LIFTING OF A CONTRACTION INTERTWINING TWO ISOMETRIES

Zoia Ceauşescu

0. Throughout this note we consider only (bounded, linear) operators on Hilbert spaces. As usual, we denote by $L(\mathscr{H}_1,\mathscr{H}_2)$ the space of all operators from $\mathscr{H}_1$ into $\mathscr{H}_2$ and by $L(\mathscr{H})$ the space $L(\mathscr{H},\mathscr{H})$. Also, for two contractions T_1 and T_2 on $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively, we shall denote by $I(T_2,T_1)$ the set of all operators $A \in L(\mathscr{H}_1,\mathscr{H}_2)$ intertwining T_2 and T_1; *i.e.*, satisfying $T_2A = AT_1$. Let $V_i \in L(\mathscr{H}_i)$ be an isometry ($i = 1,2$), $\mathscr{H}_0$ a (closed linear) subspace of $\mathscr{H}_1$ invariant for V_1 and $V_0 = V_1 | \mathscr{H}_0$. By a *contractive intertwining lifting in* $I(V_2,V_1)$ (briefly, (V_2,V_1)-CIL) of a contraction $A \in I(V_2,V_0)$ we mean any contraction $\hat{A} \in I(V_2,V_1)$ satisfying $\hat{A} | \mathscr{H}_0 = A$. In case $V_i = S_i$ ($i = 1,2$) is a unilateral shift, necessary and sufficient conditions for the existence and the uniqueness of such a (S_2,S_1)-CIL were given in [3, Theorem 2] and [4, Proposition 3.1] Also, in case $V_i = U_i$ ($i = 1,2$) is a unitary operator and $\mathscr{H}_0$ is a reducing subspace for U_1, three equivalent conditions for the uniqueness of a (U_2,U_1)-CIL of A (which obviously, in this case always exists) were given in [2, Corollary 2.3]. In the present note we extend the result of [3] to the case of arbitrary isometries V_1 and V_2 (see Thm. 1.1 below), and also, adapting the quoted results of [2] and [4], we give a necessary and sufficient condition for the uniqueness of (V_2,V_1)-CIL of A (see Section 3, Thm. 3.1).

I take this opportunity to express my thanks to Professor C. Foiaş for the helpful discussions concerning the subject of this note and also for suggesting the example given in Section 4.

1. THEOREM 1.1. *Let $V_i \in L(\mathscr{H}_i)$ be an isometry ($i = 1,2$), let $\mathscr{H}_0$ be a subspace of $\mathscr{H}_1$ invariant for V_1, let $V_0 = V_1 | \mathscr{H}_0$, and let A be a contraction belonging to $I(V_2,V_0)$. Then, there exists a (V_2,V_1)-CIL $\hat{A}$ of A if and only if the condition*

$$\|(I - V_2^n V_2^{*n})Ah_0\| \leq \|(I - V_1^n V_1^{*n})h_0\| \tag{1.1}$$

holds for all $n = 1,2,\ldots$ and $h_0 \in \mathscr{H}_0$.

Proof. Since the necessity of the condition (1.1) is obvious, it remains to prove its sufficiency. For this purpose we adapt the original proof of [3] to the present more general situation.

Let $U_i \in L(\mathscr{K}_i)$ be the minimal unitary dilation of V_i ($i = 0,1,2$) (see [9, Ch.I, Sec. 4]); obviously we can and shall identify $\mathscr{K}_0$ with the space $\bigvee_{n=0}^{\infty} U_1^{-n}\mathscr{H}_0$ and $U_0 = U_1 | \mathscr{K}_0$. Also, let us denote by P_i the orthogonal projection of $\mathscr{K}_i$ onto $\mathscr{H}_i$ ($i = 0,1,2$), and let us set $\mathscr{G}_i = (I - P_i)\mathscr{K}_i$ ($i = 1,2$) and $\mathscr{G}_0 = ((I - P_1)\mathscr{K}_0)^-$.

Received April 1, 1978. Revision received September 15, 1978.

Michigan Math. J. 26 (1979).

First, we define by the formula

$$A_0 k_0 = s - \lim_{n\to\infty} U_2^{-n} A P_0 U_1^n k_0 \qquad (k_0 \in \mathscr{K}_0) \tag{1.2}$$

(see [6, Thm. 2 and Cor. 5.1]) the unique contraction $A_0 \in I(U_2, U_0)$ satisfying

$$A_0 \mid \mathscr{H}_0 = A \tag{1.3}$$

Since we also have $A_0 U_1^{-m} \mid \mathscr{H}_0 = U_2^{-m} A$ $(m = 0,1,2,\ldots)$, it follows, by (1.1), that

$$\begin{aligned}\|(I - P_2) A_0 U_1^{-m} h_0\|^2 &= \|A_0 U_1^{-m} h_0\|^2 - \|P_2 A_0 U_1^{-m} h_0\|^2 \\ &= \|U_2^{-m} A h_0\|^2 - \|V_2^{*m} A h_0\|^2 = \|(I - V_2^m V_2^{*m}) A h_0\|^2 \\ &\le \|(I - V_1^m V_1^{*m}) h_0\|^2 \le \|U_1^{-m} h_0\|^2 - \|V_1^{*m} h_0\|^2 \\ &= \|(I - P_1) U_1^{-m} h_0\|^2 \qquad (h_0 \in \mathscr{H}_0).\end{aligned}$$

From this and from the fact that $U_1^{-m} h_0$ $(m = 0,1,2,\ldots,\ h_0 \in \mathscr{H}_0)$ span $\mathscr{K}_0$, we infer that there exists a unique contraction $A_1 \colon \mathscr{G}_0 \mapsto \mathscr{G}_2$ such that

$$A_1 (I - P_1) k_0 = (I - P_2) A_0 k_0 \qquad (k_0 \in \mathscr{K}_0). \tag{1.4}$$

Also, denoting by W_i the isometry $U_i^{-1} \mid \mathscr{G}_i$ $(i = 1,2)$ and remarking that since $\mathscr{K}_0$ reduces U_1, $\mathscr{G}_0$ is invariant to W_1^*, we have, by (1.4),

$$\begin{aligned}W_2^* A_1 (I - P_1) k_0 &= (I - P_2) U_2 (I - P_2) A_0 k_0 = (I - P_2) U_2 A_0 k_0 \\ &= (I - P_2) A_0 U_1 k_0 = A_1 (I - P_1) U_1 k_0 = A_1 W_1^* (I - P_1) k_0\end{aligned}$$

$(k_0 \in \mathscr{K}_0)$, so that $A_1 \in I(W_2^*, W_1^* \mid \mathscr{G}_0)$. Then, by the lifting theorem for coisometries (see [8], [7], [9]), it follows that there exists a contraction $B_1 \in I(W_2^*, W_1^*)$ such that $B_1 \mid \mathscr{G}_0 = A_1$.

Now, let us consider the subspace $\mathscr{M}_i = \bigvee_{n=0}^{\infty} U_i^n \mathscr{G}_i$ of $\mathscr{K}_i$ and let us denote by $P_{\mathscr{M}_i}$ the orthogonal projection of $\mathscr{K}_i$ onto $\mathscr{M}_i$ $(i = 1,2)$; obviously $U_i P_{\mathscr{M}_i} = P_{\mathscr{M}_i} U_i$ $(i = 1,2)$. Since the strong limit of $\{U_2^n B_1 (I - P_1) U_1^{-n} \mid \mathscr{M}_i\}_{n \in \mathbb{N}}$ exists, we can define by the formula

$$\tilde{B}_1 = s - \lim_{n\to\infty} U_2^n B_1 (I - P_1) U_1^{-n} \mid \mathscr{M}_i \tag{1.5}$$

an operator from $\mathscr{M}_1$ into $\mathscr{M}_2$ which is (for any B_1) the unique contraction belonging to $I(U_2 \mid \mathscr{M}_2, U_1 \mid \mathscr{M}_1)$ such that $B_1 (I - P_1) \mid \mathscr{M}_1 = (I - P_2) \tilde{B}_1$ (see [1, Thm. 1.1] and [9, Ch. VII, Prop. 3.2]). Also, by the definition of $\mathscr{M}_i$, we have

$$\begin{aligned}s - \lim_{n\to\infty} U_i^n (I - P_i) U_i^{-n} k_i &= s - \lim_{n\to\infty} U_i^n (I - P_i) U_i^{-n} P_{\mathscr{M}_i} k_i \\ &= P_{\mathscr{M}_i} k_i - (s - \lim_{n\to\infty} U_i^n P_i U_i^{-n} P_{\mathscr{M}_i} k_i) = P_{\mathscr{M}_i} k_i \\ &\qquad (k_i \in \mathscr{K}_i,\quad i = 1,2),\end{aligned}$$

whence, by (1.4),

$$(1.6)\quad \begin{aligned} B_1 P_{\mathcal{M}_1} k_0 &= s - \lim_{n\to\infty} U_2^n B_1 (I - P_1) U_0^{-n} k_0 = s - \lim_{n\to\infty} U_2^n A_1 (I - P_1) U_0^{-n} k_0 \\ &= s - \lim_{n\to\infty} U_2^n (I - P_2) A_0 U_0^{-n} k_0 = s - \lim_{n\to\infty} U_2^n (I - P_2) U_2^{-n} A_0 k_0 \\ &= P_{\mathcal{M}_2} A_0 k_0 \qquad (k_0 \in \mathcal{K}_0). \end{aligned}$$

Now, if we denote $\mathcal{K}_1' = \mathcal{M}_1 \vee \mathcal{K}_0$ and

$$(1.7)\qquad B_2 = \tilde{B}_1 (P_{\mathcal{M}_1} | \mathcal{K}_1'),$$

then B_2 is obviously a contraction belonging to $I(U_2 | \mathcal{M}_2, U_1 | \mathcal{K}_1')$ which, by (1.6), satisfies

$$(1.8)\qquad B_2 | \mathcal{K}_0 = P_{\mathcal{M}_2} A_0 .$$

Also, by (1.8), we have

$$\|(I - P_{\mathcal{M}_2}) A_0 k_0\| \le \|D_{B_2} k_0\| \qquad (k_0 \in \mathcal{K}_0),$$

(where as usual, for a contraction C $\in L(\mathcal{H}, \mathcal{H}')$ we denote $D_C = (I - C^* C)^{1/2} \in L(\mathcal{H})$ and $\mathcal{D}_C = (D_C \mathcal{H})^-$), whence it follows that there exists a unique contraction A_2 from $(D_{B_2} \mathcal{K}_0)^-$ into $(I - P_{\mathcal{M}_2}) \mathcal{K}_2$ such that

$$(1.9)\qquad A_2 D_{B_2} k_0 = (I - P_{\mathcal{M}_2}) A_0 k_0 \qquad (k_0 \in \mathcal{K}_0).$$

Because $U_1 D_{B_2} = D_{B_2} U_1 | \mathcal{K}_1'$, we obtain at once, by (1.8), that

$$A_2 \in I(U_2 | \mathcal{K}_2 \ominus \mathcal{M}_2, U_1 | (D_{B_2} \mathcal{K}_0)^-).$$

Now, let $\hat{A}_2$ be a $(U_2 | \mathcal{K}_2 \ominus \mathcal{M}_2, U_1 | \mathcal{D}_{B_2})$-CIL of A_2 (such an A_2 is for instance $A_2 Q$, where Q denotes the orthogonal projection of $\mathcal{D}_{B_2}$ onto $(D_{B_2} \mathcal{K}_0)^-$). Then, if we set

$$(1.10)\qquad \hat{A}_0' = B_2 + \hat{A}_2 D_{B_2}$$

it is clear that $\hat{A}_0' \in I(U_2, U_1 | \mathcal{K}_1')$ and that, by (1.8) and (1.9),

$$(1.11)\qquad \hat{A}_0' | \mathcal{K}_0 = A_0 .$$

Moreover, since the ranges of B_2 and $\hat{A}_2$ are orthogonal and $\hat{A}_2$ is a contraction, we easily infer that $\hat{A}_0'$ is also a contraction. Hence $\hat{A}_0'$ is a $(U_2, U_1 | \mathcal{K}_1')$-CIL of A_0. We note that, since

$$\begin{aligned} (I - P_2) B_2 &= (I - P_2) \tilde{B}_1 P_{\mathcal{M}_1} | \mathcal{K}_1' = B_1 (I - P_1) | \mathcal{K}_1' \\ &= (I - P_2) \tilde{B}_1 (I - P_1) | \mathcal{K}_1' = (I - P_2) B_2 (I - P_1) | \mathcal{K}_1', \end{aligned}$$

we also have

$$(I - P_2)\hat{A}'_0 = (I - P_2)\hat{A}'_0(I - P_1) \mid \mathscr{K}'_1. \tag{1.12}$$

Now, let $\hat{A}_0$ be a (U_2, U_1)-CIL of $\hat{A}'_0$ satisfying

$$(I - P_2)\hat{A}_0 = (I - P_2)\hat{A}_0(I - P_1). \tag{1.13}$$

Note that such a (U_2, U_1)-CIL of $\hat{A}'_0$ exists because we can set $\hat{A}_0 = \hat{A}'_0 P_{\mathscr{K}'_1}$ where $P_{\mathscr{K}'_1}$ denotes the orthogonal projection of $\mathscr{K}_1$ onto $\mathscr{K}'_1$. Obviously, by (1.11), $\hat{A}_0$ is a (U_2, U_1)-CIL of A_0 and also, by (1.13), $\hat{A}_0 \mid \mathscr{H}_1 \in L(\mathscr{H}_1, \mathscr{H}_2)$, so that the operator

$$\hat{A} = \hat{A}_0 \mid \mathscr{H}_1 \tag{1.14}$$

is a (V_2, V_1)-CIL of A. This completes the proof of theorem.

2. Let the operators A, V_i, $U_i (i = 0,1,2)$, the spaces $\mathscr{G}_i (i = 0,1,2)$, $\mathscr{M}_i (i = 1,2)$, $\mathscr{K}'_1$ and the orthogonal projections P_i $(i = 0,1,2)$, $P_{\mathscr{M}_i} (i = 1,2)$, $P_{\mathscr{K}'_1}$ be as in Section 1; also, let A_0 be the contraction defined by (1.2). In the sequel we shall assume that A satisfies the condition (1.1) and A_1 will be the contraction defined by the formula (1.4). In the following three lemmas we shall give some simple facts concerning the CIL's of the contractions A and A_0.

LEMMA 2.1. *The formulas* (1.14) *and*

$$\hat{A}_0 = s - \lim_{n\to\infty} U_2^{-n} \hat{A} P_1 U_1^n \tag{2.1}$$

establish a one-to-one correspondence between all the (U_2, U_1)-CIL's $\hat{A}_0$ *of* A_0 *satisfying* (1.13) *and all the* (V_2, V_1)-CIL's $\hat{A}$ *of* A.

Proof. This lemma follows directly from Theorem 2 and Corollary 5.1 of [6].

LEMMA 2.2. *Let* $\hat{A}'_0$ *be a contraction in* $I(U_2, U_1 \mid \mathscr{K}'_1)$ *satisfying* (1.12). *The formulas*

$$\hat{A}_0 = \hat{A}'_0 P_{\mathscr{K}'_1} + D_{\hat{A}'_0{}^*}\Gamma(I - P_{\mathscr{K}'_1}) \tag{2.2}$$

and

$$D_{\hat{A}'_0{}^*}\Gamma = \hat{A}_0 \mid \mathscr{K}_1 \ominus \mathscr{K}'_1 \tag{2.3}$$

establish a one-to-one correspondence between all the contraction

$$\Gamma \in I(U_2 \mid \mathscr{D}_{\hat{A}'_0{}^*}, U_1 \mid \mathscr{K}_1 \ominus \mathscr{K}'_1)$$

satisfying

$$(I - P_2) D_{\hat{A}'_0{}^*}\Gamma = 0 \tag{2.4}$$

and all the (U_2, U_1)-CIL's $\hat{A}_0$ *of* $\hat{A}'_0$ *satisfying* (1.13).

Proof. It is known (see [5, Lemma 3.1]) that for any contraction

$$A'_0 \in L(\mathscr{K}'_1, \mathscr{K}_1)$$

the formulas (2.2) and (2.3) establish a one-to-one correspondence between all the contractions $\Gamma \in L(\mathscr{K}_1 \ominus \mathscr{K}'_1, \mathscr{D}_{A'^*_0})$ and all the contractions $\hat{A}_0 \in L(\mathscr{K}_1, \mathscr{K}_2)$ satisfying $\hat{A}_0 \mid \mathscr{K}'_1 = \hat{A}'_0$. Also, if $\hat{A}'_0 \in I(U_2, U_1 \mid \mathscr{K}'_1)$ and satisfies (1.12), then, for any contraction $\Gamma \in I(U_2 \mid \mathscr{D}_{A'^*_0}, U_1 \mid \mathscr{K}_1 \ominus \mathscr{K}'_1)$ satisfying (2.4) the contraction $\hat{A}_0$ defined by (2.2) lies in $I(U_2, U_1)$ and satisfies (1.13); on the other hand, for any (U_2, U_1)-CIL $\hat{A}_0$ of $\hat{A}'_0$ satisfying (1.13), the formula (2.3) defines a contraction belonging to $I(U_2 \mid \mathscr{D}_{A'^*_0}, U_1 \mid \mathscr{K}_1 \ominus \mathscr{K}'_1)$ which also satisfies

$$\begin{aligned}(I - P_2) D_{A'^*_0} \Gamma (I - P_{\mathscr{K}'_1}) &= (I - P_2) \hat{A}_0 (I - P_{\mathscr{K}'_1}) = (I - P_2)(\hat{A}_0 - \hat{A}'_0 P_{\mathscr{K}'_1}) \\ &= (I - P_2) \hat{A}_0 (I - P_1) - (I - P_2) \hat{A}'_0 (I - P_1) = 0.\end{aligned}$$

Remark 2.1. If, in Lemma 2.2, $\hat{A}'_0$ is a $(U_2, U_1 \mid \mathscr{K}'_1)$-CIL of A_0, then the (U_2, U_1)-CIL $\hat{A}_0$ of $\hat{A}'_0$ defined by (2.2) is a (U_2, U_1)-CIL of A_0.

LEMMA 2.3. *Let $\hat{A}'_0$ be a $(U_2, U_1 \mid \mathscr{K}'_1)$-CIL of A_0 satisfying* (1.12). *Then $(I - P_2) \hat{A}'_0 \mid \mathscr{G}_1$ is a (W^*_2, W^*_1)-CIL of A_1, and $\hat{A}'_0$ satisfies also*

$$P_{\mathscr{M}_2} \hat{A}'_0 = P_{\mathscr{M}_2} \hat{A}'_0 (P_{\mathscr{M}_1} \mid \mathscr{K}'_1). \tag{2.5}$$

Proof. First, note that

$$\begin{aligned}(I - P_2) \hat{A}'_0 (I - P_1) U_1 \mid \mathscr{G}_1 &= (I - P_2) \hat{A}'_0 U_1 \mid \mathscr{G}_1 = (I - P_2) U_2 \hat{A}'_0 \mid \mathscr{G}_1 \\ &= (I - P_2) U_2 (I - P_2) \hat{A}'_0 \mid \mathscr{G}_1\end{aligned}$$

and

$$(I - P_2) \hat{A}'_0 (I - P_1) k_0 = (I - P_2) \hat{A}'_0 k_0 = (I - P_2) A_0 k_0 = A_1 (I - P_1) k_0 \qquad (k_0 \in \mathscr{K}_0)$$

whence it follows that $(I - P_2) \hat{A}'_0 \mid \mathscr{G}_1$ is a (W^*_2, W^*_1)-CIL of A_1. Also, since (by virtue of the last three equalities in (1.6)) we have

$$\begin{aligned}P_{\mathscr{M}_2} \hat{A}'_0 k_0 &= P_{\mathscr{M}_2} A_0 k_0 = s - \lim_{n \to \infty} U^n_2 A_1 (I - P_1) U^{-n}_1 k_0 \\ &= s - \lim_{n \to \infty} U^n_2 (I - P_2) \hat{A}'_0 (I - P_1) U^{-n}_1 k_0 = P_{\mathscr{M}_2} \hat{A}'_0 P_{\mathscr{M}_1} k_0 \qquad (k_0 \in \mathscr{K}_0)\end{aligned}$$

it follows that $\hat{A}'_0$ satisfies (2.5).

Remark 2.2. It is obvious that if $\hat{A}_0$ is a (U_2, U_1)-CIL of A_0 satisfying (1.13), then $\hat{A}_0 \mid \mathscr{K}'_1$ is a $(U_2, U_1 \mid \mathscr{K}'_1)$-CIL of A_0 satisfying (1.12). Also, any $(U_2, U_1 \mid \mathscr{K}'_1)$-CIL $\hat{A}'$ of A_0 is of the form (1.10), where $B_2 = P_{\mathscr{M}_2} \hat{A}'_0$ and $\hat{A}_2$ is a contraction in $I(U_2 \mid \mathscr{K}_2 \ominus \mathscr{M}_2, U_1 \mid \mathscr{D}_{B_2})$ such that $A_2 = \hat{A}_2 \mid (D_{B_2} \mathscr{K}_0)^-$ satisfies (1.9). From these facts and by virtue of Lemmas 2.1, 2.2 and 2.3 it is clear that:

The construction given in the proof of Theorem 1.1 yields all the (V_2, V_1)-CIL's *of a contraction A intertwining the isometries* V_2 *and* $V_0 = V_1 \mid \mathscr{K}_0$ *and satisfying* (1.1).

3. Now, we recall some results concerning the contractive intertwining dilations (CID) of a contraction $A \in I(T_2, T_1)$, where T_i is an arbitrary contraction on $\mathscr{K}_i$ $(i = 1,2)$. If we denote by U_{i+} the minimal isometric dilation of T_i on $\mathscr{K}_{i+}$ $(i = 1,2)$ (see [9, Ch. I, Sec. 10]), and by P_{i+} the orthogonal projection of $\mathscr{K}_{i+}$ onto $\mathscr{K}_i$, then by a (U_{2+}, U_{1+})-CID of A we mean any contraction $A_\infty \in I(U_{2+}, U_{1+})$ satisfying $P_{2+}A_\infty = AP_{1+}$. Also, let us denote

$$(3.1)\quad \begin{cases} \mathscr{R}(A \cdot T_1) = \{d + l_1 \in \mathscr{D}_A + \mathscr{L}_1 : T_1^* D_A d + U_{1+}^* l_1 = 0\}^- \subset \mathscr{D}_A + \mathscr{L}_1 \\ \mathscr{R}(T_2 \cdot A) = \{d \oplus l_2 \in \mathscr{D}_A \oplus \mathscr{L}_2 : D_A d + A^* U_{2+}^* l_2 = 0\}^- \subset \mathscr{D}_A \oplus \mathscr{L}_2, \end{cases}$$

where $\mathscr{L}_i = ((U_{i+} - T_i)\mathscr{K}_i)^-$ $(i = 1,2)$. By a *A-choice sequence* (see [5, Def. 3.1]) we mean a sequence of contractions $\{\Gamma_n\}_{n=1}^\infty$ such that $\Gamma_1 \in L(\mathscr{R}(A \cdot T_1), \mathscr{R}(T_2 \cdot A))$ and $\Gamma_n \in L(\mathscr{D}_{\Gamma_{n-1}}, \mathscr{D}_{\Gamma_{n-1}^*})$ $(n \geq 2)$. In [5, Propositions 2.2 and 3.1] the following result is established:

There exists a one-to-one correspondence between all the (U_{2+}, U_{1+})-CID *of A and all the A-choice sequences* $\{\Gamma_n\}_{n=1}^\infty$.

We introduce now the following

Definition 3.1. We say that a (U_{2+}, U_{1+})-CID of A is the *distinguished* (U_{2+}, U_{1+})-CID (and we denote it by A_∞^0) if it is the CID of A corresponding to the A-choice sequence $\{\Gamma_n^0\}_{n=1}^\infty$ with $\Gamma_n^0 = 0$ for all $n = 1, 2, \ldots$

Note that, if A has a unique (U_{2+}, U_{1+})-CID, then this is the distinguished CID A_∞^0.

We shall also consider the case when $Z \in L(\mathscr{G})$ is an isometry such that $\mathscr{G} \supset \mathscr{K}_{2+}$, $\mathscr{K}_{2+}$ reduces Z and $Z \mid \mathscr{K}_{2+} = U_{2+}$. By a (Z, U_{1+})-CID of $A \in I(T_2, T_1)$ we mean any contraction $\tilde{A}_\infty \in I(Z, U_{1+})$ satisfying $\tilde{P}_2 \tilde{A}_\infty = AP_1$, where $\tilde{P}_2$ denotes the orthogonal projection of $\mathscr{G}$ onto $\mathscr{K}_2$. If we denote by $\tilde{P}_{2+}$ the orthogonal projection of $\mathscr{G}$ onto $\mathscr{K}_{2+}$ it is clear that $\tilde{A}'_\infty = \tilde{P}_{2+}\tilde{A}_\infty \in L(\mathscr{K}_{1+}, \mathscr{K}_{2+})$ is a (U_{2+}, U_{1+})-CID of A and $\tilde{A}''_\infty = (I - \tilde{P}_{2+})\tilde{A}_\infty \in L(\mathscr{K}_{1+}, \mathscr{G} \ominus \mathscr{K}_{2+})$ is of the form: $\tilde{A}''_\infty = CD_{\tilde{A}'_\infty}$, where C is a contraction in $L(\mathscr{D}_{\tilde{A}'_\infty}, \mathscr{G} \ominus \mathscr{K}_{2+})$ satisfying $CD_{\tilde{A}'_\infty} U_{1+} = ZCD_{\tilde{A}'_\infty}$. The *distinguished* (Z, U_{1+})-CID of A is by definition the following (Z, U_{1+})-CID of A:

$$(3.2)\qquad \tilde{A}_\infty^0 = \begin{bmatrix} A_\infty^0 \\ OD_{A_\infty^0} \end{bmatrix} : \mathscr{K}_{1+} \mapsto \begin{matrix} \mathscr{K}_{2+} \\ \oplus \\ \mathscr{G} \ominus \mathscr{K}_{2+} \end{matrix},$$

where A_∞^0 is the distinguished (U_{2+}, U_{1+})-CID of A and 0 is regarded as operator from $\mathscr{D}_{A_\infty^0}$ into $\mathscr{G} \ominus \mathscr{K}_{2+}$.

In the sequel A will be a contraction as in Sections 1 and 2 satisfying the condition (1.1). Also, let A_0 and A_1 be the contractions uniquely defined by the formulas (1.2) and (1.4), respectively. Using the notation of Sections 1 and 2, we denote $\mathscr{G}'_1 = \bigvee_{n=0}^{\infty} W_1^n \mathscr{G}_0$ and $W_{1+} = W_1 \mid \mathscr{G}'_1$; obviously $\mathscr{G}'_1$ reduces W_1 (since

$\mathcal{G}_0$ is invariant for W_1^*), and W_{1+} is the minimal isometric dilation of $P_{\mathcal{G}_0} W_1 | \mathcal{G}_0$ (where $P_{\mathcal{G}_0}$ denotes the orthogonal projection of $\mathcal{G}_1$ onto $\mathcal{G}_0$). Let $A_{1\infty}^{*0}$ be the distinguished (W_{1+}, W_2)-CID of $A_1^* \in I(P_{\mathcal{G}_0} W_1 | \mathcal{G}_0, W_2)$, and let $\tilde{A}_{1\infty}^{*0}$ be the distinguished (W_1, W_2)-CID of A_1^*. The adjoint operator $(\tilde{A}_{1\infty}^{*0})^* = (A_{1\infty}^{*0}) P_{\mathcal{G}'_1}$ (where $P_{\mathcal{G}'_1}$ denotes the orthogonal projection of $\mathcal{G}_1$ onto $\mathcal{G}'_1$), is a (W_2^*, W_1^*)-CIL of A_1. We denote

$$B_1^0 = (A_{1\infty}^{*0})^* P_{\mathcal{G}'_1} \tag{3.3}$$

and we say that B_1^0 is the *distinguished* (W_2^*, W_1^*)-CIL of A_1.

Let $\tilde{B}_1^0$ be the contraction defined by the formula (1.5) for $B_1 = B_1^0$, let B_2^0 be the contraction defined by the formula (1.7) for $\tilde{B}_1 = \tilde{B}_1^0$, and let A_2^0 be the contraction defined by the formula (1.9) for $B_2 = B_2^0$. If we set $\hat{B}_2^0 = \tilde{B}_1^0 P_{\mathcal{M}_1}$, then the operator

$$\hat{A}_0^0 = \hat{B}_2^0 + A_2^0 \hat{Q}^0 D_{\hat{B}_2^0} P_{\mathcal{K}'_1}, \tag{3.4}$$

where $\hat{Q}^0$ denotes the orthogonal projection of $\mathcal{D}_{\hat{B}_2^0}$ onto $(D_{\hat{B}_2^0} \mathcal{K}_0)^-$, is a (U_2, U_1)-CIL of A_0 satisfying (1.13); obviously, in the case when A_0 has a unique (U_2, U_1)-CIL, this is $\hat{A}_0^0$.

Now, by virtue of Lemma 2.2, we have the following:

LEMMA 3.1. *In order that* $\hat{A}_0'^0 | \mathcal{K}'_1 \in I(U_2, U_1 | \mathcal{K}'_1)$ *have a unique* (U_2, U_1)-CIL *satisfying* (1.13) (*which will be* $\hat{A}_0^0$) *it is necessary and sufficient that*

$$\{\Gamma \in I(U_2 | \mathcal{D}_{\hat{A}_0'^{0*}}, U_1 | \mathcal{K}_1 \ominus \mathcal{K}'_1): (I - P_2) D_{\hat{A}_0^{0*}} \Gamma = 0\} = \{0\}. \tag{a$_1$}$$

Also by virtue of [2, Corollary 2.3] we obtain at once

LEMMA 3.2. $\hat{A}_0'^0 = \hat{A}_0^0 | \mathcal{K}'_1$ *is the unique* $(U_2, U_1 | \mathcal{K}'_1)$-CIL *of* A_0 *satisfying*

$$(I - P_2) \hat{A}_0'^0 = B_1^0 (I - P_1) | \mathcal{K}'_1 \tag{3.5}$$

if and only if

$$I(U_2 | \mathcal{K}_2 \ominus \mathcal{M}_2, U_1 | (\mathcal{D}_{B_2^0} \ominus (D_{B_2^0} \mathcal{K}_0)^-)) = \{0\}, \tag{a$_2$}$$

where

$$B_2^0 = P_{\mathcal{M}_2} \hat{A}_0'^0. \tag{3.6}$$

Finally, if we denote by C_k the contraction from $((I - V_1^k V_1^{*k}) \mathcal{K}_0)^-$ into $(I - V_2^k V_2^{*k}) \mathcal{K}_2$ defined by the formula

$$C_k (I - V_1^k V_1^{*k}) h_0 = (I - V_2^k V_2^{*k}) A h_0 \qquad (h_0 \in \mathcal{K}_0) \tag{3.7}$$

(which exists by virtue of (1.1)) for every $k = 1, 2, \ldots$ and $C_0 = 0 \in L(\mathcal{K}_0, \mathcal{K}_2)$, then, by virtue of [4, Proposition 3.1] we also have the following

LEMMA 3.3. *In order that A_1 have a unique (W_2^*, W_1^*)-CIL (which will be $B_1^0: (I - P_2)\hat{A}_0^0 | \mathcal{G}_1$) it is necessary and sufficient that*

(a_3) *one of the following two conditions holds:*

(i) *for every $h_1 \in \mathcal{H}_1$ there exists $\{h_{0k}\}_{k=1}^{\infty} \subset \mathcal{H}_0$ and $\{n_k\}_{k=1}^{\infty} \subset \mathbb{N} \setminus \{0\}$ such that*

$$(3.8) \qquad \begin{cases} s - \lim\limits_{k\to\infty} (I - V_1 V_1^*) V_1^{*n_k - 1} h_{0k} = (I - V_1 V_1^*) h_1 \\ s - \lim\limits_{k\to\infty} D_{C_{n(k)-1}} (I - V_1^{n_k - 1} V^{*n_k - 1}) h_{0k} = 0 \end{cases}$$

(ii) *for every $h_2 \in \mathcal{H}_2$ there exists $\{h_{0k}\}_{k=1}^{\infty} \subset \mathcal{H}_0$ and $\{n_k\}_{k=1}^{\infty} \subset \mathbb{N} \setminus \{0\}$ such that*

$$(3.9) \qquad \begin{cases} s - \lim\limits_{k\to\infty} (I - V_2 V_2^*) V_2^{*n_k - 1} A h_{0k} = (I - V_2 V_2^*) h_2 \\ s - \lim\limits_{k\to\infty} D_{C_{n(k)-1}} (I - V_1^{n_k - 1} V_1^{*n_k - 1}) h_{0k} = 0. \end{cases}$$

It is clear, by virtue of the construction given in the proof of Theorem 1.1, by Remark 2.2, and by Lemmas 2.1, 3.1, 3.2 and 3.3, that every one of the conditions (a_1), (a_2) and (a_3) is a necessary condition for the uniqueness of the (V_2, V_1)-CIL of A, and also that all the conditions (a_1), (a_2) and (a_3) together are sufficient in order that A have a unique (V_2, V_1)-CIL; hence we conclude with the following

THEOREM 3.1. *In order that a contraction A belonging to $I(V_2, V_0 = V_1 | \mathcal{H}_0)$ and satisfying (1.1) have a unique (V_2, V_1)-CIL (which will be $\hat{A}^0 = \hat{A}_0^0 | \mathcal{H}_1$) it is necessary and sufficient that the conditions (a_1), (a_2) and (a_3) hold.*

4. In this section we illustrate Theorems 1.1 and 3.1 by a functional example. Let us consider an arbitrary (fixed) scalar valued analytic function $\theta_j(\lambda) \in H^\infty$, $0 \not\equiv |\theta_j(\lambda)| \leq 1$ for $|\lambda| \leq 1$ $(j = 1, 2)$. We define the space

$$(4.1)_j \qquad \mathcal{H}_j = H^2 \oplus (\Delta_j L^2)^-$$

where $\Delta_j(e^{it}) = (1 - |\theta_j(e^{it})|^2)^{1/2}$ $(0 \leq t \leq 2\pi)$, $j = 1, 2$, and the subspace $\mathcal{H}_0$ of $\mathcal{H}_1$ by

$$(4.1)_0 \qquad \mathcal{H}_0 = \{\theta_1 u \oplus \Delta_1 u : u \in H^2\}$$

Also, we consider the following isometry on $\mathcal{H}_j$:

$$(4.2)_j \qquad V_j(u \oplus w)(e^{it}) = e^{it} u(e^{it}) \oplus e^{it} w(e^{it}) \qquad (u \oplus w \in \mathcal{H}_j, 0 \leq t \leq 2\pi).$$

$(j = 1, 2)$. Also, we denote by U_j the minimal unitary dilation of V_j on

$$\mathcal{K}_j = L^2 \oplus (\Delta_j L^2)^-$$

(obviously, U_j is the multiplication by e^{it} on $\mathcal{K}_j$), $j = 1, 2$. Using the notation of the previous section it is obvious that

$$(4.3)_j \qquad \mathscr{G}_j = (L^2 \ominus H^2) \oplus \{0\} \subset \mathscr{K}_j, \qquad \mathscr{M}_j = L^2 \oplus \{0\} \subset \mathscr{K}_j,$$

$(j = 1,2)$ and

$$(4.3)_0 \qquad \begin{cases} \mathscr{K}_0 = \{\theta_1 v \oplus \Delta_1 v : v \in L^2\}, \\ \mathscr{G}_0 = (P_{L^2 \ominus H^2} \theta_1 L^2)^- \oplus \{0\} = (L^2 \ominus H^2) \oplus \{0\} = \mathscr{G}_1. \end{cases}$$

Also, we have

$$(4.4) \qquad \mathscr{K}_1' = (L^2 \oplus \{0\}) \vee \{\theta_1 v \oplus \Delta_1 v : v \in L^2\} = (L^2 \oplus \Delta_1 L^2)^- = \mathscr{K}_1.$$

Let A be a contraction in $I(V_2, V_1 \mid \mathscr{K}_0)$. Then, it is clear that A has a unique form

$$(4.5) \qquad A(\theta_1 u \oplus \Delta_1 u)(e^{it}) = \alpha(e^{it}) u(e^{it}) \oplus \tau(e^{it}) u(e^{it}) \qquad (u \in H^2, 0 \le t \le 2\pi)$$

where $\alpha(\lambda) \in H^\infty$, $\tau(e^{it})$ is measurable,

$$(4.6) \qquad \tau(e^{it}) = 0 \quad \text{if} \quad \Delta_2(e^{it}) = 0$$

and

$$(4.7) \qquad |\alpha(e^{it})|^2 + |\tau(e^{it})|^2 \le 1 \qquad \text{a.e. on } 0 \le t \le 2\pi.$$

Because, for any contraction A as in Theorem 1.1, the condition (1.1) is equivalent to the existence of a unique contraction A_1 (defined by (1.4)), we can easily infer in the particular case of this section that, by the functional representation of A_1, the condition (1.1) is equivalent to the existence of a function $\beta \in H^\infty$, $\|\beta\|_{H^\infty} \le 1$ and satisfying

$$(4.8) \qquad \beta(\lambda)\theta_1(\lambda) = \alpha(\lambda) \qquad (|\lambda| < 1).$$

On the other hand if $A \in I(V_2, V_1 \mid \mathscr{K}_0)$ is a contraction satisfying the condition (4.8), then, since $\mathscr{G}_0 = \mathscr{G}_1$ and $\mathscr{K}_1' = \mathscr{K}_1$, the conditions (a_3) and (a_1), respectively, are automatically satisfied, so that the uniqueness of the (V_2, V_1)-CIL A^0 of A is equivalent to the condition (a_2). In order to give an explicit interpretation of the condition (a_2), we notice that in our case this can be written

$$(4.9) \qquad I(U_2 \mid (\{0\} \oplus (\Delta_2 L^2)^-), U_1 \mid (\mathscr{D}_{B_2^0} \ominus (D_{B_2^0} \mathscr{K}_0)^-)) = \{0\},$$

where the operator B_2^0 has the same meaning as in Section 3. Denote by χ_0 the characteristic function of the set $\{e^{it}: |\beta(e^{it}) \ne 1, 0 \le t \le 2\pi\}$ and by χ_j the characteristic function of the set $\{e^{it}: \Delta_j(e^{it}) \ne 0, 0 \le t \le 2\pi\}$ $(j = 1,2)$. It is easy to see that the operator $U_1 \mid (\mathscr{D}_{B_2^0} \ominus (D_{B_2^0} \mathscr{K}_0)^-)$ is unitarily equivalent to the operator of multiplication by e^{it} on $\chi_0 \chi_1 L^2$ and that the operator $U_2 \mid (\{0\} \oplus (\Delta_2 L^2)^-)$ is unitarily equivalent to the operator of multiplication by e^{it} on $\chi_2 L^2$. From this it is clear that (4.9) is equivalent to

$$(4.10) \qquad \chi_0 \chi_1 \chi_2 = 0 \qquad \text{a.e.}$$

Now we can conclude with the following

COROLLARY 4.1. *Let* $\mathscr{H}_j$ *and* V_j *be defined by* $(4.1)_j$ $(j = 0,1,2)$, *respectively by* $(4.2)_j$ $(j = 1,2)$ *and let* A *be the operator of the form* (4.5), (4.6), (4.7) *satisfying also the condition* (4.8) *with some* $\beta \in H^\infty$, $\|\beta\|_{H^\infty} \le 1$. *Then* A *has a unique* (V_2, V_1)-CIL *if and only if Lebesgue (linear) measure of the set*

$$\{e^{it}: |\theta_1(e^{it})| \neq 1,\ |\theta_2(e^{it})| \neq 1,\ |\beta(e^{it})| \neq 1,\ 0 \le t \le 2\pi\} \tag{4.11}$$

is 0.

Finally, let us give a consequence of this corollary. Using the notation and the results of [10], we have that any contraction $X \in I(S(\theta_2), S(\theta_1))$ is of the form

$$X = P_{H(\theta_2)}\hat{A}\,|\,H(\theta_1), \tag{4.12}$$

where $\hat{A}$ is a contraction in $I(V_2, V_1)$ such that

$$\hat{A}\{\theta_1 u \oplus \Delta_1 u : u \in H^2\} \subset \{\theta_2 u \oplus \Delta_2 u : u \in H^2\}. \tag{4.13}$$

Clearly such an operator $\hat{A}$ is a (V_2, V_1)-CIL of

$$A = \hat{A}\,|\,\mathscr{H}_0 = \hat{A}\,|\{\theta_1 u \oplus \Delta_1 u : u \in H^2\} \in I(V_2, V_1\,|\,\mathscr{H}_0). \tag{4.14}$$

We say that the contraction $X \in I(S(\theta_2), S(\theta_1))$ is *associated* with A. Also, by virtue of (4.13), there exists a function $\gamma \in H^\infty$, with $|\gamma(\lambda)| \le 1$ for $|\lambda| \le 1$ such that

$$A(\theta_1 u \oplus \Delta_1 u) = \theta_2\gamma u \oplus \Delta_2\gamma u \qquad (u \in H^2). \tag{4.15}$$

Comparing (4.15) with (4.5), one easily infers that the condition (1.1) is equivalent to the fact that $\theta_2\gamma$ is divisible in H^∞ by θ_1 and that $\|\theta_2\gamma/\theta_1\|_{H^\infty} \le 1$. Thus, we have the following

COROLLARY 4.2. *Let* A *be a contraction in* $I(V_2, V_1\,|\,\mathscr{H}_0)$ *of the form* (4.15) *with* $\gamma \in H^\infty$, $\|\gamma(\lambda)\| \le 1$ *for* $|\lambda| \le 1$, *and such that* $\theta_2\gamma$ *is divisible in* H^∞ *by* θ_1 *and* $\|\theta_2\gamma/\theta_1\|_{H^\infty} \le 1$. *Then there exists a unique contraction* $X \in I(S(\theta_1), S(\theta_2))$ *associated with* A *if and only if the set*

$$\{e^{it}: |\theta_1(e^{it})| \neq 1,\ |\theta_2(e^{it})| \neq 1,\ |\theta_2(e^{it})\gamma(e^{it})| \neq |\theta_1(e^{it})|,\ 0 \le t \le 2\pi\} \tag{4.16}$$

has Lebesgue (linear) measure 0.

Corollaries 4.1 and 4.2 can be also stated in case the function θ_j $(j = 1,2)$ is operator valued. The numerical case considered above was chosen only for the sake of simplicity.

REFERENCES

1. T. Ando, Z. Ceauşescu and C. Foiaş, *On intertwining dilations.* II. Acta Sci. Math. 39 (1977), 3–14.

2. Gr. Arsene, and Z. Ceauşescu, *On intertwining dilations.* IV. Tohoku Math. J., (1978).

3. J. G. W. Carswell and C. F. Schubert, *Lifting of operators that commute with shifts.* Michigan Math. J., 22 (1975), 65–69.

4. Z. Ceauşescu and C. Foiaş, *On intertwining dilations.* III. Revue Roum. Math. Pures et Appl. 22 (1977), 1387–1396.

5. ———, *On intertwining dilations.* V. Acta Sci. Math., 40 (1978), 9–32.

6. R. G. Douglas, *On the operator equation $S^*XT = X$ and related topics.* Acta Sci. Math. (Szeged) 30 (1969), 19–32.

7. R. G. Douglas, P. S. Muhly, and C. Pearcy, *Lifting commuting operators.* Michigan Math. J. 15 (1968), 385–395.

8. B. Sz. -Nagy and C. Foiaş, *Dilatation des commutants d'opérateurs.* C. R. Acad. Sci. Paris *Sér. A-B* 266 (1968), A493–A495.

9. ———, *Harmonic analysis of operators on Hilbert space.* North Holland Pub. Co., Budapest-Amsterdam-London, 1970.

10. ———, *On the structure of intertwining operators;* Acta. Sci. Math. (Szeged) 35 (1973), 225–254.

Department of Mathematics
National Institute for Scientific and Technical Creation
B-dul Păcii 220, Bucharest, Romania