COMPLEX HOMOGENEOUS MANIFOLDS WITH TWO ENDS

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INTRODUCTION

Given a topological space, one may ask if it admits the structure of a topological group. This is *not* always possible for there are certain conditions that must be satisfied. For example its fundamental group must be abelian. In investigating conditions of this kind, Freudenthal considered the notion of end of a topological space, where intuitively an end is a "hole at infinity" of the given space (for the definition see section 1). He showed that a connected topological group which is locally connected, locally compact and second countable has at most two ends [10, Satz 15]. In the case of Lie groups one can see this directly by using the Iwasawa decomposition. And in particular the Iwasawa decomposition shows that a connected Lie group G has two ends precisely if it is homeomorphic to $K \times \mathbf{R}$, where K is a maximal compact subgroup of G.

Instead of considering the action of a given topological group G on its underlying topological space, one may consider its action as a topological transformation group on various other topological spaces X. Often it is assumed that such an action is either discontinuous or else transitive and under each of these assumptions there are results known concerning the number of ends that X may have. For example Hopf showed that a necessary condition for a noncompact topological space X to admit a discontinuous group G of homeomorphisms having a compact fundamental set, i.e. a compact subset whose transforms under G cover X, is that X has either one or two or a Cantor set of ends [13].

If a Lie group G acts transitively on a smooth manifold, then it is well known that the *homogeneous manifold* X is diffeomorphic to the coset space G/H, where $H := \{g \in G : gx_0 = x_0\}$ is a closed subgroup of G called the *isotropy subgroup* of G at the point $x_0 \in X$. For homogeneous spaces of a (real) Lie group, Borel showed that if H is *connected* then G/H has at most two ends [5]. Further, such a G/H has two ends precisely if it is homeomorphic to $K/L \times \mathbf{R}$, where K (respectively L) is a maximal compact subgroup of G (respectively of H, contained in K). If H has finitely many connected components, then the same results also hold (see section 1). As well Borel gave a class of examples of homogeneous manifolds having more than two ends. The existence of such examples points out that the situation can become quite complicated if one drops the assumption that the isotropy subgroup has finitely many connected components.

The Iwasawa decomposition for Lie groups and Borel's theorem for homogeneous manifolds with connected isotropy describe *topologically* the structure of such spaces

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having two ends. Under additional hypotheses one expects that a sharper classification ought to be possible. Indeed, this is so!. For example, Ahiezer [1] considered algebraic varieties G/H, where G is a linear algebraic group over $\mathbf{C}$ and H is an algebraic subgroup. As H has only a finite number of connected components, Borel's theorem implies G/H has at most two ends. Ahiezer proved that G/H has two ends precisely if it has a homogeneous fibration with fiber $\mathbf{C}^*$ and base a rational, i.e. precisely if H is the kernel of a non-trivial character $\phi: P \to \mathbf{C}^*$, where P is a parabolic subgroup of G. By a *homogeneous fibration* of G/H we mean the fibration given by $G/H \to G/J$, where J is some closed subgroup of G containing H. As well Hosrovjan [14] has characterized those spaces G/H with two ends, where G is a complex semisimple Lie group and H is a closed *connected* complex subgroup, in terms of a root system of the Lie algebra of G.

In this note we consider *complex homogeneous manifolds,* i.e. coset spaces G/H where G is a *complex analytic* Lie group and H is a closed *complex* subgroup. For H having a finite number of connected components, we determine the structure, *as complex homogeneous manifolds,* of such G/H having two ends. The basic philosophy is to exhibit the two ends by showing the existence of a complex homogeneous fibration having $\mathbf{C}^*$ either as fiber or as base.

The complex structure of the Lie group plays a significant role, for the main technique consists of using certain homogeneous fibrations which exist in the complex case, e.g. the Steinizer fibration [21] and the normalizer fibration [32], [6]. However, in order to be able to apply any fibration to the problem at hand, one needs to know that if the total space of the fibration has two ends, then either its fiber is compact and its base has two ends, or else its base is compact and its fiber has two ends. This is false in general. But in the cases that we consider here, such a "Fibration lemma" follows either from the Iwasawa decomposition or else from Borel's theorem. Using this Fibration lemma, we obtain the following structure theorem:

Suppose G is a connected complex Lie group and H is a closed complex subgroup with a finite number of connected components such that G/H has two ends. Then there exist closed complex subgroups I and J of G, with J containing I containing H, such that G/J is a homogeneous rational manifold and the homogeneous fibrations $G/H \to G/I \to G/J$ realize G/H either as a $\mathbf{C}^*$-bundle over a torus bundle over G/J or else as a torus bundle over a $\mathbf{C}^*$-bundle over G/J.

Adding the ends to a topological space yields a compactification of that space. But this compactification may not admit any natural complex structure. As we are interested in complex Lie group actions, this is not very satisfactory in the present setting. Instead, given a complex homogeneous space $X := G/H$, we would like to find a compact complex analytic space $\tilde{X}$, together with a holomorphic action of G on $\tilde{X}$, such that this action has an open orbit biholomorphic to X. Such a space $\tilde{X}$ is called an *almost homogeneous space* relative to the group G and it is well known that $\tilde{X} \setminus X$ is an analytic subvariety of $\tilde{X}$ (see [26]).

Now $\mathbf{C}^*$ has $\mathbf{P}^1$ as its natural compactification and in the above structure theorem one has fibrations involving compact manifolds as well as $\mathbf{C}^*$. Thus one might expect that one can always find an almost homogeneous manifold compactifying such G/H. Certainly if the $\mathbf{C}^*$-bundle is "on top," i.e. if $G/H \to G/I$ is the $\mathbf{C}^*$-bundle in the above structure theorem, then the associated $\mathbf{P}^1$-bundle does the job. If the torus bundle is "on top," then the situation is not so clear. For, there do exist nontrivial elliptic curve bundles over $\mathbf{C}^*$, e.g. certain complex solv-manifolds (see [8]), which are not compactifiable as almost homogeneous manifolds. If one could interchange the torus and $\mathbf{C}^*$ fibrations, then as before, one has the associated $\mathbf{P}^1$-bundle. Now we show that *in the fiber over each point* of G/J one can do this. Namely, the typical fiber J/H of the fibration $G/H \to G/J$ is biholomorphic to the product of $\mathbf{C}^*$ and the torus. But we give an example of a complex homogeneous space X having two ends and a connected isotropy subgroup where the torus bundle is "on top" but where no interchange is possible *in the category of complex homogeneous manifolds,* i.e. we show that one cannot realize X as a homogeneous $\mathbf{C}^*$-bundle $X = \hat{G}/\hat{H} \to \hat{G}/\hat{I}$ for the (effective) action of *any* complex Lie group $\hat{G}$. The question of finding an almost homogeneous compactification in some other way lies beyond the scope of the present paper and will not be considered.

This note is organized as follows. In the first section we recall some of the basic facts about ends and develop some of the tools needed later. We show in the second section that if a connected complex Lie group has two ends, then it is abelian and may be realized as a $\mathbf{C}^*$-bundle over a torus (Theorem 1). In the third section we consider complex homogeneous manifolds G/H having two ends, assuming H has a finite number of connected components, and prove the structure theorem quoted above (Theorem 4). Also we note that if such a space is holomorphically separable then the torus part drops out and the space may be realized as a homogeneous cone in some $\mathbf{C}^n$ (Theorem 2). In the fourth section we present an example and show that the homogeneous interchange of the torus and $\mathbf{C}^*$ fibrations in the structure theorem is not possible in this case. Finally in the fifth section we point out that, for every integer $k > 2$, there exists a discrete subgroup Γ_k of $\mathrm{SL}(2,\mathbf{C})$ such that $\mathrm{SL}(2,\mathbf{C})/\Gamma_k$ has k ends. These examples are quite similar to Borel's (see [5]), except that they are *complex.*

1. PRELIMINARIES

In this section we note some of the properties concerning ends which we use later. We begin by recalling the definition (e.g. [10]).

Definition. Let X be a connected topological space. Consider the family $\mathscr{F}$ of sequences $\{U_n\}_{n\in\mathbf{N}}$ such that

1. U_n is an open, connected subset of X with nonempty, compact boundary
2. $U_{n+1} \subset U_n$ for every $n \in \mathbf{N}$
3. $\bigcap_{n\in\mathbf{N}} \bar{U}_n = \phi$

In $\mathscr{F}$ we introduce the equivalence relation $\sim$ given by: $\{U_n\} \sim \{V_m\}$ if and only if for every $m \in \mathbf{N}$ there exists $n \in \mathbf{N}$ such that $U_n \subset V_m$. The set of equivalence classes $\mathscr{F}/\sim$ are the *ends* of X.

This definition is rather cumbersome. Also it is not entirely obvious that the relation introduced in the definition is symmetric (see [10, Satz 2, p. 695]). These criticisms aside, the definition does make precise the intuitive notion of "holes at infinity." It may help the reader to visualize an end as such a "hole at infinity" together with some representative sequence $\{U_n\}$ defining that end. As an example, the two ends of $\mathbf{C}^*$ are represented by the two sequences of complements of expanding annuli, e.g. the complements of $A_n := \{z \in \mathbf{C} : 1/n \leq |z| \leq n\}$. From this example one also sees the role that the equivalence relation plays. For it is clearly irrelevant, as far as the ends are concerned, that the annuli have circles as boundaries.

The following proposition, observed as a footnote by Serre [29, p. 59], points out that only in complex dimension one can a Stein manifold have more than one end. For domains of holomorphy in $\mathbf{C}^n$ this is obvious.

PROPOSITION. *Let X be a Stein manifold with* $\dim_{\mathbf{C}} X > 1$. *Then X has precisely one end.*

Proof. Since X is Stein, for $p > n := \dim_{\mathbf{C}} X$ one has $H_p(X) = 0$ (e.g. see [12, p. 156]). In particular $H_{2n-1}(X) = 0$ since $n > 1$. But if X had more than one end, $H_{2n-1}(X)$ would necessarily be nonzero. For there would then be $(2n-1)$-cycles which would not be boundaries, e.g. any compact hypersurface approximating the boundary of one of the open sets in an "ends sequence."

Remark. One can also see this by noting that every n-dimensional Stein manifold X contains a real n-dimensional closed CW-complex which is a strong deformation retract of X [12, p. 156].

The following Corollary plays a central role in our investigation of complex homogeneous manifolds with two ends.

COROLLARY. *Any complex homogeneous manifold which is Stein and has more than one end is biholomorphic to* $\mathbf{C}^*$.

We recall the result of Borel describing the structure of homogeneous manifolds having *connected* isotropy. In the following $X \sim Y$ denotes that the topological spaces X and Y are homeomorphic.

THEOREM. [5, Théorème 2]. *Let H be a closed connected subgroup of the connected Lie group G, K and L be maximal compact subgroups of G and H such that $K \supset L$, and s and t be integers such that $G \sim K \times \mathbf{R}^s$ and $H \sim L \times \mathbf{R}^t$. Then $s \geq t$ and one of the following holds:*

0) *$s = t$, K operates transitively on G/H and $G/H \sim K/L$ is compact.*

1) *$s > t + 1$ and G/H has one end.*

2) *$s = t + 1$, $G/H \sim K/L \times \mathbf{R}$ and has two ends.*

In section three we shall consider homogeneous spaces G/H with two ends, when H has a finite number of connected components. It turns out that we would like to consider G/H^0 as well and one wants to know that it has two ends whenever G/H does. The fibration $G/H^0 \rightarrow G/H$ is a finite covering map. But in general, finite coverings do not preserve the number of ends. For example, the Moebius band has one end and admits, as a double covering, the cylinder $S^1 \times [0,1]$ which has two ends. But this quotient is given by a homogeneous fibration of *real* Lie groups. Further examples are provided by the complex homogeneous spaces given in section five. However in the case in which we are presently interested, the number of ends is preserved. We are grateful to D. N. Ahiezer for pointing out an error in an earlier version of this proposition.

PROPOSITION 1. *Suppose G is a connected complex Lie group and H is a closed complex subgroup having finitely many connected components. Then G/H and G/H^0 have the same number of ends.*

Proof. Let K (respectively L) be a maximal compact subgroup of G (respectively of H, contained in K). Then one has the covariant fibration of Mostow [23], i.e., G/H is diffeomorphic to a real vector bundle over K/L by a K-equivariant map. Thus G/H has at most two ends. As well using Borel's theorem we have that G/H^0 has at most two ends. Since G/H is compact if and only if G/H^0 is, in order to complete the proof it suffices to consider the case when G/H^0 has two ends. In this case, the K-orbits in both G/H^0 and G/H are all real hypersurfaces, and Mostow's fibration theorem allows one to identify these spaces with the normal bundles of appropriately chosen K-orbits, Σ^0 and Σ respectively (also see [25]). In fact in the former case, this bundle is trivial [5], and thus we may assume that Σ^0 lies over Σ: $\Sigma^0 = K/L^0 \to K/L = \Sigma$. It is enough to show that Σ is *orientable,* because using this along with the orientation of the ambient complex manifold G/H we obtain a trivialization of the normal bundle of Σ. Therefore it is enough to show that L/L^0 acts on Σ^0 as a group of orientation preserving transformations. For this we note that the orientation ω for the complex manifold G/H^0 can be split along Σ^0, $\omega = \nu \wedge \tau$, where ν is a nonvanishing K-equivariant section of the normal bundle of Σ^0 and τ is an orientation on Σ^0. Since K acts holomorphically on G/H, the elements $k \in K$ are orientation preserving. Thus for all $k \in K$, one has $k_*(\omega) = f \cdot \omega$, where f is a positive smooth function. Since ν is K-invariant, it follows that $k_*(\tau) = f \cdot \tau$ (i.e., K is orientation preserving on Σ^0). Thus L/L^0 preserves the orientation defined by τ.

The following is needed to be able to use homogeneous fibrations in the study of ends.

FIBRATION LEMMA. *Suppose G is a connected complex Lie group and J and H are closed complex subgroups with $H \subset J$. Suppose J/H is connected, H has a finite number of connected components and G/H has two ends. Then either G/J is compact and J/H has two ends or else G/J has two ends and J/H is compact.*

Proof. Choose Iwasawa decompositions $G \sim K \times \mathbf{R}^s$, $J^0 \sim L \times \mathbf{R}^t$ and $H^0 \sim M \times \mathbf{R}^u$, where $M \subset L \subset K$ and $s \geq t \geq u$. Since G/H has two ends, G/H^0 does as well and Borel's theorem implies $u = s + 1$. Thus $t = s$ or $t = s + 1$. The result for G/J^0 and J^0/H^0 follows at once by using Borel's theorem again. To complete the proof, we only have to observe that J^0 acts transitively on J/H and apply Proposition 1 to the *finite* covering $J^0/H^0 \to J^0/J^0 \cap H$.

In passing we note that solv-manifolds also have at most two ends. This follows from the fact that any solv-manifold can be realized as a vector bundle over a compact solv-manifold (see [24] or [2]).

Later we shall be concerned with the possibility of "interchanging" tori and $\mathbf{C}^*$ in certain homogeneous fibrations. Of course this is *not* always possible. For there exist examples of two dimensional complex solv-manifolds which are nontrivial elliptic curve bundles over $\mathbf{C}^*$ [8]. But for complex solv-manifolds with connected isotropy and for complex Lie groups we do have such an "interchange."

PROPOSITION 2. *Let G be a connected complex Lie group and J and H be closed complex subgroups with J containing H. Suppose J/H is biholomorphic to a complex torus T and G/J is biholomorphic to $\mathbf{C}^*$. Further suppose that either of the following holds:*

a) *G is solvable and H is connected.*

b) *H is a normal subgroup of G.*

Then G/H is biholomorphic to the product $\mathbf{C}^* \times T$.

Proof. a) Since G is solvable, the structure theorem of Chevalley [7] yields the existence of a *central* compact subgroup A of G, where $A = \tilde{A}/\pi_1(G)$, $\tilde{A}$ being a central vector subgroup of the universal covering group of G containing $\pi_1(G)$. Letting $\mathfrak{a}$ denote the Lie algebra of A, we obtain a closed complex subgroup B of G corresponding to the complex Lie algebra $\mathfrak{a} + i\mathfrak{a}$. Note that B is also central.

We claim that B acts transitively on G/H. Since $G/J \cong \mathbf{C}^*$ and $(\operatorname{Aut} \mathbf{C}^*)^0 = \mathbf{C}^*$, J is a normal subgroup of G. Hence the orbit map, say $\psi: G \to G/J$, is a group homomorphism. Since H and J/H are connected by assumption, J is as well. Using the exact homotopy sequence of the fibration $G \to G/J$, we see that $\pi_1(G)$ maps *onto* $\pi_1(G/J)$. Thus $\psi(B)$ contains a generator for $\pi_1(G/J)$. Since B is central and ψ is a homomorphism, it follows that $\psi(B) = G/J$. Now consider the orbit map $\phi: G \to G/H$. If $\sigma: G/H \to G/J$ denotes the bundle projection, then $\psi = \sigma \circ \phi$. From above it follows that the action of B is transitive on the set of fibers of σ. In order to complete the proof that B acts transitively on G/H, we have only to show that $B \cap J$ acts transitively on J/H. But this follows in much the same way as before. For, as H is connected, from the exact homotopy sequence of the fibration $J \to J/H$ we get the fact that $\pi_1(J)$ maps onto $\pi_1(J/H)$. Also from the exact homotopy sequence of the fibration $G \to G/J = \mathbf{C}^*$ we get that $\pi_1(J)$ maps injectively into $\pi_1(G)$. Thus $\phi(B \cap J)$ contains generators for $\pi_1(J/H)$. Since $J/H \cong T$ and $(\operatorname{Aut} T)^0 \cong T$, we have that $\phi|_J$ is a group homomorphism. This together with the fact that $J \cap B$ is central imply that $\phi(B \cap J) = J/H$. Thus the abelian group B acts transitively on G/H. Hence G/H is biholomorphic to an abelian complex Lie group and since rank $\mathcal{O}(G/H) = 1$, this group is the direct product $\mathbf{C}^* \times T$ (e.g. [22, Theorem 3.2]).

b) Without loss of generality we may assume that $G/H = G$, i.e. $H = (e)$. Clearly $J = T$ is the Steinizer of G and is thus central [21]. As G/T is abelian, G' is contained in T. Therefore G is nilpotent, hence solvable, and we may apply a). Thus G is biholomorphic to $\mathbf{C}^* \times T$.

We now describe the normalizer fibration of a complex homogeneous manifold. For a given complex homogeneous space G/H, the Lie algebra $\mathfrak{h}$ of H is a vector subspace of the Lie algebra $\mathfrak{g}$ of G and may thus be considered as a point in the Grassmann manifold $G_{k,n}$ of k-planes in n-space, where $k := \dim_{\mathbf{C}} H$ and $n := \dim_{\mathbf{C}} G$. For $g \in G$, $\operatorname{ad}(g)$ acts holomorphically on $G_{k,n}$ where $\operatorname{ad}: G \to \operatorname{Hom}(\mathfrak{g}, \mathfrak{g})$ is the adjoint representation of G. The orbit of this action can be canonically identified with G/N, where $N := N_G(H^0)$, the normalizer in G of the connected component of the identity H^0 of H. Since one always has $H \subset N$, one has the so-called normalizer fibration $G/H \to G/N$.

PROPOSITION 3. *Suppose G is a connected complex Lie group acting on $\mathbf{P}^n$ and $X := G/H$ is an orbit. Suppose I is a connected normal complex subgroup of G and V is an I-stable algebraic variety in $\mathbf{P}^n$ whose intersection $\mathscr{A} := V \cap X$ is nonempty and minimal. Then $\{g\mathscr{A} : g \in G\}$ is a G-equivariant partition of G/H. Moreover, there exists a closed complex subgroup J of G, containing H and I, such that the partition $\{g\mathscr{A} : g \in G\}$ is precisely the partition of G/H given by the homogeneous fibration $G/H \to G/J$.*

Proof. Fix $g \in G$ and let $\mathscr{A}' := g\mathscr{A} \cap \mathscr{A}$. Since I is a normal subgroup of G, for any $h \in I$, one has $h' := g^{-1}hg \in I$. Thus

$$h\mathscr{A}' = hg\mathscr{A} \cap h\mathscr{A} = gh'\mathscr{A} \cap h\mathscr{A} = g\mathscr{A} \cap \mathscr{A} = \mathscr{A}'$$

and $\mathscr{A}'$ is I-invariant. Because of the minimality of $\mathscr{A}$, either $\mathscr{A}' = \mathscr{A}$ or $\mathscr{A}' = \emptyset$. Thus $g\mathscr{A} \cap \mathscr{A} = \emptyset$ or $g\mathscr{A} = \mathscr{A}$. Thus $\{g\mathscr{A} : g \in G\}$ is a G-equivariant partition since for any $g_1, g_2 \in G$, one has

$$\begin{aligned} g_1\mathscr{A} \cap g_2\mathscr{A} \neq \emptyset \quad & \text{if and only if } g_2^{-1}g_1\mathscr{A} \cap \mathscr{A} \neq \emptyset \\ & \text{if and only if } g_2^{-1}g_1\mathscr{A} = \mathscr{A} \\ & \text{if and only if } g_1\mathscr{A} = g_2\mathscr{A}. \end{aligned}$$

A remark of Remmert and van de Ven [27, p. 144] verifies that the stabilizer J in G of $\mathscr{A}$ is a closed complex subgroup, containing H and I, such that the fibration $G/H \to G/J$ yields the same partition of G/H as the partition $\{g\mathscr{A} : g \in G\}$

Remark. The same results also hold for any minimal I-invariant *analytic* subset of an arbitrary complex homogeneous manifold, the proof being the same as in Proposition 3.

PROPOSITION 4. *Suppose G is a connected complex Lie group acting linearly on $\mathbf{P}^n$ and $X := G/H$ is an orbit. If $G = S \cdot R$ is a Levi-Malcev decomposition of G, then any minimal R-invariant algebraic set $\mathscr{A}$ in X is holomorphically separable. Moreover, if $G/H \to G/J$ is the homogeneous fibration realizing the partition $\{g\mathscr{A} : g \in G\}$, then G/J may be written as S/L, where L is an algebraic subgroup of S.*

Proof. Since G is acting linearly on $\mathbf{P}^n$, there exists an R-invariant flag $\mathbf{P}^n = L_n \supset L_{n-1} \supset \dots L_0 = \{p\}$, where $\dim_{\mathbf{C}} L_k = k$. If $\mathscr{A}$ is any minimal R-invariant algebraic set in X and $\mathscr{A} \cap L_k \neq \emptyset$ for any k, then $\mathscr{A} \subset L_k$. Hence there exists k such that $\mathscr{A} \subset L_k \setminus L_{k-1}$. Since $L_k \setminus L_{k-1}$ is holomorphically separable, so is $\mathscr{A}$.

Suppose $\mathscr{A} = X \cap V$ where V is an R-stable algebraic variety in $\mathbf{P}^n$. Then $I := \mathrm{Stab}_{\mathrm{GL}(n+1,\mathbf{C})}(V)$ is an algebraic group. But $J := \mathrm{Stab}_G(\mathscr{A}) = I \cap G$ and contains R. Thus S acts transitively on G/J with isotropy $S \cap J = S \cap I =: L$, an algebraic subgroup.

2. COMPLEX LIE GROUPS

Freudenthal [10, Satz 15] showed that a connected topological group which is locally compact, locally connected and second countable has at most two ends.

For Lie groups one can see this using the E. Cartan-Malcev-Iwasawa decomposition (see [19]): Every connected Lie group is homeomorphic to the direct product of a maximal compact subgroup and a Euclidean space. Thus a connected Lie group has two ends precisely if a maximal compact subgroup has real codimension one. In this section we point out that a *complex* Lie group has two ends only if it is abelian and is a $\mathbf{C}^*$-bundle over a torus. Clearly this is false for real Lie groups.

LEMMA 1. *Let Γ be a discrete subgroup of rank $2n-1$ of the vector group $\mathbf{C}^n$. Then $\mathbf{C}^n/\Gamma$ is a $\mathbf{C}^*$-bundle over a compact complex torus.*

Proof. Since Γ has rank $2n-1$, without loss of generality we may write $\Gamma = \langle e_1, \ldots, e_n, v_1, \ldots, v_{n-1} \rangle_{\mathbf{Z}}$ where e_i is the i-th standard basis vector of $\mathbf{C}^n$ and $\{e_1, \ldots, e_n, v_1, \ldots, v_{n-1}\}$ is linearly independent over $\mathbf{R}$. Let

$$\mathbf{R}_\Gamma^{2n-1} := \langle e_1, \ldots, e_n, v_1, \ldots, v_{n-1} \rangle_{\mathbf{R}}$$

be the real vector subspace of $\mathbf{C}^n$ spanned by Γ. Since

$$v_j = \operatorname{Re} v_j + i \operatorname{Im} v_j \qquad \text{and} \qquad \operatorname{Re} v_j \in \langle e_1, \ldots, e_n \rangle_{\mathbf{R}}$$

for $1 \leq j \leq n-1$, it is clear that the maximal *complex* vector subspace of $\mathbf{R}_\Gamma^{2n-1}$ can be written as $\mathbf{C}_\Gamma^{n-1} := \langle \operatorname{Im} v_1, \ldots, \operatorname{Im} v_{n-1} \rangle_{\mathbf{C}}$. Further we may assume that $e_1 \notin \mathbf{C}_\Gamma^{n-1}$, i.e. $\{e_1, \operatorname{Im} v_1, \ldots, \operatorname{Im} v_{n-1}\}$ is a basis of $\mathbf{C}^n$ over $\mathbf{C}$. With respect to this basis, the i-th column in the following matrix represents the i-th generator of Γ (in the order $e_1, \ldots, e_n, v_1, \ldots, v_{n-1}$):

$$\begin{bmatrix} 1 & a_{12} & a_{13} & \cdots & a_{1n} & b_{11} & b_{12} & \cdots & b_{1n-1} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} & b_{21}+i & b_{22} & \cdots & b_{2n-1} \\ 0 & a_{32} & a_{33} & \cdots & a_{3n} & b_{31} & b_{32}+i & \cdots & b_{3n-1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & a_{n2} & a_{n3} & \cdots & a_{nn} & b_{n1} & b_{n2} & \cdots & b_{nn-1}+i \end{bmatrix}$$

where $a_{ij}, b_{ij} \in \mathbf{R}$. Note that the $(n-1) \times (n-1)$ matrix (a_{ij}), $2 \leq i, j \leq n$, is non-singular since it has the same determinant as the $n \times n$ matrix expressing the set of linearly independent vectors $\{e_1, \ldots, e_n\}$ in terms of the given basis (expansion by cofactors!). Thus the columns of the matrix (a_{ij}), $2 \leq i, j \leq n$, are linearly independent. It thus follows that if $V := \langle e_1 \rangle_{\mathbf{C}}$, then $V + \Gamma$ is a closed complex subgroup of $\mathbf{C}^n$. Hence the fibration $\mathbf{C}^n/\Gamma \to \mathbf{C}^n/V + \Gamma$ realizes $\mathbf{C}^n/\Gamma$ as a bundle with fiber $V/\Gamma \cap V = \mathbf{C}^*$ and base $\mathbf{C}^n/V + \Gamma$, a compact complex torus of complex dimension $n-1$.

THEOREM 1. *Let G be a connected complex Lie group with two ends. Then G is abelian and there is a complex torus T such that one of the following holds:*

a) *$\mathcal{O}(G) = \mathbf{C}$ and G is a topologically trivial $\mathbf{C}^*$-bundle over T, such that no power of the associated line bundle is holomorphically trivial.*

b) *rank $\mathcal{O}(G) = 1$ and $G = \mathbf{C}^* \times T$.*

Proof. First suppose $\mathcal{O}(G) = \mathbf{C}$. Then, as is well-known, G is abelian and $G = \mathbf{C}^n/\Gamma_k$, where Γ_k is a lattice in the vector group $\mathbf{C}^n$ of rank k, $n < k \leq 2n$ [21]. Since $\mathbf{C}^n/\Gamma_k$ is homeomorphic to $(S^1)^k \times \mathbf{R}^{2n-k}$, G has two ends precisely if

$k = 2n - 1$. Lemma 1 shows how to realize G as a $\mathbf{C}^*$-bundle over a torus T. Also it is well known that such bundles are topologically trivial but no power of the associated line bundle is holomorphically trivial (e.g. [15, Theorem 7]).

Now if $\mathcal{O}(G) \neq \mathbf{C}$, then there exists a connected closed central complex subgroup G_0 of G, the Steinizer of G, such that G/G_0 is Stein and $\mathcal{O}(G_0) = \mathbf{C}$ [21]. Since G has two ends, one sees, either by using the Fibration lemma or directly from the Iwasawa decomposition, that G_0 is compact and thus a central torus and that G/G_0 has two ends. Thus by the Corollary to the Footnote of Serre, $G/G_0 = \mathbf{C}^*$. By Proposition 2 we have that $G = \mathbf{C}^* \times T$ and G is abelian in this case as well. Clearly rank $\mathcal{O}(G) = 1$.

3. THE MAIN STRUCTURE THEOREM

In this section we determine the structure of complex homogeneous manifolds G/H with two ends, when H has a finite number of connected components. Before turning to the general case, we first consider such spaces which satisfy the maximal rank condition.

Definition. Let X be a complex manifold. Two points p and q in X are defined to be equivalent if $f(p) = f(q)$ for every $f \in \mathcal{O}(X)$. The codimension of an equivalence class at $x \in X$ is called $\mathrm{rank}_x\, \mathcal{O}(X)$ and we say that X satisfies the *maximal rank condition* if $\mathrm{rank}_x\, \mathcal{O}(X) = \dim_{\mathbf{C}} X$ for some $x \in X$.

THEOREM 2. *Suppose G is a connected complex Lie group and H is a closed complex subgroup with finitely many connected components such that G/H has two ends and satisfies the maximal rank condition. Then there exists a closed complex subgroup J of G, containing H, such that the homogeneous fibration $G/H \to G/J$ realizes G/H as a $\mathbf{C}^*$-bundle over the rational manifold G/J.*

Proof. Consider the Mostow covariant fibration [23] $X := G/H \to K/L$, where K (respectively L) is a maximal compact subgroup of G (respectively H, contained in K). Since X has two ends and is orientable, the K-orbits in G/H are all real analytic hypersurfaces. Let M be the zero section. Then the elements of K are biholomorphisms of X which leave M invariant and thus the Levi form of M has the same signature at every point of M. We claim that because G/H satisfies the maximal rank condition, it follows that the Levi form of M is (positive) definite at every point of M. For, if the Levi form of M had any zero eigenvalues, then M would be foliated by positive dimensional complex submanifolds (see [30] or [9]). But M is compact and has analytic dimension equal to the complex dimension of X. One knows [15, Theorem 2] that in this setting X does not satisfy the maximal rank condition. This contradiction rules out the existence of zero eigenvalues.

As well the Levi form of M cannot have eigenvalues of opposite sign. For suppose that it would have. Then a small deformation of M does not change the sign of any of the eigenvalues. Since M is compact we can deform M into a hypersurface M' which still has a Levi form having eigenvalues of opposite sign at every point. But then X is a pseudoconcave manifold since the relatively compact open set between M and M' has at least one negative eigenvalue of its Levi form at each of its boundary points. Then $\mathcal{O}(X) = \mathbf{C}$, contradicting the assumption that X satisfies the maximal rank condition.

Thus M is a strongly pseudoconvex homogeneous hypersurface in the sense of Rossi [28]. Rossi showed in this setting that G/H is a $\mathbf{C}^*$-bundle over a homogeneous projective algebraic manifold. In fact, using the results of [17] one can explicitly realize G/H as a linear cone in some $\mathbf{C}^n$.

Remark. It is now obvious that any G/H which has two ends and satisfies the maximal rank condition, where H has finitely many connected components, has an envelope of holomorphy. Just add the vertex to the cone! As far as we know it is still an open question whether every holomorphically separable complex homogeneous manifold has an envelope of holomorphy.

THEOREM 3. (Ahiezer [1]). *Suppose G is a linear algebraic group over $\mathbf{C}$ and H is an algebraic subgroup such that G/H has two ends. Then there exists a parabolic subgroup P of G, containing H, such that the homogeneous fibration $G/H \to G/P$ realizes G/H as a $\mathbf{C}^*$-bundle over the homogeneous rational manifold G/P.*

Proof. Assume $G = S \cdot R$ is a Levi-Malcev decomposition of G. Since G is algebraic, so is R and thus the R-orbits in G/H are closed. Hence there is a homogeneous fibration $G/H \to G/RH$ with connected fiber, since R acts transitively on that fiber. Clearly $G/RH = S/L$ for some algebraic subgroup L of S. If S/L is Stein (precisely if L is reductive [20]) then the Fibration lemma implies that S/L has two ends and thus is $\mathbf{C}^*$ and that the fiber is compact. But then the fiber must be trivial since R acts transitively on it. Thus $G/H = \mathbf{C}^*$.

Now suppose S/L is not Stein and so L is *not* reductive. Then there exists a parabolic subgroup P such that $L \subsetneq P \subsetneq S$. The method of constructing such a P is well known, so we only remind the reader of how it goes. Let U be the unipotent radical of the connected component of the identity of L, i.e. $U := R_u(L^0)$. Since L is not reductive, $U \neq (e)$. Then set $N_1 := N_G(U)$, $U_1 := U \cdot R_u(N_1)$ and inductively define $N_k := N_G(U_{k-1})$, $U_k := U_{k-1} \cdot R_u(N_k)$. Since unipotent radicals are connected and G is finite-dimensional, these two sequences stabilize, i.e. $U_k = U_{k+1} = \ldots$ and $N_k = N_{k+1} = \ldots$ for some k. Then $P := N_k$ is the desired parabolic subgroup (e.g., see [18, Section 30.3] for more details). Now since R acts trivially on G/RH, the fibration $G/RH = S/L \to S/P$ is G-equivariant and thus one has a fibration $G/H \to G/RH \to G/J := S/P$. Since S/P is rational and thus compact, the Fibration lemma implies J/H has two ends. Proceeding by induction, we may assume the existence of a fibration $J/H \to J/I$, where $I/H = \mathbf{C}^*$ and J/I is rational. But then $G/H \to G/I$ is the fibration we want, since $G/I \to G/J$ is a rational bundle over a rational and thus G/I is also rational.

THEOREM 4. *Suppose G is a connected complex Lie group and H is a closed complex subgroup with a finite number of connected components such that G/H has two ends. Then there exist closed complex subgroups I and J of G, with J containing I containing H, such that G/J is a homogeneous rational manifold and the homogeneous fibrations $G/H \to G/I \to G/J$ realize G/H either as a $\mathbf{C}^*$-bundle over a torus bundle over G/J or else as a torus bundle over a $\mathbf{C}^*$-bundle over G/J.*

Proof. We consider the normalizer fibration $G/H \to G/N$. If G/N is compact, then N is connected and G/N is a homogeneous rational manifold (Borel-Remmert [6, Satz 7]). Now by Proposition 1, G/H^0 has two ends. Applying the Fibration lemma to the fibration $G/H^0 \to G/N$, we conclude that its fiber N/H^0 is a connected complex Lie group with two ends. Thus by Theorem 1, N/H^0 is abelian and hence H/H^0 is normal in N/H^0. Using the Iwasawa decomposition, we see that N/H has two ends. Thus by Theorem 1 again, N/H is a $\mathbf{C}^*$-bundle over a torus.

Next suppose G/N is not compact. In this case N is not necessarily connected. However, if we denote by $\hat{N}$ the set of connected components of N which meet H, then $\hat{N}$ is a closed complex subgroup of G having only finitely many connected components. Clearly $G/\hat{N}$ is not compact for it covers G/N. Also $\hat{N}/H$ is a connected complex torus since N^0/H^0 as a compact complex Lie group is a torus. By the Fibration lemma $G/\hat{N}$ has two ends.

We claim that there exists a closed complex subgroup J of G, containing $\hat{N}$, such that $J/\hat{N}$ is $\mathbf{C}^*$ and G/J is rational. To prove this we distinguish three cases. First if $G/\hat{N}$ satisfies the maximal rank condition, then Theorem 2 provides us with the appropriate J. Second if the semisimple part S of G in some Levi-Malcev decomposition $G = S \cdot R$ acts transitively on $G/\hat{N}$, then S also acts transitively on G/N. Thus we may write G/N as S/L, where $L := S \cap N$. Let $\hat{L} := S \cap \hat{N}$. Since S is acting via the adjoint representation as an algebraic group on S/L and L is the stabilizer of an algebraic set, namely a point, it follows that L is an algebraic subgroup of S (see [26, Satz 1.3.6]). Thus L has a finite number of connected components and thus $\hat{L}$ does too. Since $S/\hat{L}$ has two ends, by Proposition 1 it follows that S/L also has two ends. Hence there exists a closed complex subgroup P of S containing L such that $S/L \to S/P$ is a $\mathbf{C}^*$-bundle over a rational (see [1] or Theorem 3). Thus we have a $\mathbf{C}^*$-bundle $G/\hat{N} = S/\hat{L} \to S/P$, but we still need to find a G-equivariant fibration. We get this by noting that $S/\hat{L}$ is Zariski-open in its closure X on which G acts. Taking stabilizers we see that the algebraic hull of G acts on $S/\hat{L}$ and we may apply the result of Ahiezer to this group (which contains G!).

The remaining case occurs when the semisimple part S of G in any Levi-Malcev decomposition $G = S \cdot R$ does not act transitively on $G/\hat{N}$ and $G/\hat{N}$ does not satisfy the maximal rank condition. Then G/N also does not satisfy the maximal rank condition. As well the radical R of G acts nontrivially on $G/\hat{N}$ and thus on G/N. Now consider any minimal R-invariant algebraic set $\mathscr{A}$ in G/N. Clearly such $\mathscr{A}$ exist, since G/N itself is an R-invariant algebraic set. And by Proposition 4 $\mathscr{A}$ is holomorphically separable, thus $0 < \dim \mathscr{A} < \dim G/N$. By Proposition 3 there exists a closed complex subgroup J of G, containing N and R, such that the fibers of $G/N \to G/J$ are precisely the sets of the partition $\{g\mathscr{A} : g \in G\}$. Let $\hat{J}$ be the subgroup consisting of those (finitely many) connected components of J which meet $\hat{N}$. Applying the Fibration lemma to $G/\hat{N} \to G/\hat{J}$, we see that $G/\hat{J}$ is compact and $\hat{J}/\hat{N}$ has two ends. Thus G/J is compact and since by Proposition 4 $G/J = S/L$ is the quotient of algebraic groups, G/J is rational. Thus J is connected, $\hat{J} = J$ and $J/\hat{N}$ has two ends. Also $J/\hat{N}$ satisfies the maximal rank condition since it covers J/N. By Theorem 2 there exists a closed complex subgroup I of J, containing $\hat{N}$, such that $J/\hat{N} \to J/I$ is a $\mathbf{C}^*$-bundle over a rational. Thus I is the subgroup we want because $I/\hat{N} = \mathbf{C}^*$ and G/I, as a rational over a rational, is itself rational.

4. INTERCHANGING HOMOGENEOUS FIBRATIONS

The main structure theorem assures the existence of the homogeneous fibration $G/H \to G/J$, where G/J is rational. In this section we look at the fiber J/H of this fibration in the case when it is a torus bundle over a $\mathbf{C}^*$-bundle and consider the possibility of interchanging these fibrations. We show first that the typical fiber is biholomorphic to the product of $\mathbf{C}^*$ and the torus. Then we present an example where it is not possible to interchange the $\mathbf{C}^*$ and torus fibrations *homogeneously.*

Suppose that G/H has two ends, where H has a finite number of connected components, and we are in the second case in the proof of Theorem 4, i.e. suppose we have homogeneous fibrations $G/H \to G/N \to G/J$ where N/H is a torus and $J/N = \mathbf{C}^*$. Since G/J is a rational manifold, it is simply connected. Thus J is connected and its orbit J/H in G/H is a torus bundle over $\mathbf{C}^*$ which is given by the fibration $J/H \to J/N$. Since H has a finite number of connected components and N/H is connected, it follows that N has a finite number of connected components. Thus the fibration $J/H^0 \to J/N^0$ is also a torus bundle over $\mathbf{C}^*$. Now J/N^0 abelian implies $J' \subset N^0$ and N^0/H^0 abelian implies $(N^0)' \subset H^0$. Hence $(J')' \subset H^0$ and the solvable group $J/(J')'$ acts transitively on J/H^0 with connected isotropy $H^0/(J')'$. It follows from Proposition 2, a) that J/H^0 is biholomorphic to the product of $\mathbf{C}^*$ and the torus N^0/H^0. Thus J/H is biholomorphic to $\mathbf{C}^* \times N/H$.

Next we present an example which cannot be interchanged homogeneously. Let $G := \mathrm{SL}(2,\mathbf{C}) \times T$, where T is an elliptic curve. To simplify the notation, write

$$G = \left\{ (\alpha,\beta,\gamma,\delta,w) : \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2,\mathbf{C}),\ w \in T \right\};$$

understood with the appropriate group structure. Let

$$J := \{(\alpha,\beta,0,\alpha^{-1},w)\},\ N := \{(1,\beta,0,1,w)\} \qquad \text{and} \qquad H := \{1,\beta,0,1,\Phi(\beta)\},$$

where $\Phi : \mathbf{C} \to T$ is the usual universal covering homomorphism. Then $X := G/H$ has two ends and H is connected. Further $\pi : G/H \to G/N$ is a nontrivial torus bundle over the bundle $\mathbf{C}^2 \setminus \{0\} = G/N \to G/J = \mathbf{P}^1$ and π is the holomorphic separation map of X. Since $\pi_j(\mathbf{C}^2 \setminus \{0\}) = 0$ for $j = 1, 2$, and T is an elliptic curve, it follows that $\pi_1(X) = \mathbf{Z}^2$. Also one should note that in constructing this example we have taken the group G in order to have connected isotropy. Certainly $\mathrm{SL}(2,\mathbf{C})$ also acts transitively on X, but with infinite discrete isotropy. It is of note that in some cases one can change the group acting so as to have isotropy with a finite number of connected components.

Now suppose that $X = G/H$, where G is any connected complex Lie group. Then there exists a closed complex subgroup J of G, containing H, such that the fibration $G/H \to G/J$ is the holomorphic separation fibration of G/H [11, Theorem 1]. But $\pi : X \to \mathbf{C}^2 \setminus \{0\}$ is *the* holomorphic separation map of X. Thus $G/J = \mathbf{C}^2 \setminus \{0\}$ and $J/H = T$, i.e. one always has a homogeneous fibration $X = G/H \to G/J = \mathbf{C}^2 \setminus \{0\}$ with $J/H = T$.

For the moment consider *any* homogeneous fibration $G/H \to G/J$, where $G/J = \mathbf{C}^2 \setminus \{0\}$ and J/H is an elliptic curve T. Since $\mathbf{C}^2 \setminus \{0\}$ is simply connected, J is connected and the bundle is principal. Now without loss of generality we may assume G is simply connected and we may choose a Levi-Malcev decomposition $G = S \times R$. Since no solvable group can act transitively on $\mathbf{C}^2 \setminus \{0\}$, S has nontrivial orbits in $\mathbf{C}^2 \setminus \{0\}$. But no S-orbit can be 1-dimensional, for then this orbit would be $\mathbf{P}^1$. Thus S has only open orbits and fixed points. But fixed points are ruled out by the Cone Theorem [17] and thus S acts transitively on $\mathbf{C}^2 \setminus \{0\}$. Its isotropy is algebraic and not reductive and so is contained in a parabolic. Thus there is an S-equivariant fibration $\mathbf{C}^2 \setminus \{0\} \to \mathbf{P}^1$ and we can split off an $\mathrm{SL}(2,\mathbf{C})$ from S which also acts transitively on $\mathbf{C}^2 \setminus \{0\}$. For simplicity let $S := \mathrm{SL}(2,\mathbf{C})$ in the rest of this section. Now consider the S-orbits in G/H. Either they are open or they are sections of the principal bundle $G/H \to G/J$. In the latter case, the bundle is trivial, i.e., $G/H = \mathbf{C}^2 \setminus \{0\} \times T$. So we conclude that either S acts transitively on G/H or else the universal covering manifold of G/H is $\mathbf{C}^2 \setminus \{0\} \times \mathbf{C}$ and thus is *not* Stein.

But the complex manifold X constructed above has the universal covering manifold $\mathrm{SL}(2,\mathbf{C})$ which is Stein. Thus we have shown that if any connected complex Lie group G acts transitively on X, then there exists an $S = \mathrm{SL}(2,\mathbf{C})$ in G and S-equivariant fibrations

$$\begin{array}{ccccc} S/\Gamma & \xrightarrow{T} & S/U & \xrightarrow{\mathbf{C}^*} & S/B \\ \| & & \| & & \| \\ X & & \mathbf{C}^2 \setminus \{0\} & & \mathbf{P}^1 \end{array}$$

and by conjugating we may suppose $B = \left\{ \begin{pmatrix} \lambda & \mu \\ 0 & \lambda^{-1} \end{pmatrix} \right\}$ is the "standard Borel" and $U = \left\{ \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \right\}$ is the "standard maximal nilpotent." Also it is clear that Γ is the lattice of T in $U \cong \mathbf{C}$, i.e. $U/\Gamma = T$.

Finally we claim that X cannot be realized as a homogeneous fibration $X = G/H \to G/I$ with $I/H = \mathbf{C}^*$ and G/I compact, where G is *any* connected complex Lie group acting transitively on X. For assume that such a fibration exists. Then the fibration is also S-equivariant, i.e., we have

$$X = S/\Gamma \to S/\hat{I} = G/I, \qquad \text{where} \qquad \hat{I} := S \cap I.$$

But then $S/\hat{I}$, as a compact complex homogeneous surface, is one of the following: T^2, $T^1 \times \mathbf{P}^1$, $\mathbf{P}^2$, $\mathbf{P}^1 \times \mathbf{P}^1$ or a homogeneous Hopf surface (e.g. [32]). The first two are ruled out since S cannot act transitively on them and the second two are easily ruled out by looking at the exact homotopy sequence of the bundle $G/H \to G/I$ and recalling $\pi_1(G/H) = \mathbf{Z}^2$. Thus $S/\hat{I}$ is a homogeneous Hopf surface and has universal covering $S/\hat{I}^0 = \mathbf{C}^2 \setminus \{0\}$. Hence $\mathbf{C}^* = \hat{I}^0 / \hat{I}^0 \cap \Gamma$ and $\hat{I}^0 \cap \Gamma$ is a rank one lattice. Since $\hat{I}^0$ is algebraic and $\mathbf{C}^*$ is holomorphically separable, $\hat{I}^0$ contains the algebraic closure of $\hat{I}^0 \cap \Gamma$ (see Barth-Otte [3]), i.e.,

$\hat{I}^0 \supset \left\{\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}\right\}$. Thus $\hat{I}^0 = U$ and $\hat{I}^0/\Gamma = \mathbf{C}^*$, contradicting the fact that $U/\Gamma = T$. This contradiction rules out the existence of any homogeneous $\mathbf{C}^*$-fibration.

5. ARBITRARILY MANY ENDS

For each integer $k > 2$, Borel [5] has shown the existence of discrete subgroups Γ_k of $\mathrm{SL}(2,\mathbf{R})$ such that $\mathrm{SL}(2,\mathbf{R})/\Gamma_k$ has k ends. In this section we point out, using the same idea as Borel, how to construct *complex* homogeneous manifolds with arbitrarily many ends.

Let $S := \mathrm{SL}(2,\mathbf{C})$ and let K be a maximal compact subgroup of S. One model of the space $K\backslash S$ is hyperbolic 3-space and any discrete subgroup Γ of S acts discontinuously on H^3. One way of determining the number of ends of such an S/Γ would be to construct a fundamental domain for Γ. However we shall take a different approach. Recently Thurston [31] has shown that the complements of certain knots in S^3 are complete hyperbolic and thus have H^3 as their universal covering surface. This then gives a method for proving the existence of discrete subgroups Γ_k in S such that the double coset space $K\backslash S/\Gamma_k$ has k ends, even though one does not know explicitly in all cases what the groups Γ_k are. For example, the complement in S^3 of the k-link chain is complete hyperbolic with k ends and thus can be written as $K\backslash S/\Gamma_k$, for some discrete Γ_k in S. Since the natural map $S/\Gamma_k \to K\backslash S/\Gamma_k$ has connected *compact* fibers, S/Γ_k also has k ends and is a *complex* homogeneous manifold.

We note some particular examples. For $k = 3$ (respectively $k = 4$) one can take $\mathrm{SL}(2,R)$, where R is the ring of integers of the quadratic imaginary number field $\mathbf{Q}(\sqrt{-7})$ (respectively $\mathbf{Q}(\sqrt{-3})$). For these examples and others, Bianchi [4] computed explicit fundamental domains and initiated the study of reduction theory for Hermitian forms under the action of $\mathrm{SL}(2,R)$.

As pointed out by Thurston [31, 6.38] if k divides l then there is a covering map of degree l/k from $K\backslash S/\Gamma_l$ to $K\backslash S/\Gamma_k$ and such a map can be lifted to a map from S/Γ_l to S/Γ_k. As we noted in section one, such finite covering maps do not preserve the number of ends.

In closing we would like to point out that, for any $k > 2$, there are no non-constant holomorphic functions on $X := \mathrm{SL}(2,\mathbf{C})/\Gamma_k$. For consider the possibilities for rank $\mathscr{O}(X)$. First X is not holomorphically separable. For, if it were, then Γ_k would be algebraic (see Barth-Otte [3]). But then $S \to S/\Gamma_k$ would be a finite covering and S/Γ_k would be Stein and thus have only one end. Also rank $\mathscr{O}(X)$ cannot be positive. For denote by $S/\Gamma_k \to S/J$ the holomorphic separation fibration [11, Theorem 1]. If rank $\mathscr{O}(X) = 1$, then $\dim_{\mathbf{C}} S/J = 1$ and thus $S/J = \mathbf{P}^1$, which is clearly absurd. And if rank $\mathscr{O}(X) = 2$, one has an explicit list of the possibilities for S/J: namely the affine quadric, the complement of the quadric in $\mathbf{P}^2$ or some power $\mathscr{H}^n$, for $n > 0$, of the hyperplane section bundle over $\mathbf{P}^1$ (see [16]). But these are all easily ruled out. For example by comparing their fundamental groups with that of S/Γ_k using the exact homotopy sequence of the fibration $S/\Gamma_k \to S/J$. Thus $\mathscr{O}(X) = \mathbf{C}$.

REFERENCES

1. D. N. Ahiezer, *Dense orbits with two ends.* Izv. Akad. Nauk SSSR Ser. Mat. 41 (1977), no. 2, 308–324; English transl. in Math. USSR Izvestija 11 (1977), 293–307.
2. L. Auslander and R. Tolimieri, *Splitting theorems and the structure of solvmanifolds.* Ann. of Math. (2) 92 (1970), 164–173.
3. W. Barth and M. Otte, *Invariante holomorphe Funktionen auf reduktiven Liegruppen.* Math. Ann. 201 (1973), no. 2, 97–112.
4. L. Bianchi, *Sui gruppi di sostituzioni lineari con coefficienti appartenenti a corpi quadratici imaginari.* Math. Ann. 40 (1892), 332–412 (also 38 (1891), 313–333).
5. A. Borel, *Les bouts des espaces homogènes de groupes de Lie.* Ann. of Math. (2) 58 (1953), 443–457.
6. A. Borel and R. Remmert, *Über kompakte homogene Kählersche Mannigfaltigkeiten.* Math. Ann. 145 (1961/62), 429–439.
7. C. Chevalley, *On the topological structure of solvable groups.* Ann. of Math. (2) 42 (1941), 668–675.
8. J. Erdman-Snow, *Solv-manifolds of dimension two and three.* University of Notre Dame, Ph.D. Thesis, 1979.
9. M. Freeman, *Local complex foliations of real submanifolds.* Math. Ann. 209 (1974), 1–30.
10. H. Freudenthal, *Über die Enden topologischer Räume und Gruppen.* Math. Z. 33 (1931), 692–713.
11. B. Gilligan and A. Huckleberry, *On non-compact complex nil-manifolds.* Math. Ann. 238 (1978), no. 1, 39–49.
12. H. Grauert and R. Remmert, *Theory of Stein spaces,* Springer, 1979.
13. H. Hopf, *Enden offener Räume und unendliche diskontinuierliche Gruppen.* Comment. Math. Helv. 16 (1943/4), 81–100.
14. O. M. Hosrovjan, *Complex homogeneous spaces with two ends.* Akad. Nauk Armjan. SSR Dokl. 65 (1977), no. 5, 263–265 (Russian).
15. A. Huckleberry, *The Levi problem on pseudoconvex manifolds which are not strongly pseudoconvex.* Math. Ann. 219 (1976), no. 2, 127–137.
16. A. Huckleberry and L. Livorni, *A classification of homogeneous surfaces,* to appear.
17. A. Huckleberry and E. Oeljeklaus, *A characterization of complex homogeneous cones.* Math. Z. 170 (1980), no. 2, 181–194.
18. J. Humphreys, *Linear algebraic groups,* Springer, New York, 1975.
19. K. Iwasawa, *On some types of topological groups.* Ann. of Math. (2) 50 (1949), no. 3, 507–558.
20. Y. Matsushima, *Espaces homogènes de Stein des groupes de Lie complexes.* I, II. Nagoya Math. J. 16 (1960), 205–218; 18 (1961) 153–164.
21. A. Morimoto, *Non-compact complex Lie groups without non-constant holomorphic functions.* Proc. Conf. Complex Analysis (Minneapolis, 1964), pp. 256–272, Springer, Berlin, 1965.
22. ———, *On the classification of noncompact complex abelian Lie groups.* Trans. Amer. Math. Soc. 123 (1966), 200–228.

23. G. Mostow, *On covariant fiberings of Klein spaces.* I. II. Amer. J. Math. 77 (1955), 247–278; 84 (1962), 466–474.

24. ———, *Some applications of representative functions to solvmanifolds.* Amer. J. Math. 93 (1971), 11–32.

25. T. Nagano, *Transformation groups with $(n-1)$-dimensional orbits on non-compact manifolds.* Nagoya Math. J. 14 (1959), 25–38.

26. E. Oeljeklaus, *Über fasthomogene kompakte komplexe Mannigfaltigkeiten.* Schr. Math. Inst. Univ. Münster (2) Heft 1 (1970).

27. R. Remmert and A. van de Ven, *Zur Funktionentheorie homogener komplexer Mannigfaltigkeiten.* Topology 2 (1963), 137–157.

28. H. Rossi, *Homogeneous strongly pseudoconvex hypersurfaces.* Rice Math. Studies 59 (1973), no. 1, 131–145.

29. J-P. Serre, *Quelques problèmes globaux relatifs aux variétés de Stein.* Colloque sur les fonctions de plusieurs variables, Bruxelles, 1953, pp. 57–68, Masson, Paris, 1953.

30. F. Sommer, *Komplex-analytische Blätterung reeller Hyperflächen im* $\mathbf{C}^n$. Math. Ann. 137 (1959), 392–411.

31. W. Thurston, *The geometry and topology of 3-manifolds.* Lecture Notes, Princeton Univ., Princeton, N.J., 1977.

32. J. Tits, *Espaces homogènes complexes compacts.* Comment. Math. Helv. 37 (1962), no. 2, 111–120.

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