering which is a Haken manifold. Now we construct some examples of almost Haken manifolds. Let $\bar{\alpha}=\begin{bmatrix}1&1\\0&1\end{bmatrix}$ and $\beta=\begin{bmatrix}1&0\\1&1\end{bmatrix}$. Consider M_ϕ – punctured-torus bundle over S^1 with hyperbolic monodromy map ϕ. $H_1(\partial M_\phi)$ has a natural system of coordinates. The first generator, a longitude, is determined by the boundary of a fiber with the clockwise orientation (see Figure 3.0). To define the second generator, a meridian, we have to consider two cases:

(a) $\phi: F\to F$ is given by matrix $A\in \mathrm{SL}(2,\mathbf{Z})$ with $\operatorname{tr} A>0$; so A has two positive eigenvalues. Then the restriction of A to ∂F (∂F is understood to be the set of angles, to omit technical problems) has four fixed points, say $\pm\alpha_1$ and $\pm\alpha_2$ (see Figure 3.0). Now, under projection $F\times\mathbf{R}\to M_\phi$, the image of the straight line in

$\partial F \times \mathbf{R}$ which joins $(\alpha_1, 0)$ and $(\alpha_1, 1)$ is a circle which determines the second generator of $H_1(\partial M_\phi)$.

(b) $\phi: F \to F$ is given by matrix A with $\operatorname{tr} A < 0$; so A has two negative eigenvalues. Then the restriction of $-A$ to ∂F has four fixed points, say $\pm\alpha_1$ and $\pm\alpha_2$, so $A(\alpha_1) = -\alpha_1$ and $A(-\alpha_1) = \alpha_1$. Let γ be the curve in $\partial F \times \mathbf{R}$ given by the equation $z = e^{\pi i t}$ where $z \in \partial F$ and $t \in \mathbf{R}$ (so γ joins $(\alpha_1, 0)$ and $(-\alpha_1, 1)$ with negative half twist with respect to the chosen orientation of ∂F). The image of γ under projection $F \times \mathbf{R} \to M_\phi$ determines the second generator of $H_1(\partial M_\phi)$. Each noncontractible curve $\gamma \in \partial M_\phi$ is defined, up to isotopy, by a pair (a, b) (or a slope a/b). Let $M_\phi(a, b)$ denote the manifold obtained from M_ϕ by Dehn surgery along the curve of slope a/b (Definition 1.2).

PROPOSITION 3.1. *Consider M_{ϕ^k}, where $\phi = \bar{\alpha}\beta$ (if $k=1$, M_{ϕ^k} is the complement of the figure-eight knot). If $(a, b) = 1$ and either $0 < |a| \leqslant (|b|k - 5)/4$ or $|a| = (|b|k - 3)/4$ or $|a| = (|b|k)/4$, then the manifold $M_{\phi^k}(a, b)$ is almost sufficiently large (i.e., almost Haken).*

In particular:

PROPOSITION 3.2. *Let M_ϕ ($\phi = \bar{\alpha}\beta$) be the complement of the figure-eight knot. Let l, m be co-prime numbers satisfying either $0 < |l| \leqslant (|m| - 5)/4$ or $|l| = (|m| - 3)/4$ or $l/m = \pm 1/4$ or $l/m = 1/0$ or $l/m = \pm 1/1$ or $l/m = \pm 1/2$ or $l/m = \pm 1/3$. Then $M_\phi(l, m)$ is almost Haken. Furthermore, if $l/m = \pm 1/1$, $\pm 1/2$, or $\pm 1/3$, then $M_\phi(l, m)$ is a Seifert fibered manifold; if $l/m = \pm 1/4$ or $1/0$, then $M_\phi(l, m)$ is a Haken manifold; and in other cases (considered in Proposition 3.2) $M_\phi(l, m)$ is neither Haken nor a Seifert fibered manifold but almost Haken and a hyperbolic manifold.*

The almost Haken manifold $M_\phi(1, 16)$ was independently discovered by Culler et al. [1].

To prove Proposition 3.1 we first establish:

PROPOSITION 3.3. *Consider all incompressible, boundary incompressible, orientable surfaces in M_{ϕ^k} ($\phi = \bar{\alpha}\beta$). Then a/b is a slope of the boundary of such a surface if and only if $a/b = 1/0$ or $a/b = M/4$, where either $0 \leqslant |M| \leqslant k - 5$ or $|M| = k - 3$ or $|M| = k$.*

We prove Proposition 3.3 using the main theorem of [2] and computing from the diagram (Figure 3.1) the slopes of the boundaries of surfaces associated with ϕ-invariant minimal edge-paths (see [2, Table 1]).

Proof of Proposition 3.1. Consider the b-sheeted covering of $M_{\phi^k}(a, b)$ associated with the cyclic, b-sheeted covering of M_{ϕ^k} (projection $M_{\phi^k} \to S^1$ gives a natural epimorphism $\pi_1(M_{\phi^k}) \to \mathbf{Z}$). Now we use Proposition 3.3 (for $M_{\phi^{kb}}$) and Corollary 2.2. □

To prove Proposition 3.2 we use additionally Theorem 4.7 of [12].

We can ask whether $M_\phi(a, b)$ allows a branched covering, of a given specific type, which is a Haken manifold. We solve this problem for "cyclic", branched

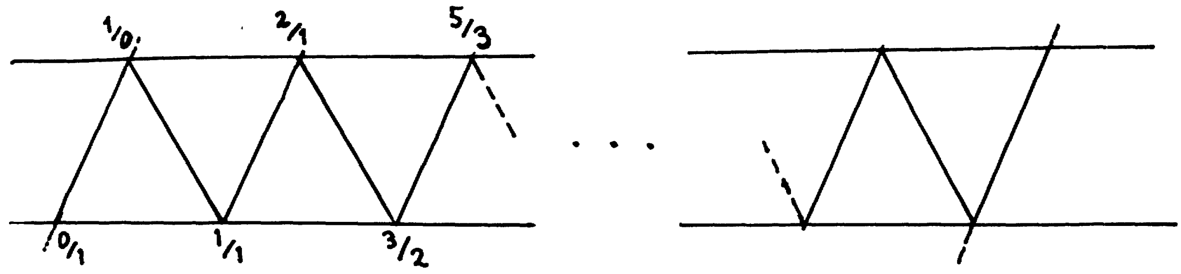

Figure 3.1

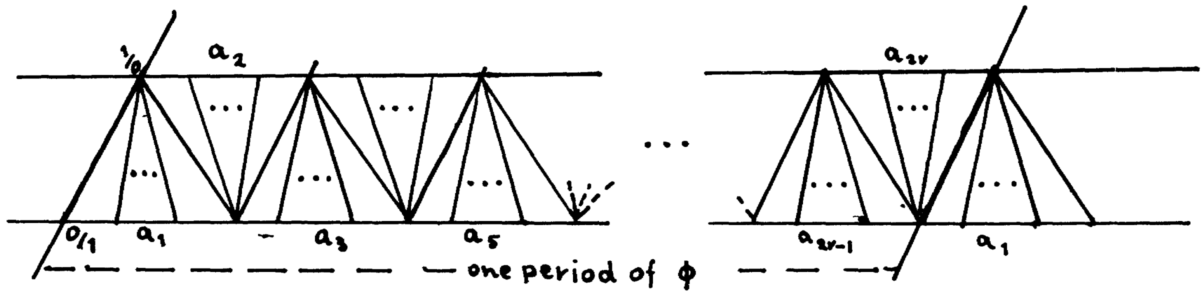

Figure 3.2

coverings. By "cyclic" coverings of $M_\phi(a,b)$ we understand the coverings associated with cyclic coverings of M_ϕ (these last are determined by the fiber projection $M_\phi \to S^1$).

PROPOSITION 3.4. *Let M_ϕ be a punctured torus bundle over S^1 with a hyperbolic monodromy map $\phi=\pm\bar{\alpha}^{a_1}\beta^{a_2}\ldots\bar{\alpha}^{a_{2r-1}}\beta^{a_{2r}}$, where $a_1,a_2,\ldots,a_{2r-1},a_{2r}$, $r>0$. Let the slope a/b satisfy:*

(a)
$$-\frac{1}{4}\sum_{i=1}^{r}a_{2i}\leqslant\frac{a}{b}\leqslant\frac{1}{4}\sum_{i=1}^{r}a_{2i-1}$$

if ϕ has positive eigenvalues; or

(b)
$$-\frac{1}{4}\left(\left(\sum_{i=1}^{r}a_{2i}\right)-2\right)\leqslant\frac{a}{b}\leqslant\frac{1}{4}\left(\left(\sum_{i=1}^{r}a_{2i-1}\right)+2\right)$$

if ϕ has negative eigenvalues. Then $M_\phi(a,b)$ allows a finite sheeted, "cyclic", branched covering which is a Haken manifold. Furthermore, if a/b does not satisfy (a) *or* (b) *and $a/b\neq 1/0$, then $M_\phi(a,b)$ does not allow a finite, "cyclic", branched covering which is a Haken manifold.*

The case of $\phi=\bar{\alpha}\beta$ and $a/b=0/1$ (so $M_\phi(a,b)=S^3$) was described in [1].

Proof. Consider the diagram of ϕ (i.e., the infinite strip associated with ϕ (Figure 3.2)); see [2]. By the slope of a minimal, ϕ-invariant edge-path in the strip, we mean the slope of the boundary of the associated incompressible, ∂-incompressible surface in M_ϕ. That is: Let $\gamma=\gamma_1\cup\cdots\cup\gamma_k$, where γ_i is the ith edge of γ (we consider one period of ϕ). Then

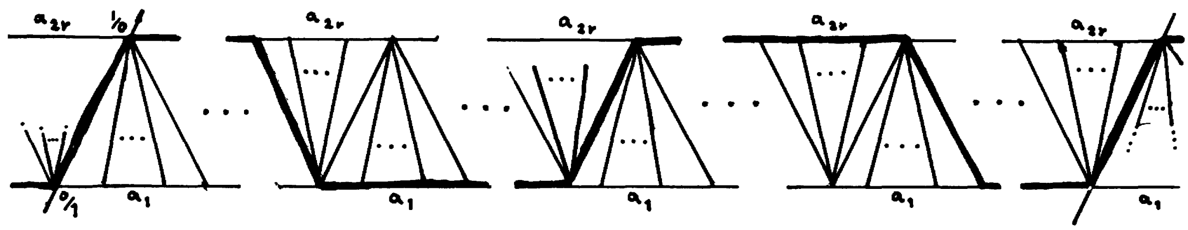

Figure 3.3

$$\mathrm{sl}(\gamma)=\begin{cases}\frac{1}{4}\sum_{i=1}^{k} q_i & \text{if eigenvalues of } \phi \text{ are positive,}\\ \frac{1}{4}((\sum_{i=1}^{k} q_i)+2) & \text{if eigenvalues of } \phi \text{ are negative,}\end{cases}$$

where

$$q_i=\begin{cases}1 & \text{if } \gamma_i \text{ has the beginning on the bottom of the strip,}\\ -1 & \text{if } \gamma_i \text{ has the beginning on the top of the strip.}\end{cases}$$

First we prove Proposition 3.4 for ϕ with positive eigenvalues and $r>1$.

Step 1. Consider the diagram of ϕ^k. There exists a k such that in the diagram of ϕ^k there is a minimal, ϕ^k-invariant edge-path of slope 0/1. To prove this, first we show the following fact.

FACT 3.5. Let $\phi = \bar{\alpha}^{a_1}\beta^{a_2}\bar{\alpha}^{a_3}\ldots\bar{\alpha}^{a_{2r-1}}\beta^{a_{2r}}$ where $r>1$, $A_1=\sum_{i=1}^{r} a_{2i-1}$, and $A_2=\sum_{i=1}^{r} a_{2i}$. Then there exists a minimal ϕ^k-invariant edge-path γ of slope $\frac{1}{4}(j_0(A_1-a_1)+(i_0+j_1)A_1-i_0(A_2-a_{2r})-(j_0+i_1)A_2)$, where $i_1,j_1\geqslant 0$, $i_0,j_0\geqslant 1$ and $2i_0+2j_0+i_1+j_1=k$, which starts from (goes through) the vertex 1/0. The same is true for the vertex 0/1.

Proof of Fact 3.5 follows from the "combinatorics" of the strip for ϕ^k; see Figure 3.3 for $i_1=j_1=0$, $i_0=j_0=1$, $k=4$ and start from 1/0.

To prove Step 1, it is enough to put

$$i_0=A_1-a_1,\quad j_0=A_2-a_{2r},\quad i_1=A_2A_1^2-(A_2-a_{2r}),$$
$$j_1=A_1A_2^2-(A_1-a_1)\quad\text{and}\quad k=A_2A_1^2+A_2-a_{2r}+A_1A_2^2+A_1-a_1.$$

Step 2. Let N_0 be a number such that the diagram of ϕ^{N_0} has ϕ^{N_0}-invariant, minimal edge-paths of slope 0/1 (one going through the vertex 1/0 and the second through 0/1). Assume that $0\leqslant a/b\leqslant A_1/4$ (for a negative a/b the proof is similar). Let $N=N_0A_1b$. Consider M_{ϕ^N}, the N-sheeted, cyclic covering of M_ϕ. One period of the diagram of ϕ^N contains $N=N_0A_1b$ segments, which are equivalent to periods of the diagram of ϕ. Let us construct a ϕ^N-invariant, minimal edge-path, γ, in a period of the diagram of ϕ^N, as follows: γ starts at 0/1 and is the bottom curve through the first $4N_0a$ segments (we call this part γ_1). We have still left $N_0A_1b-4N_0a=N_0(A_1b-4a)$ segments (this number is not negative because of the assumption $a/b\leqslant A_1/4$). However, we know that N_0 segments contain a curve of slope 0/1, so $N_0(A_1b-4a)$ segments also contain a curve of slope 0/1. We can assume that this curve starts from the bottom of the strip, which allows us to extend γ_1 to γ; Σ has the slope $(4N_0aA_1)/4=A_1N_0a=Na/b$

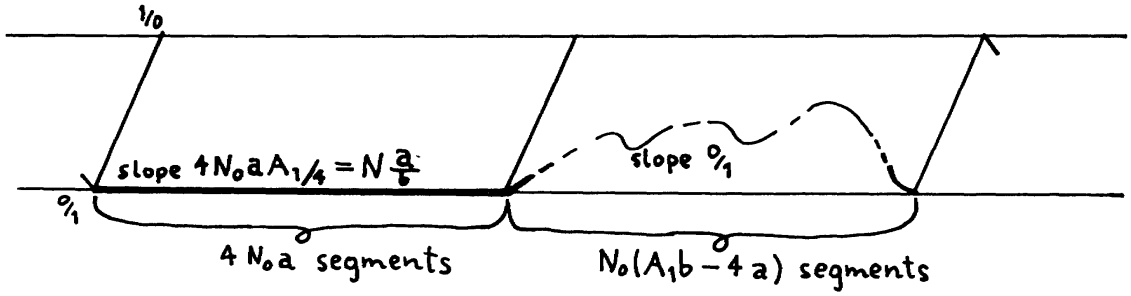

Figure 3.4

(see Figure 3.4). Hence $M_{\phi^N}(Na/b)$ is a Haken manifold. For this we use Corollary 2.2 ($N>2$ guarantees that $M_{\phi^N}(Na/b)$ is irreducible). So M_ϕ satisfies the conclusion of Proposition 3.4.

Now let $r=1$ or the eigenvalues of ϕ be negative. We have two possibilities:

(i) Let $\phi=\bar{\alpha}^{a_1}\beta^{a_2}$. Let α be a simple, closed curve in ∂M_ϕ of slope a/b. Then a lift of α to M_{ϕ^2} has the slope $2a/b$. For M_{ϕ^2} Proposition 3.4 is proven. Condition (a) for ϕ, $-\frac{1}{4}a_2 \leqslant a/b \leqslant \frac{1}{4}a_1$, is equivalent to Condition (a) for ϕ^2, $-\frac{1}{4}(a_2+a_2) \leqslant 2a/b \leqslant \frac{1}{4}(a_1+a_1)$, so Proposition 3.4 in the case (i) is proved.

(ii) Let $\phi=-\bar{\alpha}^{a_1}\beta^{a_2}\ldots\bar{\alpha}^{a_{2r-1}}\beta^{a_{2r}}$ so that the eigenvalues of ϕ are negative. Let α be a simple, closed curve in ∂M_ϕ of slope a/b. Then a lift of α to M_{ϕ^2} has slope $(2a/b)-1$. For M_{ϕ^2} Proposition 3.4 is proven. Condition (b) for ϕ,

$$-\frac{1}{4}\left(\left(\sum_{i=1}^{r} a_{2i}\right)-2\right) \leqslant \frac{a}{b} \leqslant \frac{1}{4}\left(\left(\sum_{i=1}^{r} a_{2i-1}\right)+2\right),$$

is equivalent to Condition (a) for ϕ^2,

$$-\frac{1}{4}\left(\left(\sum_{i=1}^{r} a_{2i}\right)+\left(\sum_{i=1}^{r} a_{2i}\right)\right) \leqslant \frac{2a}{b}-1 \leqslant \frac{1}{4}\left(\left(\sum_{i=1}^{r} a_{2i-1}\right)+\left(\sum_{i=1}^{r} a_{2i-1}\right)\right),$$

so Proposition 3.4 is proved in the case (ii).

The last statement of Proposition 3.4 follows immediately from considerations of diagrams of ϕ^k (cf. [2]). This ends the proof of Proposition 3.4. □

4. Unknotted surfaces. In this part we establish some conditions which guarantee that a surface in a 3-manifold is unknotted.

We will use the results of Jaco concerning hierarchies [6]. First, start from Jaco's modification of the existence of a hierarchy ("I prove that any Haken-manifold has a hierarchy of length no more than four—a result that I never been able to use" [6]).

THEOREM 4.1 [6]. *Let M be a Haken-manifold. If the 2-manifolds in a hierarchy are not required to be connected, then M has a hierarchy (M,F_1), (M_2,F_2), (M_3,F_3), (M_4,F_4) of length four.*

The results which follow (Propositions 4.2 and 4.3) relate to the proof of Theorem 4.1 (sketched in [6]) and we formulate them here without proof.

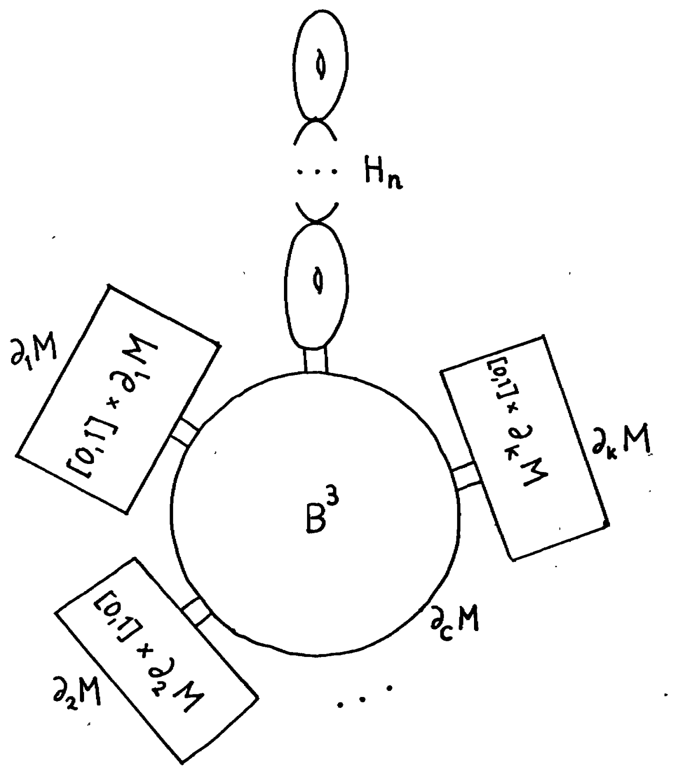

Figure 4.1

PROPOSITION 4.2. *Let M be a compact 3-manifold such that each closed, 2-sided, incompressible surface in M is parallel to the boundary of M. Then either* (i) *∂M is incompressible in M (or $\partial M \neq \varnothing$), or* (ii) *if* (i) *is not the case and $\partial_c M$ is a compressible boundary component of M, then all other boundary components of M (say, $\partial_1 M, \partial_2 M, \ldots, \partial_k M$) are incompressible, and*

$$M = H_n{}^{\Delta}[0,1] \times \partial_1 M^{\Delta} \ldots {}^{\Delta}[0,1] \times \partial_k M,$$

where H_n is an appropriate handlebody (n is the rank of the free factor of $\pi_1(M)$) and the disk sum is formed as in Figure 4.1.

PROPOSITION 4.3. *Let M be a compact 3-manifold with incompressible boundary such that each closed, 2-sided, incompressible surface in M is parallel to the boundary. Let F be a properly embedded, 2-sided, incompressible surface in M, not parallel to ∂M. Then M cut open along F can be described as follows:*

1. *Let F disconnect M (into M_1 and M_2). Let $\partial_{a_1} M, \partial_{a_2} M, \ldots, \partial_{a_k} M$ be a collection of boundary components of M which are disjoint from F and lie in M_1, and let $\partial_{b_1} M, \partial_{b_2} M, \ldots, \partial_{b_{k'}} M$ be a collection of boundary components of M which are disjoint from F and lie in M_2. Then*

$$M_1 = H_n{}^{\Delta}[0,1] \times \partial_{a_1} M^{\Delta} \ldots {}^{\Delta}[0,1] \times \partial_{a_k} M \quad \textit{and}$$

$$M_2 = H_n{}^{\Delta}[0,1] \times \partial_{b_1} M^{\Delta} \ldots {}^{\Delta}[0,1] \times \partial_{b_{k'}} M.$$

2. *Let F not disconnect M (M cut open along F gives M_1) and $\partial_1 M, \partial_2 M, \ldots, \partial_k M$ be a collection of boundary components of M which are disjoint from F. Then*

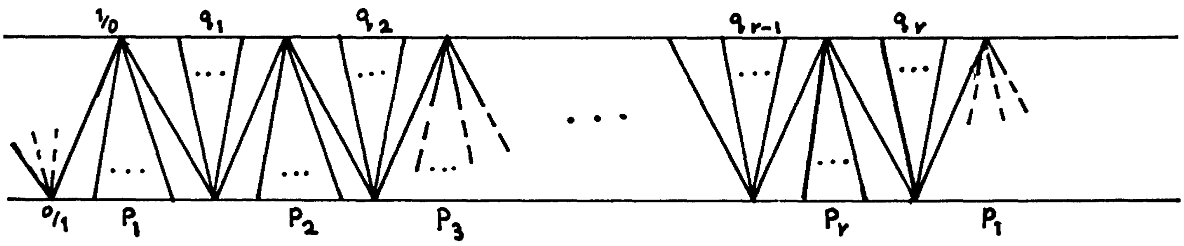

Figure 5.1

$$M_1=H_n{}^{\Delta}[0,1]\times\partial_1 M^{\Delta}\ldots{}^{\Delta}[0,1]\times\partial_k M.$$

DEFINITION 4.4. Let L be a link in a compact 3-manifold M. We shall say that (M,L) is sufficiently large if $M-L$ is irreducible and $M-\operatorname{int} V_L$ contains a closed, two-sided, incompressible surface of genus greater than 0 which is not parallel to the boundary (V_L is a regular neighborhood of L in M).

COROLLARY 4.5. *Let L be a link in a closed 3-manifold M, where $M-\operatorname{int} V_L$ is irreducible and (M,L) is not sufficiently large. In addition, let F be a 2-sided, incompressible surface which is not parallel to the boundary in $M-\operatorname{int} V_L$. Then the following conditions are equivalent:*

1. *F is unknotted.*
2. *F meets each component of L.*

Proof. $M-\operatorname{int} V_L$ satisfies assumptions of Proposition 4.3 or L is a trivial knot in S^3. □

5. Non sufficiently large links. The Gordon and Litherland result [3] can sometimes be used to recognize whether the assumptions of Corollary 4.5 are satisfied.

THEOREM 5.1 [3]. *Let L be a prime, sufficiently large link in a closed 3-manifold M, and let $\tilde{M}$ be a regular branched covering of (M,L). Then either $\tilde{M}$ is sufficiently large or M and $\tilde{M}$ both contain a non-separating 2-sphere.*

COROLLARY 5.2 [3]. *The complement of a 2-bridge link contains no closed, non-parallel to the boundary, 2-sided, incompressible surface.*

This is also proved in [4] without the assumption that a surface is 2-sided.

Now we use results of [2] and the notation and terminology of [9] to prove:

COROLLARY 5.3. *Let $\gamma\in B_3$ be a 3-braid of type Ω_6* [9, p. 7] *($\pi(\gamma)$ is hyperbolic, where $\pi: B_3\to \mathrm{PSL}_2(\mathbf{Z})$) so, up to conjugacy,*

$$\gamma=\Delta^{2n}\sigma_1^{-p_1}\sigma_2^{q_1}\sigma_1^{-p_2}\sigma_2^{q_2}\ldots\sigma_1^{-p_r}\sigma_2^{q_r},$$

where p_i and q_i are positive integers and $r>0$. Let Figure 5.1 represent the diagram of $\pi(\gamma)$ (see [2]*).*

If the diagram does not contain any invariant, minimal edge-path such that $L-R=2n$, then the link $\tilde{\gamma}$, defined by the braid γ, is prime, irreducible (i.e., the

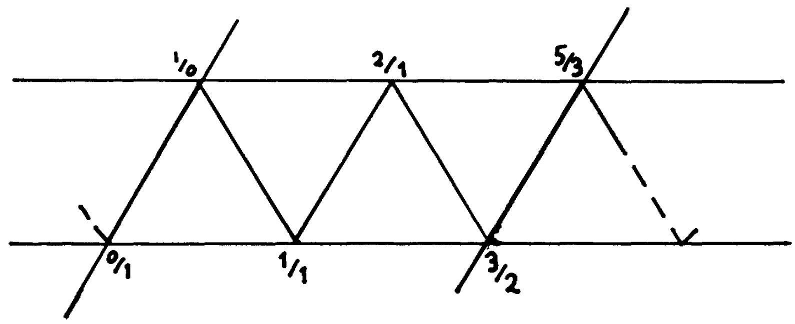

Figure 5.2

complement of γ in S^3 is irreducible), and not sufficiently large. L is the number of vertices of the edge path in one period, at which it turns left on the $\mathrm{PSL}_2(Z)$-diagram (i.e., it starts on the bottom of Figure 5.1), and R is the corresponding number of right turns.

REMARK 5.3.1. If we use the terminology of the proof of Proposition 3.2 then, for a given minimal, invariant edge-path λ, $L-R=4\,\mathrm{sl}(\lambda)$.

Proof of Corollary 5.3. We will show that in our case $\tilde{M}$ — the 2-sheeted covering of S^3 branched over $\tilde{\gamma}$ is an irreducible, non-Haken manifold. Then Corollary 5.3 follows from Theorem 5.1 and the irreducibility of $\tilde{M}$ (cf. [3]).

It is not difficult to see that $\tilde{M}$ is obtained by a Dehn surgery on a punctured torus bundle over S^1 with monodromy defined by $\pm\pi(\gamma)$ (+ if and only if n is even). To find the slope of the surgery, we have to recognize the significance of the coefficient Δ^{2n} in γ. We can compute that the slope of the surgery is equal to $(2n+\epsilon)/4$, where $\epsilon=0$ if n is even and $\epsilon=2$ if n is odd (for an appropriate system of coordinates in the ∂(bundle), see the definition before Proposition 3.1). Now, $\tilde{M}$ can be sufficiently large or not irreducible if and only if $L-R=2n$ for some invariant, minimal edge-path (see Corollary 2.2 and [2]). □

EXAMPLE 5.4. Consider $\gamma=\Delta^{2n}\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2$. Figure 5.2 represents the diagram of $\pi(\gamma)$. We have two possibilities for $L-R$: $L-R=2$ or -2. So for $n\neq\pm1$, $\tilde{\gamma}$ is a prime, irreducible, not sufficiently large knot. If $n=0$, we deal with the figure-eight knot.

EXAMPLE 5.5. Consider $\gamma=\Delta^{2n}(\sigma_1^{-1}\sigma_2)^4$. Figure 5.3 represents the diagram of $\pi(\gamma)$. We have four possibilities for $L-R$: $L-R=\pm1$ or ±4. So for $n\neq\pm2$, $\tilde{\gamma}$ is a prime, irreducible, not sufficiently large knot. If $n=0$, we deal with the 8_{18} knot [10].

EXAMPLE 5.6. Consider $\gamma=\Delta^{2n}\sigma_1^{-4}\sigma_2\sigma_1^{-1}\sigma_2^2$. Figure 5.4 represents the diagram of $\pi(\gamma)$. We have four possibilities for $L-R$: $L-R=\pm1$, -3 or 5. Thus $\tilde{\gamma}$ is a prime, irreducible, not sufficiently large knot. If $n=0$ we deal with the 8_7 knot, and if $n=1$ with the 10_{143} knot [10].

6. Property R. A knot in S^3 has *property R* if every surgery along this knot yields a manifold other than $S^1\times S^2$. Jaco has pointed out that property R

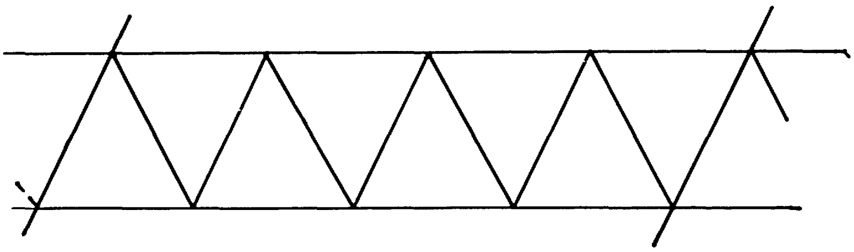

Figure 5.3

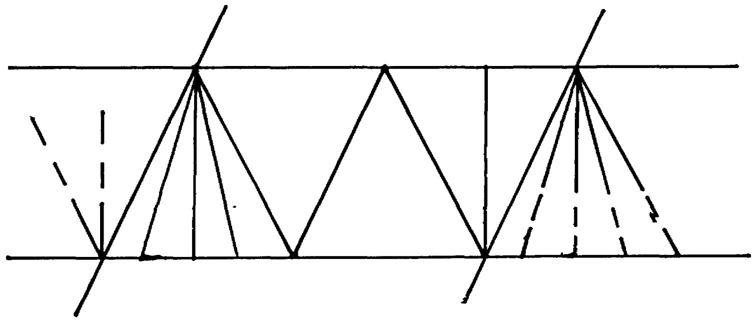

Figure 5.4

follows from Theorem 1.4 for nontrivial knots which possess an unknotted, incompressible Seifert surface (e.g., nontrivial knots which are not sufficiently large). Birman has informed me that: "Prof. Jaco has generalized it and is writing a short paper on the subject" (*Adding a* 2*-handle to a* 3*-manifold: an application to property R*).

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