DISCRETE QUADRATURE AND BOUNDS ON t-DESIGNS

Charles F. Dunkl

The theory of Chebyshev systems on discrete sets has a close relationship with the linear programming bounds of coding and design theory. We will sketch the idea of representing measures, as developed by Krein and Rehtman [11], and then consider specific examples related to some classical discrete orthogonal polynomials. One of the main problems is to find the maximal mass at a given point among all representing measures, dually to minimize the values of a positive functional on a certain set of polynomials. The common solution to these problems provides a lower bound for the cardinality of t-designs.

Calculations with orthogonal polynomials lead to exact results in the continuous case, but generally give only bounds in the discrete case. This is because in the discrete case the extremal nonnegative polynomials do not have to be squares as they essentially are in the continuous case. The exception to this is when the appropriate orthogonal polynomial has all of its zeros on the discrete set, making it extremal. It was Lloyd [12] who noticed this phenomenon in connection with perfect codes.

We will discuss the relationship between generalized t-designs (Delsarte [3]) in finite homogeneous spaces, and representing measures with maximal mass at a given point. We are able to give exact solutions for t = 1,2,3 for an arbitrary positive functional. We also state the bounds obtainable from orthogonal polynomials (these are due to Schoenberg and Szegö [15]). For certain parameter values of the classical discrete distributions we can construct the desired representing measures explicitly, which situation corresponds to "equality in the Singleton bound," as Delsarte [2,p.54] has named it.

The paper is organized as follows:

1. Representing measures and extremal polynomials: an outline of the existence and properties of such measures, the characterization of extremal nonnegative polynomials, and the relations between the two concepts.

2. Orthogonal polynomials: the results of the continuous-type theory.

3. Principal representations for n = 1,2,3: specific constructions for arbitrary positive functionals on spaces of polynomials of these degrees.

4. Finite homogeneous spaces: the definition of a t-design and its relationship to representing measures.

Received December 23, 1977. Revision received May 26, 1978.

During the preparation of this paper the author was partially supported by NSF Grant MCS76-07022, and also held a Sesquicentennial Research Associateship from the University of Virginia.

Michigan Math. J. 26 (1979).

5. Examples: positive functionals corresponding to the binomial and the hypergeometric distributions, and their q-analogues; the associated homogeneous spaces; specific bounds and conditions for equality.

1. REPRESENTING MEASURES AND EXTREMAL POLYNOMIALS

Fix a finite subset Ω of real numbers of the form $\{a_0,a_1,\ldots,a_N\}$ with

$$0 = a_0 < a_1 < a_2 \ldots < a_N.$$

For each $n = 0,1,\ldots,N$ let V_n be the space of real polynomials of degree less than or equal to n, considered as functions on Ω. Here is a list of some notations and definitions:

$V_n^+ = \{p \in V_n : p(a) \geq 0, (a \in \Omega)\}$;

D_n : the dual space of V_n;

$D_n^+ = \{\phi \in D_n : \phi(p) \geq 0, (p \in V_n^+)\}$, the dual cone of positive functionals; a functional $\phi \in D_n^+$ is called strictly positive if $p \in V_n^+, p \neq 0$ implies $\phi(p) > 0$, otherwise is called singular.

M_+ : the set of positive measures on Ω (naturally isomorphic to D_N^+);

$\mu_j = \mu\{a_j\}$, for $\mu \in M_+$, $0 \leq j \leq N$;

spt $\mu = \{a \in \Omega : \mu\{a\} > 0\}$, the support of $\mu \in M_+$;

δ_a is the element of M_+ with mass 1 at a, 0 elsewhere, $a \in \Omega$;

deg p is the degree of the polynomial p;

Z(p) is the set of real zeros of $p \in V_N$;

LC(p) is the leading coefficient (highest power) of $p \in V_N$.

1.1 *Definition.* Let $E \subsetneq \Omega$, and define the index of E (denoted ind E) to be $\min\{\deg p : p \neq 0,\ p \in V_N^+, Z(p) \supset E\}$. Further let $p \in V_N^+$ with $Z(p) \supset E$ and $\deg p = \operatorname{ind} E$; then say that E is of upper or lower type according to LC(p) being negative or positive, respectively.

A set cannot be of both types, for if there exist $p_1, p_2 \in V_N^+$ with $LC(p_1) = 1$, $LC(p_2) = -1$, $\deg p_1 = \deg p_2 = \operatorname{ind} E$, $Z(p_1) \supset E$, $Z(p_2) \supset E$, then the polynomial $p_1 + p_2$ is a nonzero element of V_N^+, vanishes on E and has degree less than ind E, a contradiction. This definition and the following ones on representing measures are due to Krein and Rehtman [11]. Another presentation of their work can be found in the book [10] of Karlin and Studden.

The following statements can be easily proved. To calculate the index of a set, consider E as a union of "intervals," that is, subsets of Ω of the form $\{a_i, a_{i+1}, \ldots, a_j\}$ which are separated by at least one point of Ω. A maximal interval of E of cardinality m contributes m to the index if m is even or the interval contains a_0 or a_N, and contributes $m + 1$ otherwise. The set E if of upper type exactly when the maximal interval containing a_N is of odd cardinality. The index is an increasing set function. For $E \subsetneq \Omega$, there exist $p \in V_N^+$ with $\deg p = \operatorname{ind} E$

and $Z(p) = E$, (not necessarily unique; the set $\{a_1, a_2, a_3\}$ is the zero set of

$$(x - a_1)^2(x - a_2)(x - a_3),\ (x - a_1)(x - a_2)^2(x - a_3) \quad \text{and} \quad (x - a_1(x - a_2)(x - a_3)^2,$$

all in V_4^+).

Any $\phi \in D_n^+$ can be extended to an element of D_N^+, that is, there exist $\mu \in M_+$ such that $\phi(p) = \sum_i \mu_i p(a_i)$, $p \in V_n$. Such a measure is said to represent ϕ. We want to find representing measures whose supports have minimal index. Indeed μ is called a canonical representation if ind (spt μ) $\leq n + 2$, and principal if ind (spt μ) $= n + 1$.

1.2 LEMMA. *Let $\phi \in D_n^+$ and let μ represent ϕ, then* ind (spt μ) $\leq n$ *if and only if ϕ is singular, and in this case μ is unique.*

Proof. Indeed ϕ is singular if and only if there exists $p \in V_n^+$, $p \neq 0$ such that $0 = \phi(p) = \sum_i \mu_i p(a_i)$, if and only if $Z(p) \supset$ spt μ. For the uniqueness, let $E = Z(p)$ (where $\phi(p) = 0$, $p \neq 0$, $p \in V_n^+$) and let $a \in E$, then there exist $q \in V_n^+$ with $Z(q) = E \setminus \{a\}$ (by the above remarks) and

$$\phi(q) = q(a)\mu\{a\} \qquad \text{for any } \mu \text{ representing } \phi.$$

The fundamental theorem for strictly positive functionals (Krein and Rehtman [11]) is that there exist exactly two principal representing measures for each such functional, one supported by a set of upper type (called "upper principal"), and the other by a set of lower type (called "lower principal"). We will outline the steps leading to this result (proofs can be found explicitly in, or adapted from, Ch. 7 in Karlin and Studden [10]).

Fix $\phi \in D_n^+$, and for each $a \in \Omega$ define $\rho(a) = \inf\{\phi(p) : p \in V_n^+, p(a) = 1\}$. Note that $0 \leq \rho(a) \leq \phi(1)$, and the inequality $p(a)\rho(a) \leq \phi(p)$ holds for all $p \in V_n^+$.

1.3 THEOREM. $\rho(a) = \sup\{\mu\{a\} : \mu \in M_+$ *and* μ *represents* $\phi\}$.

Proof. If μ represents ϕ, then $\phi(p) = \sum_j \mu_j p(a_j) \geq \mu\{a\} p(a)$ for all $p \in V_n^+$. Conversely the functional $\psi = \phi - \rho(a)\delta_a$ is positive, thus can be represented by $\nu \in M_+$, but then $\nu + \rho(a)\delta_a$ represents ϕ and has mass greater than or equal to $\rho(a)$ at a.

1.4 THEOREM. *Under the above hypotheses, there exists* $p \in V_n^+$ *such that* $p(a) = 1$ *and* $\phi(p) = \rho(a)$, (*and thus* $\rho(a) > 0$ *for all* $a \in \Omega$ *if ϕ is strictly positive*).

Proof. If ϕ is strictly positive, use an appropriate sequence of polynomials and the local compactness of V_n^+. If ϕ is singular, deal directly with the unique representation of ϕ (see 1.2).

Henceforth fix ϕ strictly positive in D_n^+.

1.5 THEOREM. *For each* $a \in \Omega$ *there exists a unique canonical representation* μ *of* ϕ *with* $\mu\{a\} = \rho(a)$.

Proof. Consider the positive functional $\psi = \phi - \rho(a)\delta_a$. By 1.4 there exists $p \in V_n^+$ with $p(a) = 1$ and $\phi(p) = \rho(a)$, but then $\psi(p) = 0$, and so ψ being singular is uniquely represented by $\nu \in M_+$ with $\operatorname{ind}(\operatorname{spt}\nu) \leq n$ and $\operatorname{spt}\nu \subset Z(p)$ (thus $\nu\{a\} = 0$). Then $\mu = \nu + \rho(a)\delta_a$ represents ϕ and is supported by $\operatorname{spt}\nu \cup \{a\}$ which has index less than or equal to $n + 2$ (consider $p(x)(x - a)^2$) and index equal to $n + 1$ if a is an endpoint (consider $xp(x)$ or $(a_N - x)p(x)$).

1.6 COROLLARY. *For any* $p \in V_n^+$ *with* $p(a) = 1$, *equality in* $\phi(p) \geq \rho(a)$ *occurs exactly when* $Z(p) \supset (\operatorname{spt}\mu)\setminus\{a\}$, *where* μ *is the representing measure constructed in* 1.5.

The above procedure gives the construction of the upper principal representation if n is odd and $a = 0$ or a_N, or if n is even and $a = a_N$, and the lower principal representation if n is even and $a = 0$. If n is odd, the way to obtain the lower principal representation is to use the given construction on the set

$$\Omega' = \{a_j, a_{j+1}, \dots, a_N\}$$

with $a = a_{j+1}$, where $j = \max\{i\colon p \in V_n, p(a_r) \geq 0, i \leq r \leq N \text{ implies } \phi(p) \geq 0\}$. Notice that the principal representation having maximal mass at a_N is the upper one (the reason for the name), but the one with maximal mass at 0 is either the upper (n odd) or the lower (n even) one. Because we will be concerned mostly with the representation with maximal mass at 0, we will name it the 0-principal representation.

The corollary showed the relation between a certain extremal problem for polynomials, and canonical representations. Under the assumption that ϕ has been extended to be a positive functional on V_{n+1}, here is another extremal problem related to principal representations.

1.7 THEOREM. *Let* μ *be a principal representation of* ϕ, *then* $\operatorname{spt}\mu \subset Z(p^*)$ *where* p^* *solves: minimize* $\phi(p)$ *subject to* $p \in V_{n+1}^+$, $LC(p) = \varepsilon$, *where* $\varepsilon = 1$ *if* μ *is lower,* $\varepsilon = -1$ *if* μ *is upper, and* $\operatorname{spt}\mu$ *is the minimal zero set of such polynomials* p^*.

Proof. Let $p(x) = \varepsilon x^{n+1} + p_1(x)$ with $p_1 \in V_n$, then

$$\begin{aligned}\phi(p) &= \varepsilon\phi(x^{n+1}) + \phi(p_1) = \varepsilon\phi(x^{n+1}) + \sum_j \mu_j p_1(a_j)\\ &= \varepsilon\phi(x^{n+1}) + \sum_j \mu_j p(a_j) - \sum_j \varepsilon\mu_j a_j^{n+1}\\ &\geq \varepsilon\left(\phi(x^{n+1}) - \sum_j \mu_j a_j^{n+1}\right)\end{aligned}$$

with equality exactly when $Z(p) \supset \operatorname{spt}\mu$. Since μ is principal there exists $p^* \in V_{n+1}^+$ with $\operatorname{spt}\mu = Z(p^*)$.

We now give a description of extremal positive polynomials. Since V_n^+ is a cone we consider the extreme rays, that is, polynomials $p \in V_n^+$ such that $p = p_1 + p_2$ with $p_1, p_2 \in V_n^+$ implies $p_1 = cp$ for some $c \geq 0$. For use in the description we let $P(a_{i_1}, \ldots, a_{i_r}; a_{j_1}, \ldots, a_{j_s}; x)$ (or just P) denote the polynomial

$$\prod_{k=1}^{r} (x - a_{i_k})^2 \prod_{m=1}^{s} (x - a_{j_m})(x - a_{j_m+1}),$$

where $a_{i_1} \leq a_{i_2} \leq \ldots \leq a_{i_r}$ and $a_{j_1} < a_{j_2} \ldots < a_{j_s} < a_N$. Note that $P \in V_{2(r+s)}^+$.

1.8 THEOREM. *The extreme rays in* V_n^+ *are exactly the following sets of polynomials:*

a) *for* n *odd,*

$$c_1\, x\, P(a_{i_1}, \ldots, a_{i_r}; a_{j_1}, \ldots a_{j_s}; x) \quad c_1 > 0, r + s = (n-1)/2$$

$$c_2(a_N - x)P(x), \qquad c_2 > 0, r + s = (n-1)/2;$$

b) *for* n *even,*

$$c_1 P(x), \qquad c_1 > 0, r + s = n/2$$

$$c_2\, x\, (a_N - x)P(x), \qquad c_2 > 0, r + s = (n/2) - 1.$$

Proof. To show that these are extreme, let p have one of these forms and suppose $p = p_1 + p_2$ with $p_1, p_2 \in V_n^+$. Let f be one of the following factors

$$(x - a_{i_k})^2, (x - a_{j_m})(x - a_{j_m+1}), x, (a_N - x),$$

which divides p, then p/f is a positive polynomial and $p/f = p_1/f + p_2/f$. Since $f \geq 0$ on Ω we see that p_1/f is finite on Ω and so f divides p_1. Now p/f has lower degree than p, but is of the same form, so that by induction p_1 is a scalar multiple of p.

For each $r = 0,1,2,\ldots,n$ let E_r be the set of polynomials which can be expressed as positive linear combinations of polynomials of the above type and of degree r. We want to show $E_n = V_n^+$. By the distributive law the set of products $E_r E_s \subset E_{r+s}$ (for $r + s \leq n$), and $1 \in E_r$ for all r because $1 = (1/a_N)(x + (a_N - x)) \in E_1$. Observe $x - c = x + (-c) \in E_1$ for $c \leq 0$, and $d - x = (d - a_N) + (a_N - x) \in E_1$ for $d \geq a_N$, thus $E_1 = V_1^+$. This also shows $(x - c)^2 + d^2 \in E_2$ for $c \leq 0$ or $c \geq a_N$ and real d. The other main type of element of V_2^+ is $(x - c)(x - d)$ where

$$a_j \leq c \leq d \leq a_{j+1}, \qquad \text{for some } j = 0, 1, \ldots, N+1,$$

but

$$(x - c)(x - d) = \alpha(x - a_j)^2 + \beta(x - a_j)(x - a_{j+1}) + \gamma(x - a_{j+1})^2,$$

where

$$\alpha = (a_{j+1} - c)(a_{j+1} - d)/(a_{j+1} - a_j)^2,$$
$$\beta = [(c - a_j)(a_{j+1} - d) + (d - a_j)(a_{j+1} - c)]/(a_{j+1} - a_j)^2,$$
$$\gamma = (c - a_j)(d - a_j)/(a_{j+1} - a_j)^2,$$

and so $(x - c)(x - d) \in E_2$. We will say a polynomial satisfies (E) if $p \in E_r$ where $r = \deg p$.

Let $p \in V_n^+$, then by the fundamental theorem of algebra $p = p_1 p_2 p_3$ where

$$p_1(x) = \prod_{y_i<0} (x - y_i) \prod_{y_i>a_N} (y_i - x), \quad p_2(x) = \prod_i ((x - b_i)^2 + c_i^2),$$

$$p_3(x) = c \prod_{0\le y_i \le a_N} (x - y_i),$$

(the y_i's are the real zeros of p) and each factor is in V_n^+. By the above remarks p_1 and p_2 satisfy (E), so we need to consider p_3. Divide out all possible factors of the form $(x - y_i)(x - y_k)$ (an element of E_2) where

$$a_j < y_i \le y_k < a_{j+1} \qquad \text{for some } j = 0, 1, \dots, N-1,$$

leaving a polynomial $p_4 \in V_n^+$ having at most one zero (a simple one) in each open interval (a_j, a_{j+1}) and all other zeros on Ω. We will show p_4 satisfies (E) by inductively showing for each $j = 0, 1, \dots, N$ that $p_4 = f_j g_j$ with $f_j, g_j \in V_n^+$, g_j satisfies (E) and $f_j(x) > 0$ for $x < a_j$. At $j = N$ the result is proven since f_N satisfies (E). To begin the induction suppose $p_4(x) = x^r q(x)$, $q(0) \neq 0$, then set

$$g_0(x) = x^r,\ f_r(x) = q(x) \qquad \text{for } r \text{ even},$$

or $g_0(x) = x^{r-1}$, $f_0(x) = xq(x)$ for r odd. Assume now that f_j, g_j have been constructed and suppose $f_j(x) = (x - a_j)^r q(x)$, $q(a_j) \neq 0$.

Case 1. p_4 (thus q) has no zeros in (a_j, a_{j+1}), then set

$$g_{j+1}(x) = g_j(x)(x - a_j)^r,\ f_{j+1}(x) = q(x) \qquad \text{if } r \text{ is even},$$

or $g_{j+1}(x) = g_j(x)(x - a_j)^{r-1}$, $f_{j+1}(x) = (x - a_j)q(x)$ if r is odd.

Case 2. p_r (thus q) has one zero (a sign change) at $c \in (a_j, a_{j+1})$. Since $q(a_{j+1}) \ge 0$ either (i) $q(a_j) < 0$ and r is odd, or (ii) $q(a_j) > 0$, r is even and $q(a_{j+1}) = 0$. In case (i) put $g_{j+1}(x) = (x - a_j)^r (x - c) g_j(x)$, in case (ii) put

$$g_{j+1}(x) = (x - a_j)^r (x - c)(x - a_{j+1}) g_j(x),$$

and define f_{j+1} accordingly.

This completes the induction, and the proof.

The theorem shows that the extremal problems in Theorems 1.4 and 1.7 have solutions of the above specified form. It is obvious for 1.4; for 1.7 suppose $p \in V_{n+1}^+$

minimizes $\phi(p')$ among all p' with $LC(p') = 1$ and suppose $p = p_1 + p_2$ with $LC(p_1) > 0$, $LC(p_2) < 0$, $p_1, p_2 \in V_{n+1}^+$, but then $LC(p_1) = 1 - LC(p_2) > 1$ and so $\phi(p_1/LC(p_1)) \leq \phi(p_1) \leq \phi(p)$.

For the rest of the paper we will be concerned with the 0-principal representation (maximal mass at 0). For any subset E of Ω of cardinality $n + 1$, there exists a unique measure ν (not necessarily positive) supported by E that represents ϕ, with $\nu\{a\} = \phi(p)$ where p is the unique polynomial of degree n which vanishes on $E \setminus \{a\}$ and has $p(a) = 1$, for each $a \in E$. If in addition ind $E = n + 1$ and $\nu\{a\} \geq 0$ for all $a \in E$, then ν must be one of the two principal representations. An important special case is described in the following, whose proof is obvious.

1.9 PROPOSITION. *Let* $s_n(x) = \Pi_{j=0}^{n-1}(1 - x/a_{N-j})$ *and for* $i = 0, 1, \ldots, n-1$, *let* $s_{ni}(x) = (x/a_{Ni}) \cdot \Pi_{j=0, j\neq i}^{n-1}((x - a_{N-j})/(a_{N-i} - a_{N-j}))$, *then* $\rho(0) \leq \phi(s_n)$ *with equality exactly when* $\phi(s_{ni}) \geq 0$ *for* $i = 0, 1, \ldots, n-1$. *In this case the 0-principal representation* μ *is supported by* $\{0, a_{N-n+1}, a_{N-n+2}, \ldots, a_N\}$ *with* $\mu_{N-i} = \phi(s_{ni})$, $\mu_0 = \phi(s_n)$.

Following Delsarte [2, p.54], we call $\rho(0) \leq \phi(s_n)$ the Singleton bound.

2. ORTHOGONAL POLYNOMIALS

The results of this section are mostly due to Schoenberg and Szegö [15] (see also [10,p.115]) who developed these procedures to calculate $\rho(a)$ for an arbitrary point in an interval (the continuous problem). Applied to the discrete case these will give upper bounds for $\rho(a)$.

Throughout, ϕ is a fixed strictly positive functional and n is fixed less than N. Recall that $\rho(0) = \inf\{\phi(p)\colon p(0) = 1, p \in V_n^+\}$ and also $\mu\{0\} = \rho(0)$ where μ is the 0-principal representation of ϕ. Among the appropriate polynomials in V_n^+ are the squares

$$p^2(x), \deg p \leq n/2 \quad \text{and}$$

$$p^2(x)(1 - x/a_N), \deg p \leq n/2 - 1, p(0) = 1.$$

Minimizing $\phi(p^2)$ and $\phi(p^2(x)(1 - x/a_N))$ leads to orthogonal polynomials. To state the results we introduce four families of discrete orthogonal polynomials on Ω.

2.1 *Definition.* The systems of polynomials which are orthonormal with respect to the strictly positive functionals

$$p \mapsto \phi(w_i p), \quad w_1(x) = 1, w_2(x) = x, w_3(x) = (1 - x/a_N), w_4(x) = x(1 - x/a_N),$$

and normalized so that the values at zero are positive, will be denoted by

$$\{p_{1,k}\}_{k=0}^{N}, \quad \{p_{2,k}\}_{k=0}^{N-1}, \quad \{p_{3,k}\}_{k=0}^{N-1}, \quad \{p_{4,k}\}_{k=0}^{N-2}$$

respectively.

2.2a THEOREM. *Let* $n = 2m$. *The polynomial* p *of degree less than or equal to* m *which minimizes* $\phi(p^2)$ *and has* $p(0) = 1$ *is* $p_{2,m}/p_{2,m}(0)$. *The corresponding value of* ϕ *is*

$$\left(\sum_{j=0}^{m} p_{1,j}(0)^2\right)^{-1} \geq \rho(0),$$

and equality occurs exactly when $p_{2,m}$ *has all* m *zeros on* Ω. *In this case the* 0*-principal (lower) representation is supported by* $Z(p_{2,m}) \cup \{0\}$.

2.2b THEOREM. *Let* $n = 2m + 1$. *The polynomial* p *of degree less than or equal to* m *which minimizes* $\phi((1 - x/a_N)p(x)^2)$ *and has* $p(0) = 1$ *is* $p_{4,m}/p_{4,m}(0)$. *The corresponding value of* ϕ *is* $\left(\sum_{j=0}^{m} p_{3,j}(0)^2\right)^{-1} \geq \rho(0)$, *and equality occurs exactly when* $p_{4,m}$ *has all* m *zeros on* Ω. *In this case the* 0*-principal (upper) representation is supported by* $\{0, a_N\} \cup Z(p_{4,m})$.

Proof. We discuss only 2.2a. The first two statements depend on simple orthogonality ideas, and the Christoffel-Darboux formula. From 1.6 we see that $\phi((p_{2,m}/p_{2,m}(0))^2) = \rho(0)$ exactly when $p_{2,m}$ vanishes on spt $\mu\setminus\{0\}$ where μ is the 0-principal representation of ϕ, but ind (spt μ) $= 2m + 1$ so spt μ must contain at least $m + 1$ points. Conversely if $Z(p_{2,m}) \subset \Omega$ then, as in ordinary Gaussian quadrature, there exists a measure $\nu \in M_+$ supported by $S = \{0\} \cup Z(p_{2,m})$ which represents ϕ. Now ind $S \leq 2m + 1$, (hence $= 2m + 1$) and S is of lower type, being the zero set of $x\, p_{2,m}(x)^2$, hence ν is the lower principal representation of ϕ.

We point out that $p_{2,m}(x)$ is a multiple of $(p_{1,m+1}(x)\, p_{1,m}(0) - p_{1,m}(x)\, p_{1,m+1}(0))/x$ and $p_{4,m}$ is similarly expressed in terms of $p_{3,m+1}$ and $p_{3,m}$.

Generally, the value of $\rho(0)$ is not obtained by the above methods, and indeed, knowledge of the zeros (that is, the intervals $[a_j, a_{j+1})$ containing sign changes) of $p_{2,m}$ or $p_{4,m}$ is not very helpful in finding the supports of 0-principal representations. The part of the theorem dealing with the zero sets of polynomials and equality in the bounds is a generalized Lloyd's theorem [12].

3. PRINCIPAL REPRESENTATIONS FOR $n = 1,2,3$

Thoughout we fix a strictly positive functional ϕ defined on V_N. Define an associated functional ψ by $\psi(p) = \phi(p(x)(1 - x/a_N))$, $p \in V_{N-1}$, and let

$$\phi_j = \phi(x^j),\ \psi_j = \psi(x^j) = \phi_j - \phi_{j+1}/a_N, \qquad j = 0, 1, 2, \ldots.$$

3.1 PROPOSITION. *The* 0*-principal representation* μ *for* $n = 1$ *is supported on* $\{0, a_N\}$ *and has* $\mu_0 = \psi_1 = \phi_0 - \phi_1/a_N, \mu_n = \phi_1/a_N$.

Proof. The extremal polynomial p in V_1^+ which has $p(0) = 1$ is $1 - x/a_N$.

3.2 THEOREM. *The* 0*-principal representation* μ *for* $n = 2$ *is supported on* $\{0, a_j, a_{j+1}\}$ *where* $a_j \leq \phi_2/\phi_1 = a_j + r < a_{j+1}$ (*the definition of* r) *and is given by*

$$\rho(0) = \mu_0 = \phi_0 - (\phi_1^2/\phi_2)(1 + r(a_{j+1} - a_j - r)/(a_j a_{j+1})),$$
$$\mu_j = (a_{j+1} - a_j - r)\,\phi_1/((a_{j+1} - a_j)\,a_j),$$
$$\mu_{j+1} = r\phi_1/((a_{j+1} - a_j)\,a_{j+1}).$$

Proof. By Theorem 1.7 we need to minimize $\phi(p)$ where $p \in V_3^+$ and $LC(p) = 1$. By the extreme point theorem 1.8, p must have the form

$$x(x - a_j)(x - a_{j+1}) \text{ or } x(x - a_j)^2, \quad \text{some } j.$$

If $p = x(x - a_j)^2$ is the minimum then

$$0 \le \phi\,(x(x - a_j)(x - a_{j+1})) - \phi(x(x - a_j)^2) = (a_j - a_{j+1})\,\phi(x(x - a_j)),$$

hence $\phi_2 - a_j\phi_1 \le 0$; by similarly considering $x(x - a_j)(x - a_{j-1})$ one obtains $\phi_2 - a_j\phi_1 \ge 0$ and so $a_j = \phi_2/\phi_1$.

If $p = x(x - a_j)(x - a_{j+1})$ is the minimum then

$$\begin{aligned} 0 \le \phi\,(x(x - a_{j+1})(a - a_{j+2})) - \phi(p) &= (a_j - a_{j+2})\,\phi(x(x - a_{j+1})) \\ &= (a_j - a_{j+2})(\phi_2 - a_{j+1}\phi_1), \end{aligned}$$

and similarly $0 \le (a_{j+1} - a_{j-1})(\phi_2 - a_j\phi_1)$. Thus $a_j \le \phi_2/\phi_1 \le a_{j+1}$. Trivially $0 \le \phi_2 \le \phi_1 a_N$ so there exists a unique j such that $a_j \le \phi_2/\phi_1 < a_{j+1}$. Observe that in the exceptional case $\phi_2 = a_j\phi_1$ there are at least two minimal polynomials. To obtain the masses of μ at ϕ, a_j, a_{j+1} evaluate ϕ at

$$(1 - x/a_j)(1 - x/a_{j+1}),\ x(a_{j+1} - x)/(a_j(a_{j+1} - a_j)),\ x(x - a_j)/(a_{j+1}(a_{j+1} - a_j))$$

respectively, and use the relation $\phi_2 = \phi_1(a_j + r)$.

3.3 THEOREM. *The 0-principal representation μ for $n = 3$ is supported on $\{0, a_j, a_{j+1}, a_N\}$ where $a_j \le \psi_2/\psi_1 = (a_N\phi_2 - \phi_3)/(a_N\phi_1 - \phi_2) = a_j + r < a_{j+1}$ (the definition of r) and is given by*

$$\mu_0 = \rho(0) = \psi_0 - (\psi_1^2/\psi_2)(1 + r(a_{j+1} - a_j - r)/(a_j a_{j+1})),$$
$$\mu_j = a_N(a_{j+1} - a_j - r)\,\psi_1/(a_j(a_{j+1} - a_j)(a_N - a_j)),$$
$$\mu_{j+1} = a_N r\,\psi_1/(a_{j+1}(a_{j+1} - a_j)(a_N - a_{j+1})),$$
$$\mu_N = \phi_0 - (\mu_0 + \mu_j + \mu_{j+1}).$$

Proof. The method is similar to that of 3.2. Here the extremal polynomial has the form $x(a_N - x)(x - a_j)(x - a_{j+1})$ or $x(a_N - x)(x - a_j)^2$. Replace ϕ by ψ in the above proof to obtain the stated results.

In each of these cases it does happen that the support of μ brackets the zero set of the appropriate orthogonal polynomial (see section 2). The zero sets are $\{0, a_N\}$, $\{0, \phi_2/\phi_1\}$, $\{0, \psi_2/\psi_1, a_N\}$ for $n = 1, 2, 3$ respectively. However this trick of bracketing the zeros does not in general produce the supports of principal representations.

4. FINITE HOMOGENEOUS SPACES

Suppose that G is a finite group with a subgroup H such that the Hecke algebra, namely the algebra of two-sided cosets HgH, is commutative. Suppose further that the collection of these cosets can be placed in one-to-one correspondence with a finite subset Ω of the nonnegative real numbers, in such a way that the coset H corresponds to $0 \in \Omega$ and that the spherical functions correspond to a family of orthogonal polynomials $\{P_n\}_{n=0}^{N}$ of one variable. Thus there is a family of representatives, g_0 equals the identity in G, $g_1, g_2, \ldots, g_N$ corresponding to

$$0 = a_0, a_1, \ldots, a_N \in \Omega$$

respectively. The spherical functions are given by $g \to P_n(a_i)$ where $g \in Hg_iH$. These assumptions imply that $g_i^{-1} \in Hg_iH$, each i.

The measure m on Ω furnishing orthogonality is the one induced by G, namely $m_i = m\{a_i\} = |Hg_iH| / |G|$ ($|E|$ is the cardinality of the set E). Denote the homogeneous space $G/H = \{Hg : g \in G\}$ by X, and let $L^2(X)$ be the space of complex functions on X, furnished with the inner product $\langle f_1, f_2\rangle = (1/|X|) \sum_{x\in X} f_1(x)\, \overline{f_2(x)}$. Then G is represented on $L^2(X)$ by right translation, and there is a decomposition $L^2(X) = \sum_{n=0}^{N} \oplus H_n$, where each H_n is an irreducible G-module containing exactly one H-invariant vector, namely the spherical function given by P_n. In this context, we could call H_n the space of spherical harmonics on X of degree n. Now X is an association scheme, where Hg and Hg′ are "ith associates" if

$$g' g^{-1} \in Hg_iH, \qquad 0 \le i \le N.$$

(For a reference on association schemes see Delsarte [2] or Sloane [17]). It may also happen that X is a distance-transitive graph (if $(Hg_iH)(Hg_1H)$ is a linear combination of $Hg_{i-1}H$, Hg_iH, $Hg_{i+1}H$ in the group algebra of G); see Biggs [1]. Conditions sufficient for X to be a distance-transitive graph have been given by D. G. Higman [19]. This property is called P-polynomial by Delsarte [2, p.56].

4.1. *Definition.* For $t = 1, 2, \ldots, N-1$, a (generalized) t-design is a probability measure μ on X which annihilates $\sum_{n=1}^{t} \oplus H_n$.

This definition is due to Delsarte [3] in the context of association schemes. If the scheme is the Johnson scheme (see section 5.2.6) and μ has all its nonzero masses equal then the support of μ is a classical t-design. In the context of non-discrete compact homogeneous spaces (like the sphere), μ is a cubature rule exact for spherical harmonics of degree less than or equal to t. Note that the definition is equivalent to

$$\sum_{x\in X} f(x)\, \mu\{x\} = (1/|X|) \sum_{x\in X} f(x) \qquad \text{for all } f \in \sum_{n=0}^{t} \oplus H_n.$$

The interesting designs are those which have small supports or equal nonzero masses.

Any measure μ on X can be considered as a measure on G, constant on the cosets Hg ($g \in G$), and thus μ has an adjoint μ^* given by $\mu^*\{g\} = \overline{\mu\{g^{-1}\}}$, constant on the cosets gH ($g \in G$). The convolution $\mu * \mu^*$ is bi-invariant and corresponds to a measure ω on Ω, defined by

$$\omega\{a_i\} = \sum \{\mu\{x\}\,\overline{\mu\{y\}} : x, y \in X, xy^{-1} = Hg_i H\} \qquad 0 \leq i \leq N,$$

(we interpret $y \in X$ as a coset $Hg \subset G$ and so $y^{-1} = g^{-1}H$). Note that

$$\omega\{0\} = \sum_{x \in X} |\mu\{x\}|^2.$$

4.2 THEOREM. *Let μ be a t-design, and let $\nu \in M_+$ correspond to $\mu * \mu^*$ as above. Then ν represents the strictly positive functional* $\phi(p) = \sum_{i=0}^{N} m_i p(a_i)$, $p \in V_N$, *for* V_t. *Further* $|\text{spt}\,\mu| \geq 1/\rho(0)$ *where* $\rho(0)$ *is as used in* 1.3 *with* $n = t$.

Proof. The orthogonality relations for spherical functions (see Dunkl and Ramirez [9,Ch.9]) imply the relations $\sum_{i=0}^{N} m_i P_k(a_i)P_n(a_i) = 0$, $k \neq n$; in particular $\sum_{i=0}^{N} m_i P_k(a_i) = 0$, $k \geq 1$. Any $p \in V_t$ can be written as $\sum_{j=0}^{t} c_j P_j$ and so $\phi(p) = c_0$. On the other hand μ, and hence $\mu * \mu^*$, annihilate $\sum_{k=1}^{t} \oplus H_k$, thus

$$\sum_{i=0}^{N} P_k(a_i)\,\omega\{a_i\} = 0, \qquad 1 \leq k \leq t.$$

Also $\sum_{i=0}^{N} \omega\{a_i\} = \left| \sum_{x \in X} \mu\{x\} \right|^2 = 1$, and so ω represents ϕ on V_t, showing $\rho(0) \geq \nu\{0\}$ (by 1.3). By the Cauchy-Schwarz inequality

$$1 = \sum_{x \in S} \mu\{x\} \leq \left(\sum_{x \in S} |\mu\{x\}|^2 \right)^{1/2} (|S|)^{1/2}$$

where $S = \text{spt}\,\mu$, and so $|\text{spt}\,\mu| \geq 1/\nu\{0\} \geq 1/\rho(0)$.

The bound on $|\text{spt}\,\mu|$ was obtained by Delsarte for association schemes [3], and by Delsarte, Goethals and Seidel [5] for the real spheres.

4.3 THEOREM. *A necessary condition that there exist a t-design μ with*

$$|\text{spt}\,\mu| = 1/\rho(0)$$

is that $\lambda\{a_j\}/\lambda\{0\}^2$ *be an integer for each* j, *where* λ *is the* 0-*principal representation for* ϕ.

Proof. Suppose μ is a t-design with $|\operatorname{spt}\mu| = 1/\rho(0)$. Let $S = \operatorname{spt}\mu$, then by 4.2, $\phi(0) = 1/|S| \le \sum_{x\in S} \mu\{x\}^2 \le \rho(0)$, implying that $\mu\{x\} = 1/|S|$, $x \in S$. Further $\lambda = \mu * \mu^*$ is the 0-principal representation (by the uniqueness result 1.5) and

$$|S|^2\lambda\{a_j\} = |S|^2 \sum \{\mu(x)\mu(y)\colon xy^{-1} \in Hg_jH\}$$

which is an integer.

5. EXAMPLES

For each of four types of weight functions, the binomial, the hypergeometric, and their respective q-analogs, we will discuss the specific bounds from sections 2 and 3, the associated homogeneous spaces and orthogonal polynomials, and conditions on the parameters which imply equality in the Singleton bound (see 1.9). We emphasiz8 that "equality in the Singleton bound" only provides the value of $\rho(0)$, and does not imply the existence of a t-design with cardinality $1/\rho(0)$.

In each case we describe Ω and the weight function m. The corresponding functional ϕ is given by $\phi(p) = \sum_{i=0}^{N} m_i p(a_i)$, $p \in V_N$.

5.1 *The binomial distribution.* Fix parameters $N = 1, 2, 3 \ldots, k > 0$, and let $\Omega = \{0, 1, 2 \ldots, N\}$, $m_i = (k+1)^{-N}\binom{N}{i}k^i$.

5.1.1. Upper principal, $n = 1$: $\rho(0) = 1/(k+1)$.

5.1.2. Lower principal, $n = 2$: $\operatorname{spt}\mu = \{0, j, j+1\}$, where

$$j = [(kN+1)/(k+1)], \quad r = (kN+1)/(k+1) - j,$$

($[x]$ is the largest integer less than or equal to x), and

$$\rho(0) = \mu_0 = \frac{1}{kN+1} - \frac{kNr(1-r)}{(kN+1)j(j+1)},$$

$$\mu_j = (1-r)kN/(j(k+1)), \quad \mu_{j+1} = rkN/((j+1)(k+1)).$$

5.1.3. Upper principal, $n = 3$: $\operatorname{spt}\mu = \{0, j, j+1, N\}$, where $j = [\alpha]$, $r = \alpha - [\alpha]$, $\alpha = (kN+1-k)/(k+1)$, and

$$\rho(0) = \frac{1}{(k+1)(kN+1-k)} - \frac{k(N-1)r(1-r)}{(k+1)(kN+1-k)j(j+1)},$$

$$\mu_j = \frac{(1-r)kN(N-1)}{(k+1)^2 j(N-j)}, \quad \mu_{j+1} = \frac{rkN(N-1)}{(k+1)^2(j+1)(N-j-1)},$$

$$\mu_N = 1 - (\rho(0) + \mu_j + \mu_{j+1}).$$

5.1.4. Associated orthogonal polynomials: The families orthogonal with respect to m_x, xm_x, $(1 - x/N)m_x$, $x(1 - x/N)m_x$ are

$$x \leftrightarrow K_j\left(x; \frac{k}{k+1}, N\right), \quad K_j\left(x-1; \frac{k}{k+1}, N-1\right),$$
$$K_j\left(x; \frac{k}{k+1}, N-1\right), \quad K_j\left(x-1; \frac{k}{k+1}, N-2\right)$$

respectively, (not normalized), where K_j is the Krawtchouk polynomial defined by

$$\begin{aligned} K_j(x; p, N) &= \binom{N}{j}^{-1} \sum_{i=0}^{j} \left(\frac{p-1}{p}\right)^i \binom{N-x}{j-i}\binom{x}{i} \\ &= {}_2F_1\left(\begin{matrix} -j, -x \\ -N \end{matrix}; \frac{1}{p}\right), \qquad p > 0, j = 0, 1, \ldots, N. \end{aligned}$$

The corresponding bounds for $\rho(0)$ are:

(a) $n = 2m$, $\rho(0) \le \left(\sum_{j=0}^{m} \binom{N}{j} k^j\right)^{-1}$, the sphere-packing bound; equality if and only if $x \mapsto K_m\left(x-1; \frac{k}{k+1}, N-1\right)$ has m zeros in $\{1, 2, \ldots, N\}$; (Lloyd [12] found this necessary condition for the existence of perfect codes, in the case $k = 1$).

(b) $n = 2m + 1$, $\rho(0) \le \left((k+1)\sum_{j=0}^{m} \binom{N-1}{j} k^j\right)^{-1}$, equality if and only if

$$x \mapsto K_m\left(x-1; \frac{k}{k+1}, N-2\right)$$

has m zeros in $\{1, 2, \ldots, N-1\}$.

5.1.5. The Singleton bound: The bound is $\rho(0) \le 1/(k+1)^n$, with equality if and only if $k \ge N - n$. Here

$$s_n(x) = (x-N)_n/(-N)_n \quad \text{and}$$

$$s_{ni}(x) = (-1)^i x(x-N)_i(x-N+i+1)_{n-i-1}/((N-i)i!(n-i-1)!);$$

where $(a)_0 = 1$, $(a)_{m+1} = (a)_m (a + m)$. Direct calculation shows that

$$\phi(s_{ni}) = \binom{N}{i} \frac{k}{(k+1)^{i+1}} S(N - i - 1, n - i - 1; k), \quad \text{where}$$

$$S(M, r;k) = \sum_{y=0}^{M} \binom{M}{y} k^{M-y} (k+1)^{-M} (1-y)_r / r! \qquad (M = 1, 2, \ldots, r = 0, 1, \ldots, M).$$

To calculate $S(M,r;k)$ observe $S(M,0;k) = 1$ and

$$S(M,r;k) - S(M, r-1;k) = \sum_{y} \binom{M}{y} k^{M-y} (k+1)^{-M} \frac{(-y)^r}{r!} = \frac{(-M)_r}{(k+1)^r r!}$$

and hence $S(M,r;k) = \sum_{j=0}^{r} \frac{(-M)_j}{j!} \left(\frac{1}{k+1}\right)^j$, a truncated ${}_1F_0$. Set $r = n - 1 - i$ then $\phi(s_{n,n-r-1})$ is a positive multiple of

$$S(N - n + r, r; k) = \lim_{\varepsilon \to 0} {}_2F_1 \left(\begin{matrix} -r, -(N-n) - r \\ \varepsilon - r \end{matrix} ; \frac{1}{k+1} \right).$$

There is a three-term recurrence relation (for $r - 1, r, r + 1$) that follows from the differential equation for the ${}_2F_1$ function (see Bailey [18, p.1]), namely

$$\begin{aligned} (r+1) S(N - n + r + 1, r + 1; k) \\ &= (r + 1 - (N - n + 2r + 1)/(k+1)) S(N - n + r, r; k) \\ &\quad + (N - n + r)(k/(k+1)^2) S(N - n + r - 1, r - 1; k). \end{aligned}$$

But $S(N - n + 1, 1; k) \geq 0$ if and only if $k \geq N - n$, and this condition implies all the recurrence coefficients are positive. Thus

$$S(N - n + r, r; k) \geq 0 \qquad \text{for } r = 1, 2, \ldots$$

if and only if $k \geq N - n (\geq 1)$. (The author thanks the referee for the suggestion to use a recurrence relation.) The condition for equality was stated by Delsarte [2, p.55].

5.1.6. The homogeneous space: For $N, k = 1, 2, 3 \ldots$ let X be the set of ordered N-tuples $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ with $\alpha_i = 0, 1, 2, \ldots k$ (for $1 \leq i \leq N$) and let G be the wreath product of S_{k+1} with S_N (the semidirect product of $(S_{k+1})^N$ with S_N), where S_n is the symmetric group on n letters. Thus a given S_{k+1} factor acts on one coordinate, and S_N permutes the coordinates. Pick the base-point $\omega = (0, 0, \ldots, 0)$, then the isotropy subgroup H is isomorphic to the wreath product of S_k with S_N. The G-invariant distance on X is given by $d(\alpha,\beta) = |\{i: \alpha_i \neq \beta_i\}|$, and two points are in the same H-orbit (corresponds to a two-sided coset in G) if and only if $d(\alpha,\omega) = d(\beta,\omega)$. With this metric, X is called the Hamming scheme H(N, k + 1), and X satisfies the hypotheses of section 4, with $\Omega = \{0, 1, \ldots, N\}$,

$$m_i = \binom{N}{i} k^i (k+1)^{-N},$$

and spherical functions $g \mapsto K_j\left(d(\omega,\omega g); \frac{k}{k+1}, N\right)$, $(g \in G)$ (see Dunkl [6]). A t-design with equal nonzero masses is the same as an "array of strength t" (Rao [13]). The bounds from 5.14, $|\operatorname{spt}\mu| \geq \sum_{j=0}^{m} \binom{N}{j} k^j$ for a 2m-design μ, and

$$|\operatorname{spt}\mu| \geq (k+1) \sum_{j=0}^{m} \binom{N-1}{j} k^j = \sum_{j=0}^{m} \binom{N}{j} k^j + \binom{N-1}{m} k^{m+1}$$

for a (2m + 1)-design are due to Rao [13]. The bound from 5.1.5, $|\operatorname{spt}\mu| \geq (k+1)^t$ for a t-design μ is essentially due to Singleton [16].

5.2. *The hypergeometric distribution.* Fix parameters a,b,N with N = 1,2,3, ... and $a \geq b \geq N$, and let $\Omega = \{0, 1, 2, \ldots, N\}$,

$$m_i = \binom{a}{i}\binom{b}{N-i} \Big/ \binom{a+b}{N} = \binom{N}{i} (-a)_i (-b)_{N-i} \Big/ (-a-b)_N.$$

5.2.1. Upper principal, n = 1: $\rho(0) = b/(a+b)$.

5.2.2. Lower principal, n = 2: $\operatorname{spt}\mu = \{0, j, j+1\}$, where $j = [\alpha]$, $r = \alpha - [\alpha]$, $\alpha = (N(a-1)+b)/(a+b-1)$, and

$$\rho(0) = \frac{b(a+b-N)}{(a+b)(N(a-1)+b)} - \frac{aNr(1-r)(a+b-1)}{(a+b)(N(a-1)+b)j(j+1)},$$

$$\mu_j = (1-r)Na/(j(a+b)), \quad \mu_{j+1} = rNa/((j+1)(a+b)).$$

5.2.3. Upper principal, n = 3: $\operatorname{spt}\mu = \{0, j, j+1, N\}$, where $j = [\alpha]$, $r = \alpha - [\alpha]$, $\alpha = (a(N-1) - N + b)/(a+b-2)$, and

$$\rho(0) = \frac{b}{a+b} - \frac{ab(N-1)(a+b-2)}{(a+b)(a+b-1)(a(N-1)-N+b)} \left(1 + \frac{r(1-r)}{j(j+1)}\right),$$

$$\mu_j = \frac{(1-r)abN(N-1)}{(a+b)(a+b-1)j(N-j)}, \quad \mu_{j+1} = \frac{rabN(N-1)}{(a+b)(a+b-1)(j+1)(N-j-1)},$$

$$\mu_N = 1 - (\rho(0) + \mu_j + \mu_{j+1}).$$

5.2.4. Associated orthogonal polynomials: The families orthogonal with respect to m_x, xm_x, $(1 - x/N)m_x$, $x(1 - x/N)m_x$ are

$$x \mapsto Q_j(x; -a-1, -b-1, N), \quad Q_j(x-1; -a, -b-1, N-1),$$
$$Q_j(x; -a-1, -b, N-1), \quad Q_j(x-1, -a, -b, N-2)$$

respectively (not normalized), where Q_j is the Hahn polynomial defined by

$$Q_j(x; \alpha, \beta, N) = {}_3F_2\left(\begin{matrix}-j, N+\alpha+\beta+1, -x; 1\\ -N, \alpha+1\end{matrix}\right).$$

The corresponding bounds for $\rho(0)$ are:

(a) $n = 2m$, $\rho(0) \le \left(\sum_{j=0}^{m} \frac{(-a)_j(-N)_j}{(-b)_j(N-a-b)_j}\binom{a+b}{j}\left(\frac{a+b-2j+1}{a+b-j+1}\right)\right)^{-1}$,
with equality if and only if $x \leftrightarrow Q_m(x-1; -a, -b-1, N-1)$ has m zeros in $\{1, 2, \ldots, N\}$;

(b) $n = 2m + 1$,

$$\rho(0) \le \left(\frac{a+b}{a}\sum_{j=0}^{m} \frac{(-a)_j(1-N)_j}{(1-b)_j(N-a-b)_j}\binom{a+b-1}{j}\left(\frac{a+b-2j}{a+b-j}\right)\right)^{-1}$$

with equality if and only if $x \leftrightarrow Q_m(x-1; -a, -b, N-2)$ has m zeros in $\{1, 2, \ldots, N-1\}$.

5.2.5. The Singleton bound: The bound is $\rho(0) \le (-b)_n/(-a-b)_n$. Direct calculation shows that

$$\phi(s_{ni}) = a\binom{b}{i}\frac{(-1)(-N)_i}{(-a-b)_{i+1}} S(a-1, b-i, N-1-i, n-i-1),$$

where $S(\alpha, \beta, M, r) = \sum_{j=0}^{M}\binom{\alpha}{j}\binom{\beta}{M-j}(j-M+1)_r \Big/ \left(r!\binom{\alpha+\beta}{M}\right)$. As in 5.1.5 we note $S(\alpha, \beta, M, 0) = 1$ and that $S(\alpha, \beta, M, r) - S(\alpha, \beta, M, r-1)$ can be summed and found to be $\frac{(-\beta)_r(-M)_r}{(-\alpha-\beta)_r r!}$, thus $S(\alpha,\beta, M, r) = \sum_{j=0}^{r}\frac{(-\beta)_j(-M)_j}{(-\alpha-\beta)_j j!}$, a truncated ${}_2F_1$.

Set $r = n - 1 - i$, then $\phi(s_{n,n-r-1})$ is a positive multiple of

$$S(a-1, b-n+1+r, N-n+r, r)$$
$$= \lim_{\varepsilon\to 0} {}_3F_2\left(\begin{matrix}-r, -(b-n+1)-r, -(N-n)-r\\ \varepsilon-r, -(a+b-n)-r\end{matrix}; 1\right) = S_r$$

(for short). There is a third-order differential equation for the general ${}_3F_2$ function (see Bailey [18, p.8]; setting the variable equal to one leads to a three-term recurrence for S_r (the third derivative disappears), which is

$$\begin{aligned}(r+1)(a+b-n+r)(a+b-n+r+1)\,S_{r+1} &= (a+b-n+r)((r+1)(a-1) \\ &\quad -(N-n+r)(b-n+2+2r))S_r \\ &\quad +(b-n+1+r)(N-n+r)(a-N+n-r)S_{r-1}\end{aligned}$$

(with $S_0 = 1$, $S_{-1} \equiv 0$). The coefficient of S_r has a negative second derivative, so its minimum value is at one of the end-points of the range of r, $0 \le r \le n-2$. A necessary condition for $S_r \ge 0$, $1 \le r \le n-1$ is $S_1 \ge 0$, that is,

$$a \ge (N-n)(b-n+2)+1.$$

If $n \le (1/2)(N+b+1-((b-N)(b-N+6)+8N-7)^{1/2})$ then this condition ($S_1 \ge 0$) is also sufficient, otherwise (larger n),

$$a \ge 1 + (N-2)(b+n-2)/(n-1)$$

(from the value at $r = n-2$) is sufficient.

5.2.6. The homogeneous space: For M, N = 1, 2, 3 ... and $M \ge 2N$ let X be the collection of subsets of size N of a given set of M elements; specifically let [a, b] denote the set of integers $\{a, a+1, a+2, \ldots, b\}$ and take

$$X = \{\xi \subset [1, M] : |\xi| = N\}.$$

Let $G = S_M$, the symmetric group on [1, M], and let H be the subgroup fixing the base-point $\omega = [1, N]$. The G-invariant distance on X is given by

$$d(\xi, \eta) = N - |\xi \cap \eta|,$$

and two points $\xi, \eta \in X$ are in the same H-orbit exactly when $d(\xi, \omega) = d(\eta, \omega)$. Now X is called the Johnson scheme J(M, N), and X satisfies the hypotheses of section 4 with $\Omega = \{0, 1, \ldots, N\}$, $m_i = \binom{M-N}{i}\binom{N}{N-i}\Big/\binom{M}{N}$, and spherical functions $g \mapsto Q_j(d(\omega, \omega g); -(M-N)-1, -N-1, N)$. (see Dunkl [7]). A t-design with equal nonzero masses is exactly a "classical t-design." The bounds of the previous paragraphs can be simplified slightly by setting $a = M - N$, $b = N$.

(a) Lower principal, t = 2: let $N(M-N) = j(M-1) + R$ with integers j, R such that $0 \le R \le M-2$, then

$$\rho(0) = \frac{1}{M} - \frac{R(M-1-R)}{j(j+1)M(M-1)},$$

$$\mu_j = \frac{N(M-N)}{Mj}\left(\frac{M-1-R}{M-1}\right), \quad \mu_{j+1} = \frac{N(M-N)R}{M(j+1)(M-1)},$$

(this is an improvement on Fisher's inequality, $|\operatorname{spt}\mu| \gtrsim M$).

(b) Upper principal, $t = 3$: let $(N - 1)(M - N) = j(M - 2) + R$ with integers j, R such that $0 \le R \le M - 3$, then

$$\rho(0) = \frac{N}{M(M-1)} - \frac{NR(M-2-R)}{M(M-1)(M-2)j(j+1)},$$

$$\mu_j = \frac{(M-N)N^2(N-1)(M-2-R)}{jM(M-1)(M-2)(N-j)},$$

$$\mu_{j+1} = \frac{(M-N)N^2(N-1)R}{(j+1)M(M-1)(M-2)(N-j-1)},$$

$$\mu_N = 1 - (\rho(0) + \mu_j + \mu_{j+1}).$$

(c) The orthogonal polynomial bounds: The sums in 5.2.4 are of telescoping form for $b = N$. For a 2m-design μ, $|\mathrm{spt}\,\mu| \ge \binom{M}{m}$, (the Wilson and Ray-Chaudhuri bound [14]), with equality only if $x \leftrightarrow Q_m(x - 1; -(M - N), -N - 1, N - 1)$ has m zeros in $\{1, 2, \ldots, N\}$. For a $(2m + 1)$-design μ, $|\mathrm{spt}\,\mu| \ge \frac{M}{N}\binom{M-1}{m}$, with equality only if $x \leftrightarrow Q_m(x - 1; -(M - N), -N, N - 2)$ has m zeros in $\{1, 2, \ldots, N - 1\}$.

(d) The Singleton bound is $\rho(0) \le (-N)_t/(-M)_t$. If $t \le N - ((8N - 7)^{1/2} - 1)/2$, then equality holds if and only if $M \ge N + (N - t + 1)^2$. Otherwise (for larger t), the latter is necessary, and $M \ge N + 1 + (N - 2)(N + t - 2)/(t - 1)$ is sufficient (for equality).

5.3. *The q-binomial distribution:* Fix integer parameters M, N with $M \ge N \ge 1$ and a real number $q > 1$, and let $\Omega = \{1 - q^{-j}: j = 0, 1, \ldots, N\}$ (so that $a_j = 1 - q^{-j}$). Define the symbol $(a;q^{-1})_m$ by $(a; q^{-1})_0 = 1$, $(a;q^{-1})_{m+1} = (a; q^{-1})_m(1 - aq^{-m})$. The weight function is $m_j = (q^M; q^{-1})_j(q^N; q^{-1})_j/((q^{-1}; q^{-1})_j\, q^{j+MN})$. Note $\sum_{j=0}^{N} m_j = 1$ (the q-analogue of the binomial sum).

The homogeneous spaces which give rise to this distribution all have q equal to a power of a prime number, thus $q \ge 2$. We will show that for $q \ge 2$, $\rho(0)$ is always given by the Singleton bound.

5.3.1. Here $s_n(1 - q^{-j}) = (q^{N-j};q^{-1})_n/(q^N;q^{-1})_n$ and

$$s_{ni}(1 - q^{-j}) = \frac{(q^{N-j}; q^{-1})_i(q^{N-i-1-j}; q^{-1})_{n-i-1}(1 - q^{-j})}{(q^i; q^{-1})_i\,(q^{-1}; q^{-1})_{n-i-1}(1 - q^{i-N})} \qquad 0 \le i \le n - 1, 0 \le j \le N.$$

Thus $\rho(0) = \sum_{j=0}^{N} m_j s_n(1 - q^{-j}) = q^{-Mn}$. Further

$$\phi(s_{ni}) = (q^M - 1)\frac{(q^N; q^{-1})_i}{(q^i; q^{-1})_i} q^{-M(i+1)} S(M-1, N-i-1, n-i-1; q)$$

where

$$S(\alpha, \beta, r; q) = \sum_{j=0}^{\beta} \frac{(q^\alpha; q^{-1})_j (q^\beta; q^{-1})_j}{(q^{-1}; q^{-1})_j} q^{-j-\alpha\beta} \frac{(q^{\beta-1-j}; q^{-1})_r}{(q^{-1}; q^{-1})_r}$$

(and $\beta = 0, 1, 2, \ldots$). Note $S(\alpha, \beta, 0; q) = 1$ and

$$S(\alpha, \beta, r; q) - S(\alpha, \beta, r-1; q) = (q^\beta; q^{-1})_r q^{-r(\alpha+1)} / (q^{-1}; q^{-1})r$$

(by the q-binomial sum), so that $S(\alpha, \beta, r; q) = \sum_{i=0}^{r} \frac{(q^\beta; q^{-1})_i}{(q^{-1}; q^{-1})_i} q^{-i(\alpha+1)}$, a truncated ${}_1\phi_0$ series. From the definition we see that $\alpha \geq \beta$ and $q > 1$ imply $S(\alpha, \beta, 2m; q) \geq 0$. For a nonnegative integer $j \leq (r-1)/2$ consider the sum of terms number $2j$ and $(2j+1)$ of the series, which turns out to be a nonnegative quantity times A_j, where $A_j = 1 + q^{-\alpha-1} - q^{-2j}(q^{-1} + q^{\beta-\alpha-1})$. But A_j is an increasing function of j, so

$$A_j \geq A_0 = 1 - q^{-1} + q^{-\alpha-1}(1 - q^\beta) = q^{-1-\alpha}(q-1)(q^\alpha - (q^\beta - 1)/(q-1)).$$

We see that $A_0 \geq 0$ is a sufficient condition for $S(\alpha, \beta, r,; q) \geq 0$ for all r (and for any $q > 1$). But we assume further than $q \geq 2$, which implies

$$q^\alpha - (q^\beta - 1)/(q-1) \geq q^\alpha - (q^\beta - 1)$$

which is positive for $\alpha \geq \beta$. Thus $M \geq N$, $q \geq 2$ imply that $\phi(s_{ni}) > 0$ for all i, $0 \leq i \leq n-1$, and hence $\rho(0)$ has the Singleton value.

5.3.2. The homogeneous space: Let q be a power of a prime, and let GF(q) denote the field of q elements. Let X be the collection of $M \times N$ matrices over GF(q), and let G be the subgroup of the general linear group of $GF(q)^{M+N}$ defined by

$$\left\{\begin{bmatrix} g_{11} & 0 \\ g_{21} & g_{22} \end{bmatrix} : g_{11} \text{ is invertible } N \times N,\ g_{22} \text{ is invertible } M \times M,\ g_{21} \in X\right\};$$

and let

$$H = \left\{\begin{bmatrix} g_{11} & 0 \\ 0 & g_{22} \end{bmatrix} \in G\right\}.$$

It can be shown that $\begin{bmatrix} g_{11} & 0 \\ g_{21} & g_{22} \end{bmatrix}$ and $\begin{bmatrix} g_{11}' & 0 \\ g_{21}' & g_{22}' \end{bmatrix}$ are in the same 2-sided coset HgH exactly when rank g_{21} = rank g_{21}', and that $G/H \cong X$ (there is a unique element of the form $\begin{bmatrix} I & 0 \\ g_{21} & I \end{bmatrix}$ in each coset Hg). The space X satisfies the hypotheses of Section 4 with $\Omega = \{1 - q^{-j} : j = 0, 1, \ldots, N\}$ and m_j as above (the number of $M \times N$ matrices of rank j divided by q^{MN}), and the spherical functions are expressed in terms of q-Krawtchouk polynomials (a theorem of Delsarte [4], or see Dunkl [8]).

Delsarte [4] has constructed t-designs in X of cardinality equal to $\rho(0)^{-1}$, for each q, M, N.

5.4. *The q-hypergeometric distribution:* Fix parameters a, b, N with N = 1, 2, 3, ... and a, b $\geq$ N and a real number $q > 1$. Let $\Omega = \{1 - q^{-j} : j = 0, 1, \ldots, N\}$ and let $m_j = \binom{a}{j}_q \binom{b}{N-j}_q q^{j(b-N+j)} \Big/ \binom{a+b}{N}_q$, where $\binom{m}{n}_q$ is the q-binomial (Gaussian) coefficient defined to be $\dfrac{(q^m; q^{-1})_n}{(q^n; q^{-1})_n}$ (and equalling the number of n-dimensional subspaces of an m-dimensional vector space over GF(q), prime power q). As in 5.3 we will show that equality is achieved in the Singleton bound for $q \geq 2$. The polynomials s_n and s_{ni} are as in 5.3.

Indeed $\rho(0) = \phi(s_n) = \dfrac{(q^b; q^{-1})_n}{(q^{a+b}; q^{-1})_n} = \binom{b}{n}_q \Big/ \binom{a+b}{n}_q$. Also

$$\phi(s_{ni}) = \binom{b}{i}_q \frac{(1 - q^a)(q^N; q^{-1})_i}{(q^{a+b}; q^{-1})_{i+1}} q^{b-i} S(a - 1, b - i, N - i - 1, n - i - 1; q),$$

where

$$S(\alpha, \beta, \gamma, r; q) = \sum_{j=0}^{\gamma} \binom{\alpha}{j}_q \binom{\beta}{\gamma - j}_q q^{j(\beta - \gamma + j)} \times (q^{\gamma-1-j}; q^{-1})_r \Big/ \left((q^{-1}; q^{-1})_r \binom{\alpha+\beta}{\gamma}_q \right),$$

(and $\gamma = 1, 2, 3, \ldots$). Note that $S(\alpha, \beta, \gamma, 0; q) = 1$ and

$$S(\alpha, \beta, \gamma, r; q) - S(\alpha, \beta, \gamma, r - 1; q) = \frac{(q^\beta; q^{-1})_r (q^\gamma; q^{-1})_r}{(q^{\alpha+\beta}; q^{-1})_r (q^{-1}; q^{-1})_r} q^{-r}$$

(by Heine's ${}_2\phi_1$ sum), so that $S(\alpha, \beta, \gamma, r; q) = \sum_{i=0}^{r} \dfrac{(q^\beta; q^{-1})_i (q^\gamma; q^{-1})_i}{(q^{\alpha+\beta}; q^{-1})_i (q^{-1}; q^{-1})_i} q^{-i}$, a truncated ${}_2\phi_1$ series. The sum of terms number 2j and (2j + 1) (for $0 \leq j \leq (\gamma - 1)/2$) is a positive quantity times B_j, where

$$B_j = (q^{\alpha+\beta} - q^{2j})(q^{2j+1} - 1) - (q^\beta - q^{2j})(q^\gamma - q^{2j}).$$

We see that B_j is increasing in j in the range

$$1 \leq q^{2j} \leq (q^{\alpha+\beta+1} + q^{\beta} + q^{\gamma} + 1)/(2(q+1)).$$

Under the assumptions $\alpha, \beta \geq \gamma > 1$, $q > 1$, this inequality holds for $0 \leq 2j \leq \gamma - 1$, and so to have $S(\alpha, \beta, \gamma, r; q) \geq 0$ for $0 \leq r \leq \gamma$ it suffices to have

$$B_0 = (q^{\alpha-\beta} - 1)(q - 1) - (q^{\beta} - 1)(q^{\gamma} - 1) \geq 0.$$

This always holds for $q \geq 2$ since

$$B_0 \geq q^{\alpha+\beta} - 1 - (q^{\beta} - 1)(q^{\gamma} - 1) = q^{\beta}(q^{\alpha} - q^{\gamma}) + (q^{\beta} + q^{\gamma} - 2) \geq 0.$$

Thus $\phi(s_{ni}) \geq 0$, $0 \leq i \leq n - 1$.

5.4.2. The homogeneous space: Let q be a power of a prime, and let M,N be positive integers with $M \geq 2N$. Let X be the collection of N-dimensional subspaces of $GF(q)^M$, and let G be the general linear group of $GF(q)^M$. Fix a base-point $\omega \in X$, and let H be the stabilizer of ω (this group H is isomorphic to the group G of the previous example with parameters N and $M - N$). Then $\xi, \zeta \in X$ are in the same H-orbit if and only if $\dim(\xi \cap \omega) = \dim(\zeta \cap \omega)$, and we let the space of H-orbits correspond to $\Omega = \{a_j = 1 - q^{-j} : j = 0, 1, \ldots, N\}$ by associating

$$\{\xi \in X : \dim(\xi \cap \omega) = N - j\}$$

to a_j. We obtain m_j as above with the parameters $a = M - N$, $b = N$. The spherical functions are q-Hahn polynomials (see Delsarte [3], and Dunkl [7]). The value of $\rho(0)$ is $(q^N; q^{-1})_n/(q^M; q^{-1})_n$.

REFERENCES

1. N. Biggs, *Perfect codes in graphs.* J. Combinatorial Theory (B) 15 (1973), 289–296.
2. P. Delsarte, *An algebraic approach to the association schemes of coding theory.* Philips Res. Repts. Suppl. 1973, No. 10.
3. ———, *Association schemes and* t*-designs in regular semilattices.* J. Combinatorial Theory (A) 20 (1976), 230–243.
4. ———, *Bilinear forms over a finite field, with applications to coding theory.* MBLE Report R327, April 1976, Brussels.
5. P. Delsarte, J. M. Goethals and J. J. Seidel, *Spherical codes and designs.* Geometriae Dedicata 6 (1977), 363–388.
6. C. F. Dunkl, *A Krawtchouk polynomial addition theorem and wreath products of symmetric groups.* Indiana Univ. Math. J. 25 (1976), 335–358.
7. ———, *An addition theorem for some* q*-Hahn polynomials.* Monatsh. Math. 85 (1977), 5–37.
8. ———, *An addition theorem for Hahn polynomials: the spherical functions.* SIAM J. Math. Anal. 9 (1978), 627–637.

9. C. F. Dunkl, and D. E. Ramirez, *Topics in harmonic analysis.* Appleton-Century-Crofts, New York, 1971.

10. S. Karlin and W. J. Studden, *Tchebycheff systems: With applications in analysis and statistics.* Pure and Applied Mathematics, Vol. XV. Interscience Publishers, New York, 1966.

11. M. G. Kreĭn and P. G. Rehtman, *Development in a new direction of the Čebyšev-Markov theory of limiting values of integrals.* Uspehi Mat. Nauk (N.S.) 10, no.1(63), (1955), 67–78; Amer. Math. Soc. Transl. Ser. 2, 12 (1959), 123–135.

12. S. Lloyd, *Binary block coding.* Bell System Tech. J. 36 (1957), 517–535.

13. C. Radhakrishna Rao, *Factorial experiments derivable from combinatorial arrangements of arrays.* Suppl. J. Roy. Statist. Soc. 9 (1947), 128–139.

14. D. Ray-Chaudhuri and R. M. Wilson, *Generalization of Fisher's inequality to* t-*designs.* Notices Amer. Math. Soc. 18 (1971), 805.

15. I. Schoenberg and G. Szegö, *An extremum problem for polynomials.* Compositio Math. 14 (1960), 260–268.

16. R. Singleton, *Maximum distance* q-*nary codes.* IEEE Trans. Information Theory IT-10 (1964), 116–118.

17. N. Sloane, *An introduction to association schemes and coding theory.* Theory and Application of Special Functions, Proceedings of an advanced seminar sponsored by the Mathematics Research Center, the University of Wisconsin-Madison (April, 1975), R. Askey, ed., pp. 225–260. Academic Press, New York, 1975.

18. W. Bailey, *Generalized hypergeometric series.* Cambridge University Press, Cambridge, 1935.

19. D. G. Higman, *Intersection matrices for finite permutation groups.* J. Algebra 6 (1967), 22–42.

School of Mathematics
Georgia Institute of Technology
Atlanta, Georgia 30332

Current address:
Department of Mathematics
University of Virginia
Charlottesville, Virginia 22903