SURFACE SYMMETRY II

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Introduction. In Part I of this work [2] we presented a classification of the actions of a finite abelian group on a closed, orientable surface in terms of cobordism class and fixed point data. Here we extend those results to actions of certain nonabelian finite groups. In Section 1 we examine free actions of a split metacyclic group G, i.e., a semidirect product of two cyclic groups. The main result of Section 1 is as follows.

THEOREM. *Let G be a finite nonabelian split metacyclic group which acts freely on a closed, oriented surface M, preserving orientation. Then the set of orientation-preserving equivariant homeomorphism classes of free actions of G on M is in bijective correspondence with the second homology group $H_2(G; \mathbf{Z})$.*

Although this result is certainly of a rather specialized nature, we view it as somewhat remarkable that the free actions of a nonabelian group can be classified up to equivariant homeomorphism by the abelian group $H_2(G; \mathbf{Z})$.

Since $H_2(G; \mathbf{Z})$ can be identified with the free equivariant cobordism group $\mathcal{O}_2^{\text{free}}(G)$, we have the following consequence.

COROLLARY. *Two free orientation-preserving actions of a split metacyclic group on a closed oriented surface are equivalent, by an orientation-preserving equivariant homeomorphism, if and only if they are freely cobordant.*

The analogous statement for finite abelian groups G was the main result from [2].

PROBLEM. Find invariants other than $\mathcal{O}_2^{\text{free}}(G)$ for free G actions on surfaces.

In Section 2 we take a different tack and examine "indecomposable" actions which preserve no nontrivial family of disjoint simple closed curves. These actions have exactly three singular orbits and have orbit space the sphere. By *fixed point data* for an action we mean a description of the equivariant homeomorphism class of the restriction of the action to a neighborhood of the singular orbits. This can be described as an unordered set of conjugacy classes in G, with multiplicities allowed, one for each singular orbit. (See Section 2.) Recall that two G actions are *weakly equivalent* if they are equivalent modulo automorphisms of G. Similarly we say that two sets of fixed point data are weakly equivalent if there is an automorphism of G taking one onto the other.

THEOREM. *Two indecomposable actions of a split metacyclic group G on a surface M are weakly equivalent if and only if they have weakly equivalent fixed point data.*

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A similar result is also given for the 2×2 matrix groups $\mathrm{SL}_2(\mathbf{F}_q)$. On the other hand, we construct an example of two inequivalent, cobordant, indecomposable actions of the symmetric group $\mathcal{S}_7$ with identical fixed point data.

1. Free actions of metacyclic groups. We begin by recalling some terminology and basic facts from [2; §2]. We denote by $\mathfrak{F}\mathfrak{A}(G,M)^*$ the set of orientation-preserving, equivariant homeomorphism classes of free actions of the finite group G on the surface M. All free G actions on M have the same orbit surface, say N, and we can identify $\mathfrak{F}\mathfrak{A}(G,M)^*$ with the set $\mathrm{Cov}(G,N)^*$ of equivalence classes of connected G-coverings of N. Covering space theory and basic facts from the topology of surfaces allow one to identify $\mathrm{Cov}(G,N)^*$ with the set $\mathrm{Epi}(\pi_1(N),G)^*$ of epimorphisms $\pi_1(N)\to G$ modulo the natural action of $\mathrm{Aut}\,\pi_1(N)$.

Now we have the bordism invariant

$$\mathbf{B}: \mathfrak{F}\mathfrak{A}(G,M)^* \to \mathcal{O}_2^{\mathrm{free}}(G),$$

where $\mathcal{O}_2^{\mathrm{free}}(G)\approx \Omega_2(BG)\approx H_2(G)$. If $[\phi]\in\mathfrak{F}\mathfrak{A}(G,M)^*$ corresponds to

$$\hat{\phi}: N\to BG,$$

then $\mathbf{B}[\phi]$ corresponds to $\hat{\phi}_*[N]\in H_2(G)$ where $[N]\in H_2(N)$ is the fundamental class induced from that of the oriented manifold M. Henceforth we view $\mathbf{B}$ as a homomorphism from $\mathrm{Epi}(\pi_1(N),G)^*$ to $H_2(G)$.

Our aim in this section then is to show that $\mathbf{B}: \mathrm{Epi}(\pi_1(N),G)^*\to H_2(G)$ is bijective when G is a metacyclic group and $\mathrm{Epi}(\pi_1(N),G)\neq\varnothing$.

By definition a split metacyclic group G has a presentation, which we fix once and for all, of the form

$$\langle x,y : x^m=y^n=1,\ yxy^{-1}=x^r\rangle \tag{1.1}$$

where m, n, and r are nonnegative integers such that $r^n\equiv 1 \bmod m$. To avoid the abelian case already covered by [2] we assume throughout that $m>1$, $n>1$, and $1<r<m$. (The present techniques can be easily applied to the abelian case as well, however.) We view G as an extension of $\mathbf{Z}_m=\langle x\rangle$ by $\mathbf{Z}_n=G/\langle x\rangle$, the latter generated by the image $\bar{y}$ of y.

(1.2) LEMMA. *$H_2(G)=\mathbf{Z}_d$, where $d=(m,r-1)(m,\sum_{i=0}^{n-1} r^i)/m$.*

Proof sketch. In the E^2 term of the Lyndon–Hochschild–Serre spectral sequence for the extension defining G, $E_{ij}^2=H_i(\mathbf{Z}_n;\ H_j(\mathbf{Z}_m))$. For $i+j=2$ the only possibly nonzero term is E_{11}^2, which one calculates to be $\mathbf{Z}_d$, d as above. The only possibly nontrivial differential entering or leaving E_{11} is $d_{30}^2: E_{30}^2\to E_{11}^2$. To finish the proof it suffices then to prove that d_{30}^2 vanishes.

To this end, consider the subgroup $\bar{G}=\langle x^c,y\rangle$ of G where $c=m/(m,r-1)$. Note that x^c is the smallest nontrivial power of x fixed by y, so $\langle x^c\rangle=(\mathbf{Z}_m)^y$. View $\bar{G}$ as the direct product extension of $(\mathbf{Z}_m)^y$ by $\mathbf{Z}_n$ and let $\bar{E}^2$ denote the corresponding spectral sequence. Now $\bar{E}_{30}^2=H_3(\mathbf{Z}_n;\ H_0((\mathbf{Z}_m)^y))\approx H_3(\mathbf{Z}_n;\ H_0((\mathbf{Z}_m))=E_{30}^2$. On the other hand $\bar{d}^2: \bar{E}_{30}^2\to\bar{E}_{11}^2$ is zero as a consequence of the Kunneth formula. Naturality of the spectral sequence now implies that $d_{30}^2=0$ as required. □

For more details of this and related computations see [1; §IV.2] and the references cited there.

We now proceed to a geometric description of a free action of a finite group G on a connected, closed, oriented surface M. Let $p: M \to N$ be the orbit map, which we view as before as a G-covering map of N. This covering is classified by an epimorphism $\rho: \pi_1(N, x_0) \to G$. Now $\pi_1(N, x_0)$ has a standard presentation of the form

$$\langle a_1, b_1, \ldots, a_n, b_n : [a_1, b_1] \cdots [a_n, b_n] = 1 \rangle$$

where $a_1, b_1, \ldots, a_n, b_n$ are represented by oriented simple closed curves $A_1, B_1, \ldots, A_n, B_n$ on N, suitably connected to the basepoint x_0 such that $A_i \cap A_j = B_i \cap B_j = A_i \cap B_j = \varnothing$ if $i \neq j$ and $A_i \cap B_i$ is one point of transverse intersection for all i. It follows from the classification of surfaces that any two such systems of simple closed curves and arcs in N are related by an orientation-preserving homeomorphism of N. In particular we can describe the representation $\rho: \pi_1(N, x_0) \to G$ by a sequence $(g_1, h_1; \ldots; g_n, h_n)$ of elements of G such that $[g_1, h_1] \cdots [g_n, h_n] = 1$. We call this sequence a *Hurwitz system* for the action. It follows that two actions are equivalent if and only if they possess identical Hurwitz systems, for suitable choices of simple closed curves for the two actions. We now return to our standard metacyclic group G.

(1.3) PROPOSITION. *Suppose the split metacyclic group G acts freely, preserving orientation, on the connected surface M. Then $N = M/G$ has genus greater than one, and the action has a Hurwitz system of the form*

$$(x^i y, x^j; x^k, 1; 1, 1; \ldots; 1, 1).$$

Proof. Since the corresponding representation $\rho: \pi_1(N) \to G$ is surjective and we are assuming G is nonabelian, N must have genus at least two.

First consider the composition $\pi_1(N) \to \mathbf{Z}_n$ of ρ with projection $G \to \mathbf{Z}_n$. This corresponds to a $\mathbf{Z}_n$ action with orbit space N. By the uniqueness of free cyclic actions on surfaces ([2] or [4]), we may choose a Hurwitz system for the given G action which has any desired image in $\mathbf{Z}_n$ (so long as it generates $\mathbf{Z}_n$). In particular, the action has a Hurwitz system of the form

$$(x^i y, x^j;\ x^{k_2}, x^{l_2}; \ldots; x^{k_n}, x^{l_n}).$$

Now the tail of the Hurwitz system, $(x^{k_2}, x^{l_2}; \ldots; x^{k_n}, x^{l_n})$, is a Hurwitz system for an action of a cyclic group contained in $\langle x \rangle \subset G$, with orbit surface of genus $n-1$. Again by the uniqueness of free cyclic actions we may arrange that this tail has the form $(x^k, 1; \ldots; 1, 1)$. In particular, the given G action has a Hurwitz system of the form $(x^i y, x^j;\ x^k, 1; 1, 1; \ldots; 1, 1)$. □

(1.4) COROLLARY. *If G is a split nonabelian metacyclic group acting freely on a surface M, then $\mathbf{B}: \mathfrak{F}\mathfrak{A}(G, M)^* \to H_2(G)$ is surjective.*

Proof. Since $H_2(G) \cong \Omega_2(BG)$, each element of $H_2(G)$ is represented by a free action on some surface, which can be chosen to be connected. But (1.3) shows that each element can be represented on any surface on which G acts freely. □

(1.5) PROPOSITION. *If the nonabelian split metacyclic group G acts freely on a surface M, then the action has a Hurwitz system of the form*

$$(y, x^{cl}; x, 1; 1, 1; \ldots; 1, 1), \quad \text{where } c = m/(m, r-1).$$

Proof. By (1.3) we may assume $N = M/G$ has genus 2 and that the action is given by a Hurwitz system $(x^i y, x^j;\ x^k, 1)$, where we suppress the tail string of 1's which shall subsequently be unchanged. We shall make use of two families of alterations of Hurwitz systems. (In fact the moves below are all that one needs to prove the result about cyclic actions used above; and in that case one may ignore base points.) Let $(u_1, v_1;\ u_2, v_2)$ be a Hurwitz system.

MOVE D. $(u_i, v_i) \to (u_i, v_i u_i^q)$ or $(u_i v_i^q, v_i)$, $q \in \mathbf{Z}$. To achieve this simply apply a q-fold Dehn twist about A_i or B_i.

MOVE M. $(u_1, v_1;\ u_2, v_2) \to (u_2 u_1, v_1;\ v_1 u_2 v_1^{-1}, u_2 v_2 u_2^{-1} v_1^{-1})$. To achieve this "mixing" alteration of the Hurwitz system, note that the defining relation of the surface of genus 2 still holds if u_1, v_1, u_2, and v_2 are replaced by the second 4-tuple, and the latter elements generate. A pictorial proof can be achieved by moving the u_1-curve over u_2 and suitably choosing the other curves. Fortunately in our applications $v_1 u_2 v_1^{-1} = u_2$ and $u_2 v_2 u_2^{-1} v_1^{-1} = v_2 v_1^{-1}$.

We first arrange to have the system in the form in (1.3) such that $k \mid j$. To do this, let $t = (j, k)$, and choose integers a, b such that $t = aj + bk$. Apply D with $q = b$ to achieve $(x^i y, x^j;\ x^k, x^{bk})$. Now apply M $qm - a$ times, where q is chosen so that $qm - a$ is positive. This yields $(x^{k(qm-a)} x^i y, x^j;\ x^k, x^{bk} x^{-j(qm-a)}) = (x^{i-ka} y, x^j;\ x^k, x^t)$. Since $t \mid k$, Dehn twists yield $(x^{i-ka} y, x^j;\ x^t, 1)$. Since $t \mid j$, we may now assume that in form (1.3), $k \mid j$.

Now we claim that x^{is} and x^k generate $\mathbf{Z}_m = \langle x \rangle$ where $s = \sum_{\alpha=0}^{n-1} r^\alpha$, so that (is, k) is prime to m. Since $k \mid j$, we have that $\langle x^i y, x^k \rangle = G$. Now $x^{is} = (x^i y)^n$ and $H = \langle x^{is}, x^k \rangle \subset \langle x \rangle$ is a normal subgroup of G with G/H generated by the coset of $x^i y$. Since no $(x^i y)^k$, $0 < k < n$, lies in H, it follows that H has order m, and hence that $H = \langle x \rangle$, as required.

It follows that we may choose integers a and b such that $ais + bk \equiv 1 \bmod m$. Apply D with $q = b$ in the second half of the Hurwitz system to obtain $(x^i y, x^j; x^k, x^{bk})$. Then apply D with $q = -an$ in the first half to obtain $(x^i y, x^j x^{-ais};\ x^k, x^{bk})$, where $s = \sum_{\alpha=0}^{n-1} r^\alpha$. Now apply M, obtaining

$$(x^{k+i} y, x^{j-ais}; x^k, x^{bk-j+ais}) = (x^{i'} y, x^{j'}; x^k, x^{1-j}).$$

Since $k \mid j$, $(k, 1-j) = 1$. Therefore, by applying D suitably, we can achieve $(x^{i'} y, x^{j'};\ x, 1)$. Now apply M $pm - i'$ times where p is chosen so that $pm - i' \geqslant 0$. This yields $(y, x^{j'};\ x, x^{-j'(pm-i')})$. Finally D, with $q = -i'j'$, yields the required form $(y, x^{j'};\ x, 1)$. Since $[y, x^{j'}][x, 1] = 1$, we have $[y, x^{j'}] = 1$. It follows easily that j' has the form cl where $c = m/(m, r-1)$ and l is some integer. □

(1.6) PROPOSITION. *If an action of G has a Hurwitz system $(y, x^{cl};\ x, 1)$, then it also has a Hurwitz system $(y, x^{c(l+d)};\ x, 1)$, where $c = m/(m, r-1)$, $d = (m, r-1)(m, s)/m$, and $s = \sum_{\alpha=0}^{n-1} r^\alpha$.*

Proof. Clearly $cd = (m, s)$, so there are integers a and b such that $cd = am + bs$. Applying Move M b times we obtain $(x^b y, x^{cl};\ x, x^{-bcl})$. Applying D in the second part with $q=bcl$, we get $(x^by, x^{cl};\ x, 1)$. Now applying D, with $q=n$, in the first part of the system we obtain

$$\begin{aligned}(x^by, x^{cl}x^{bs}; x, 1) &= (x^by, x^{cl+cd-am}; x, 1)\\ &= (x^by, x^{c(l+d)}; x, 1).\end{aligned}$$

Now apply M $pm-b$ times, where $pm-b\geqslant 0$, to get $(y, x^{c(l+d)};\ x, x^{bc(l+d)})$. Application of D creates the desired form. □

(1.7) THEOREM. *If the nonabelian split metacyclic group G acts freely, preserving orientation, on a connected, closed, oriented surface M, then* $\mathbf{B}\colon \mathfrak{F}\mathfrak{A}(G,M)^* \to H_2(G)$ *is a bijection.*

Proof. By (1.4), $\mathbf{B}$ is surjective. By (1.1), $H_2(G)$ has order d. But by (1.6), $\mathfrak{F}\mathfrak{A}(G,M)^*$ has cardinality at most d. □

REMARKS. (1) The results (1.3) through (1.5) apply just as well to general metacyclic groups $(y^n=x^d)$. Perhaps (1.6) can be modified to apply in this generality too. (2) Aut G acts on $\mathfrak{F}\mathfrak{A}(G,M)^*$ and on $H_2(G)$ and $\mathbf{B}$ induces a bijection of the weak equivalence classes of free G actions on M with the set $H_2(G)/\mathrm{Aut}\, G$.

2. Indecomposable actions. Let G be an arbitrary finite group, and let ϕ be any effective, orientation-preserving action of G on a connected, closed, oriented surface M. The *singular set* S_ϕ is the set of points with nontrivial isotropy group; the *branch set* B_ϕ is the image of S_ϕ in the orbit space $N=M/\phi$. The corresponding *G-branched covering* $\pi\colon M\to N$ is determined by a surjective homomorphism $\rho\colon \pi_1(N-B_\phi, x_0)\to G$; M is the end compactification of the corresponding G-covering space of $N-B_\phi$.

Let $B_\phi=\{x_1,\ldots,x_n\}$ and let $C_1,\ldots,C_n$ be small, disjoint simple closed curves in $N-B_\phi$ such that C_i bounds a disk D_i with $D_i\cap B_\phi=\{x_i\}$, for $i=1,\ldots,n$. Each C_i is to be oriented in a standard way and suitably connected to the base point x_0, so that elements $\gamma_1,\ldots,\gamma_n\in\pi_1(N-B_\phi,x_0)$ are determined in a standard way. If $N\cong S^2$, these elements generate $\pi_1(N-B_\phi,x_0)$, subject only to the relation $\gamma_1\gamma_2\ldots\gamma_n=1$. In any case the set of conjugacy classes of the elements $\rho(\gamma_1),\ldots,\rho(\gamma_n)$ of G (with multiplicities) constitutes the *fixed point data* for the action ϕ and depends, up to order, only on the action ϕ and the orientation of M. Moreover, two actions of G have the same fixed point data if and only their restrictions to invariant neighborhoods of the singular sets are equivalent.

Define an action of G on a surface M to be *indecomposable* if it preserves no family of disjoint, homotopically nontrivial, simple closed curves on M.

(2.1) LEMMA. *An action ϕ of G on M is indecomposable if and only if the orbit space $N=M/\phi$ is S^2 and the action has at most three singular orbits.*

Proof. Suppose the action of G on M is decomposable, and let $J\subset M$ denote an invariant family of nontrivial simple loops. We may suppose G acts transitively on the set of components of J. We may further assume G acts freely on J: the only other possibility is that each component of J consists of two arcs connecting points of isotropy group $\mathbf{Z}_2$; in this case replace J with the boundary of a small invariant tubular neighborhood of J. Let $\pi: M \to N$ be the orbit map. Then $\pi(J)$ is a single simple closed curve. Suppose $N \cong S^2$. If G has at most three singular orbits, then $\pi(J)$ bounds a disk D containing at most 1 branch point. It follows that each component of $\pi^{-1}(D)$ is a disk, and, hence, that J is homotopically trivial, contradicting the nontriviality of J.

For the converse, suppose that either $N \not\cong S^2$ or M contains more than three singular orbits. Then in N one can choose a simple loop C which misses the branch set, and which does not bound a disk containing less than two branch points. One easily verifies that $J = \pi^{-1}(C)$ is an invariant system of nontrivial simple loops in M. □

REMARKS. (1) The cases when $M/\phi \cong S^2$ with only two singular orbits are uninteresting. *Henceforth an indecomposable action shall be one with exactly three singular orbits.*

(2) The alteration of a decomposable group action given above produces an action cobordant to the original one. Therefore indecomposable actions generate the cobordism group $\mathcal{O}_2(G)$ of G actions on surfaces.

Now let ϕ be an indecomposable G action on M with $N = M/\phi$ homeomorphic to S^2 and branch set $B_\phi = \{x_1, x_2, x_3\} \subset N$. Let C_1, C_2, C_3 be the small circles about the branch points as before, with corresponding elements $\gamma_1, \gamma_2, \gamma_3$ of $\pi_1(N - B_\phi, x_0)$. As before, we shall refer to the triple $(\rho(\gamma_1), \rho(\gamma_2), \rho(\gamma_3))$ of elements of G as a *Hurwitz system* for the action ϕ. Of course, a given action has many different Hurwitz systems in general. One easily proves that two indecomposable G actions are equivalent by an orientation-preserving equivariant homeomorphism if and only if they have identical Hurwitz systems. From one Hurwitz system for a given action one can construct all others by altering the given one by orientation-preserving homeomorphisms of S^2 which preserve the set of branch points and fix the base point.

Recall that two G actions ϕ and ψ on M are *weakly equivalent* if there is $\alpha \in \operatorname{Aut} G$ such that $\psi = \phi \circ \alpha$ (where we view ϕ and ψ as homomorphisms from G into $\operatorname{Homeo}(M)$). Similarly ϕ and ψ have *weakly equivalent fixed point* data if there is an automorphism of G taking one set of fixed point data onto the other.

(2.2) PROPOSITION. *Let ϕ and ψ be indecomposable actions of G on a connected surface M with Hurwitz systems (g_1, g_2, g_3) and (h_1, h_2, h_3), respectively. Suppose that g_i is conjugate to h_i, $i = 1, 2, 3$.*

(i) *If there is a single element $g \in G$ such that $g g_i g^{-1} = h_i$, $i = 1, 2, 3$, then ϕ is equivalent to ψ.*

(ii) *If there is $\alpha \in \operatorname{Aut} G$ such that $\alpha(g_i) = h_i$, $i = 1, 2, 3$, then ϕ is weakly equivalent to ψ.*

(iii) *If the three conjugacy classes represented by g_1, g_2, g_3, are distinct, then the converses of* (i) *and* (ii) *hold.*

Proof. (i) An isotopy which drags the base point x_0 around a loop representing g converts the Hurwitz system for ϕ into that for ψ. (Such a loop exists since M is connected.)

(ii) The condition implies that the Hurwitz system $(\alpha(g_1), \alpha(g_2), \alpha(g_3))$ for $\phi\circ\alpha$ is a Hurwitz system for ψ.

(iii) When the three conjugacy classes are distinct, the other Hurwitz systems for ϕ are obtained by applying an orientation-preserving homeomorphism which fixes the base point and the three branch points. Such a homeomorphism is isotopic to the identity, while fixing the branch set, but perhaps letting the base point move. See [5; Exposé 2, §III], for example. Thus all Hurwitz systems for ϕ, with conjugacy classes in the same order, are of the form (g_1^g, g_2^g, g_3^g), $g \in G$; and all Hurwitz systems for weakly equivalent actions, with the same fixed point data, are of the form $(\alpha(g_1), \alpha(g_2), \alpha(g_3))$, $\alpha \in \operatorname{Aut} G$. □

Now specialize again to the case of a finite split metacyclic group G with a standard presentation of the form

$$\langle x, y : x^m = y^n = 1,\ yxy^{-1} = x^r \rangle \tag{2.3}$$

where $r^n \equiv 1 \bmod m$. If X is a collection of elements of G, then $\langle X \rangle$ denotes the subgroup of G generated by X.

(2.4) THEOREM. *Let G be a finite split metacyclic group with standard presentation* (2.3). *Suppose that $G = \langle g_1, g_2 \rangle = \langle h_1, h_2 \rangle$, and g_i is conjugate to h_i for $i = 1, 2$. Then there is an automorphism $\alpha: G \to G$ such that $\alpha(g_i) = h_i$, $i = 1, 2$.*

REMARK. We do not know whether α can always be chosen to be an inner automorphism.

(2.5) COROLLARY. *Two indecomposable actions of a finite metacyclic group G on a connected, closed, oriented surface M are weakly equivalent if and only if they have weakly equivalent fixed point data.*

Proof. Weakly equivalent actions clearly have weakly equivalent fixed point data. For the converse, let two indecomposable actions of G on M be given with weakly equivalent Hurwitz systems (g_1, g_2, g_3) and (h_1, h_2, h_3), respectively. One may alter one of these by an automorphism of G and hence assume that g_i is conjugate to h_i for $i = 1, 2, 3$. Since M is connected, $G = \langle g_1, g_2 \rangle = \langle h_1, h_2 \rangle$. By (2.4) there is an automorphism $\alpha: G \to G$ such that $\alpha(g_1) = h_1$ and $\alpha(g_2) = h_2$. Then $\alpha(g_3) = h_3$, and the two actions are weakly equivalent. □

Proof of (2.4). By conjugating h_1, h_2 we may assume $g_1 = h_1$. Let $g_1 = x^a y^b$ and $g_2 = x^c y^d$, so that $h_2 = x^e y^d$ for some integers a, b, c, d, e. We shall seek an automorphism $\alpha: G \to G$ such that $\alpha(x) = x^u$ and $\alpha(y) = x^v y$. We omit the easy verification that for each u and v this yields a well-defined homomorphism.

Let σ_k stand for $\sum_{i=0}^{k-1} r^i$. Then we require that

$$x^a y^b = \alpha(x^a y^b) = x^{au}(x^v y)^b = x^{au + \sigma_b v} y^b$$

and

$$x^e y^d = \alpha(x^c y^d) = x^{cu}(x^v y)^d = x^{cu + \sigma_d v} y^d.$$

In other words we seek simultaneous solutions to the following congruences.

$$\text{(2.6)}\qquad \begin{cases} au+\sigma_b v\equiv a \bmod m \\ cu+\sigma_d v\equiv e \bmod m. \end{cases}$$

Unfortunately the matrix of coefficients need not be invertible in $\mathbf{Z}_m$.

Let $x^e y^d = x^s y^t (x^c y^d) y^{-t} x^{-s} = x^{s+r^t c - r^d s} y^d$. Then

$$e-c\equiv (r^t-1)c-(r^d-1)s \bmod m.$$

In particular $e-c$ is divisible by $r-1$ in $\mathbf{Z}_m$.

(2.7) LEMMA. *If $G=\langle g,h\rangle$, then the commutator subgroup $[G,G]$ is generated by the commutator $[g,h]=ghg^{-1}h^{-1}$.*

Proof. Note that $[G,G]=\langle x^{r-1}\rangle$, since $G/\langle x^{r-1}\rangle$ is abelian and $x^{r-1}=x^{-1}yxy^{-1}\in[G,G]$. Let $H=\langle [g,h]\rangle\subset[G,G]=\langle x^{r-1}\rangle$. It follows that H is normal in G, since any element of G conjugates any element of H to a power of itself. Then the cosets of g and h generate G/H and commute. This implies that $[G,G]\subset H$. □

Now

$$[x^a y^b, x^c y^d] = x^a y^b x^c y^d y^{-b} x^{-a} y^{-d} x^{-c} = x^{a+r^b c - r^d a - c} = x^{a(1-r^d)-c(1-r^b)}.$$

By (2.7) there is an integer k such that $k[a(1-r^d)-c(1-r^b)]\equiv 1-r \bmod m$.

We use this to solve the system (2.6). Eliminate u by subtracting c times the first congruence from a times the second. This yields $(a\sigma_d - c\sigma_b)v\equiv a(e-c) \bmod m$. Since $\sigma_p=(1-r^p)/(1-r)$, multiplying both sides by $k(1-r)$, k as above, yields $(1-r)v\equiv k(1-r)a(e-c) \bmod m$. Thus we may set $v=ka(e-c)$. Returning to the first congruence of (2.6) and substituting for v yields $au+\sigma_b ka(e-c)\equiv a \bmod m$. Therefore we may set $u=1+\sigma_b k(c-e)$.

These values for u and v do indeed satisfy the second congruence of (2.6) as well, as one can easily verify, using the fact that $e-c$ is divisible by $r-1$ in $\mathbf{Z}_m$.

Finally, since the homomorphism α defined by u and v takes one generating set for G to another, α is onto, hence an automorphism. □

It is an intriguing problem in its own right to explore the extent to which analogs of (2.4) hold for other finite groups. The rest of this paper is devoted to one extension of (2.4) and the construction of a family of limiting counterexamples.

(2.8) THEOREM. *Let $A_1, A_2, B_1, B_2\in \mathrm{SL}_2(\mathbf{F}_q)$, the group of 2×2 matrices of determinant 1, with entries in the finite field with q elements. Suppose that*

1) $A_1\sim B_1$, $A_2\sim B_2$, and $A_1A_2\sim B_1B_2$, where $\sim$ denotes conjugacy by an element of $\mathrm{GL}_2(\mathbf{F}_q)$; and that

2) $\mathrm{SL}_2(\mathbf{F}_q)=\langle A_1,A_2\rangle=\langle B_1,B_2\rangle$.

Then there is $P\in\mathrm{GL}_2(\mathbf{F}_q)$ such that $PA_1P^{-1}=B_1$ and $PA_2P^{-1}=B_2$.

As before this implies the following result about actions of $\mathrm{SL}_2(\mathbf{F}_q)$.

(2.9) COROLLARY. *Any two indecomposable actions of $\mathrm{SL}_2(\mathbf{F}_q)$ on a connected, closed, oriented surface are weakly equivalent if and only if they have weakly equivalent fixed point data.* □

Proof of (2.8). Fix the pair (A_1, A_2) and consider all pairs (B_1, B_2) such that $B_1 \sim A_1$, $B_2 \sim A_2$, and $B_1B_2 \sim A_1A_2$ and such that $\{B_1, B_2\}$ generates $\mathrm{SL}_2(\mathbf{F}_q)$.

Now $\mathrm{GL}_2(\mathbf{F}_q)$ acts by conjugation on such pairs (B_1, B_2) and each orbit contains a pair (B_1, B_2) with $B_1 = A_1$. Consider the set $\mathcal{P}$ of such pairs (B_1, B_2) with $B_1 = A_1$. Then the centralizer $C(A_1)$ of A_1 in $\mathrm{GL}_2(\mathbf{F}_q)$ acts on $\mathcal{P}$. Moreover, since each $\{B_1, B_2\}$ is required to generate $\mathrm{SL}_2(\mathbf{F}_q)$ it follows that $C(A_1)/Z$ acts freely on $\mathcal{P}$, where Z is the center of $\mathrm{GL}_2(\mathbf{F}_q)$, which consists of the nonzero scalar matrices and has order $q-1$.

Now we shall compute that

$$|C(A_1)/Z| = \begin{cases} q-1 & \text{if } A_1 \text{ is diagonalizable} \\ q & \text{if } A_1 \text{ is triangularizable but not diagonalizable} \\ q+1 & \text{if } A_1 \text{ is not triangularizable.} \end{cases}$$

On the other hand we shall show that

$$|\mathcal{P}| \leqslant \begin{cases} q-1 & \text{if } A_1 \text{ is diagonalizable} \\ q & \text{if } A_1 \text{ is triangularizable but not diagonalizable} \\ 2q & \text{if } A_1 \text{ is not triangularizable.} \end{cases}$$

Because the action of $C(A)/Z$ on $\mathcal{P}$ is free, it follows that $C(A)/Z$ acts transitively on $\mathcal{P}$, as required.

First suppose A_1 is diagonalizable. We may assume $A_1 = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}$. Suppose $CA_1 = A_1C$. Since $\langle A_1, A_2 \rangle = \mathrm{SL}_2(\mathbf{F}_q)$, A_1 cannot be a scalar matrix. It follows that C must be diagonal, since it must preserve the eigenspaces of A_1. Since $\det C \neq 0$, there are $(q-1)^2$ possible matrices C. But $|Z| = q-1$. Therefore $|C(A_1)/Z| = q-1$.

To estimate $|\mathcal{P}|$ when $A_1 = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}$ we determine the number of matrices $C = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ such that $\det C = 1$, $\operatorname{tr} C = \operatorname{tr} A_2$, $\operatorname{tr} A_1C = \operatorname{tr} A_1A_2$ and $\langle A_1, C \rangle = \mathrm{SL}_2(\mathbf{F}_q)$. Let $\operatorname{tr} A_2 = t$ and $\operatorname{tr} A_1A_2 = s$. Then these three equations become

$$\begin{cases} xw - yz = 1 \\ x + w = t \\ ax + a^{-1}w = s. \end{cases}$$

Because $A_1 \neq \pm I$, $a - a^{-1} \neq 0$. Therefore the second two equations may be solved uniquely, for x and w. The first equation then has $(q-1)$ solutions unless $xw = 1$. But if $xw = 1$, one of y or z is 0; and then C is triangular and $\langle A_1, C \rangle \neq \mathrm{SL}_2(\mathbf{F}_q)$. Thus $|\mathcal{P}| \leqslant q-1$, as required.

Next suppose that A_1 is triangularizable, but not diagonalizable. We may then assume that $A_1 = \pm\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, $a \neq 0$. To determine $|C(A_1)/Z|$, suppose that $CA_1 = A_1C$, where $\det C \neq 0$. It follows that C has the form $\begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$, $x \neq 0$. Because $\det C \neq 0$, there are exactly $(q-1)q$ such matrices. Since $|Z| = q-1$, $|C(A_1)/Z| = q$.

To estimate $|\mathcal{P}|$ when $A_1 = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, $a \neq 0$, we determine the number of matrices $C = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ such that $\det C \neq 0$, $\operatorname{tr} C = \operatorname{tr} A_2 = t$, and $\operatorname{tr} A_1 C = \operatorname{tr} A_1 A_2 = s$, and $\langle A_1, C \rangle = \mathrm{SL}_2(\mathbf{F}_q)$.

These three equations become

$$\begin{cases} xw - yz = 1 \\ x + w = s \\ x + az + w = t. \end{cases}$$

The second and third equations imply that $z = (t-s)/a$. Let $x \in \mathbf{F}_q$ be arbitrary. Then the second equation yields $w = s - x$; and the first equation then yields

$$x(s-x) - y(t-s)/a = 1.$$

This has exactly one solution y in terms of x, provided we show that $s \neq t$—that is $|\mathcal{P}| \leqslant q$. To see that $s \neq t$, suppose to the contrary that $s = t$. Then $z = 0$ and both C and A_1 are upper triangular, so that $\langle A_1, C \rangle \neq \mathrm{SL}_2(\mathbf{F}_q)$, contrary to hypothesis.

Finally suppose A_1 is not triangularizable. Then A_1 is conjugate to a matrix of the form $\begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix}$. We may assume A_1 has this form. Since the characteristic polynomial has no roots in $\mathbf{F}_q$, it follows that $a^2 - 4$ is not a square in $\mathbf{F}_q$. Suppose $CA_1 = A_1 C$ where $C = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ and $\det C \neq 0$. We then find that $z = -y$ and $x - ya = w$. Then $\det C = x(x - ya) + y^2$. We claim that $\det C = 0$ if and only if $x = y = 0$. If $y \neq 0$, then $x^2 - yax + y^2 = 0$ if and only if $\Delta = a^2 y^2 - 4y^2$ is a square in $\mathbf{F}_q$. But Δ is a square if and only if $a^2 - 4$ is; and we assumed $a^2 - 4$ is not a square. Similarly there are no solutions with $x \neq 0$. Therefore there are precisely $q^2 - 1$ matrices as above. So $|C(A)| = q^2 - 1$; and hence $|C(A)/Z| = q + 1$.

As before, we estimate $|\mathcal{P}|$ when $A_1 = \begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix}$, by estimating the number of matrices $C = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ such that $\det C = 1$, $\operatorname{tr} C = \operatorname{tr} A_2 = t$, and $\operatorname{tr} A_1 C = \operatorname{tr} A_1 A_2 = s$. These three equations become

$$\begin{cases} xw - zy = 1 \\ x + w = t \\ aw + y - z = s. \end{cases}$$

Using the second and third equations to eliminate z and w from the first, we obtain

$$-x^2 + axy - y^2 + tx + (s - at)y = 1.$$

For each value of x there are 0, 1, or 2 values of y which solve the equation. We conclude that $|\mathcal{P}| \leqslant 2q$. □

REMARKS. (1) In the statement of (2.8) one cannot in general choose P in $\mathrm{SL}_2(\mathbf{F}_q)$, even if the conjugacies in Hypothesis 1) are by elements of $\mathrm{SL}_2(\mathbf{F}_q)$. This is why it is necessary to use the notion of weak equivalence.

(2) Minor changes in the proof show that the analogue of (2.8) in which $SL_2(\mathbf{F}_q)$ is replaced by $GL_2(\mathbf{F}_q)$ also holds true. It follows that indecomposable actions of $GL_2(\mathbf{F}_q)$ are equivalent if and only if they have the same fixed point data.

(3) If one attempts to carry through the proof of (2.8) with $SL_2(\mathbf{F}_q)$ replaced by $PSL_2(\mathbf{F}_q)$, one discovers weakly inequivalent actions of $PSL_2(\mathbf{F}_q)$ with weakly equivalent fixed point data. See (2.10) and (2.11) for simpler examples of this phenomenon.

(4) We have learned that the computations in the proof of (2.8) are similar to ones appearing in [6].

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In conclusion we shall construct some indecomposable actions of symmetric groups which are inequivalent but have the same fixed point data.

(2.10) PROPOSITION. *There exist two inequivalent, indecomposable actions of the symmetric group $\mathcal{S}_5$ on the surface of genus* 16, *having identical fixed point data.*

Proof. Let $B \subset S^2 - \{x_0\}$ be a set of three points. Identify $\pi_1(S^2 - B, x_0)$ as the free group on two generators with three-generator presentation $\langle x, y, z : xyz = 1\rangle$. Define homomorphisms $\rho_1, \rho_2 \colon \pi_1(S^2 - B, x_0) \to \mathcal{S}_5$ by

$$\begin{array}{lcl} \rho_1(x) = (145)(23) & & \rho_2(x) = (145)(23) \\ \rho_1(y) = (123) & \text{and} & \rho_2(y) = (124) \\ \rho_1(z) = (1254) & & \rho_2(z) = (2543). \end{array}$$

Left-to-right multiplication of permutations is understood. One easily checks that ρ_1 and ρ_2 are well-defined epimorphisms. Actions ϕ_1 and ϕ_2 are defined by taking the corresponding regular $\mathcal{S}_5$-branched coverings. The Riemann-Hurwitz formula says that the two actions are both on the surface M of genus 16. Clearly the two actions have the same fixed point data. By (2.2) the actions are equivalent if and only if ρ_1 and ρ_2 are conjugate by an element μ of $\mathcal{S}_5$. But such an element μ would have to be a power of (145) times a power of (23), to fix (145)(23); and such an element cannot conjugate (123) to (124). $\square$

REMARKS. (1) Since every automorphism of $\mathcal{S}_5$ is inner, the two actions in (2.10) are not weakly equivalent.

(2) Since $H_2(\mathcal{S}_n; \mathbf{Z}) \approx \mathbf{Z}_2$, $n \geqslant 4$ [3], it is possible that these two actions are distinguished by cobordism considerations as in [2; §5] for abelian groups. That is, it is possible that ϕ_1 and ϕ_2 are not cobordant by an action with branch set consisting of three arcs. To eliminate this possibility we make a similar construction for $\mathcal{S}_7$.

(2.11) PROPOSITION. *There exist three indecomposable actions of* $\mathcal{S}_7$ *on a surface of genus* 1321, *having identical fixed point data, which are not weakly equivalent to one another.* (*At least two of these actions are cobordant by a* $\mathcal{S}_7$*-cobordism with just three branch arcs, since* $H_2(\mathcal{S}_7;\, \mathbf{Z}) = \mathbf{Z}_2$.)

Proof. As in (2.10) define three homomorphisms $\rho_1, \rho_2, \rho_3 : \pi_1(S^2 - B, x_0) \to \mathcal{S}_7$ by

$$\begin{array}{llll} \rho_1(x) = (1234)(567) & \rho_2(x) = (1234)(567) & & \rho_3(x) = (1234)(567) \\ \rho_1(y) = (1567) \quad , & \rho_2(y) = (1523) \quad , & \text{and} & \rho_3(y) = (1526) \\ \rho_1(z) = (1675432) & \rho_2(z) = (1276543) & & \rho_3(z) = (1543276). \end{array}$$

One argues just as before. □

REFERENCES

1. F. Beyl and J. Tappe, *Group extensions, representations, and the Schur multiplicator,* Lecture Notes in Math., 958, Springer, New York, 1982.
2. A. Edmonds, *Surface symmetry I,* Michigan Math. J. 29 (1982), 171–183.
3. M. Nakaoka, *Decomposition theorem for homology groups of symmetric groups,* Ann. of Math. (2) 71 (1960), 16–42.
4. J. Nielsen, *Die Struktur periodischer Transformationen von Flächen,* Danske Vid. Selsk., Mat.-Fys. Medd. 15 (1937), 1–77.
5. V. Poénaru, et al., *Travaux de Thurston sur les surfaces,* Astérisque, 66–67, Soc. Math. France, Paris, 1979.
6. C. H. Sah, *Groups related to compact Riemann surfaces,* Acta Math. 123 (1969), 13–42.

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