THE BEZOUT PROBLEM FOR A SPECIAL CLASS OF FUNCTIONS

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INTRODUCTION

For a holomorphic mapping $f: \mathbb{C}^2 \to \mathbb{C}P^2$ Cornalba and Shiffman [1] have shown that it is, in general, impossible to estimate the growth of $f^{-1}(W)$ in terms of the growth of f. Here $W \in \mathbb{C}P^2$ and $f^{-1}(W)$ is assumed discrete. The growth of $f^{-1}(W)$ is measured by counting the number of points in $f^{-1}(W) \cap \{|z| \leq r\}$. In this paper we give a class of functions E for which it is possible to measure the growth of $f^{-1}(W)$ in terms of the growth of f and another function M(r). It is hoped that M(r) will be easier to estimate than the error term S(r) of Griffiths [2].

1. NOTATION

Let $\omega = dd^c \log \|Z\|^2$ be the standard Kähler metric on $\mathbb{C}P^2$. Let $\tau = \log|z|^2$ be the exhaustion function on $\mathbb{C}^2$. If $\xi \in \mathbb{C}P^1$ let $\mathbb{C}_\xi$ be the corresponding line through the origin in $\mathbb{C}^2$. If $f: \mathbb{C}^2 \to \mathbb{C}P^2$ let $f_\xi = f|\mathbb{C}_\xi$. Let $W \in \mathbb{C}P^2$. Then $W = A \cap B$ (= intersection of perpendicular lines in $\mathbb{C}P^2$). Let

$$\omega_o = dd^c \log(|\langle Z,A\rangle|^2 + |\langle Z,B\rangle|^2)$$

in $\mathbb{C}P^2$. Let $\Lambda_W = \log[|Z|^2/(|\langle Z,A\rangle|^2 + |\langle Z,B\rangle|^2)](\omega + \omega_o)$. If $f: \mathbb{C}^2 \to \mathbb{C}P^2$ is holomorphic and $f^{-1}(W)$ discrete then we have the following functions from Nevanlinna theory:

$$n(W,r) = \operatorname{card}(\{|z| \leq r\} \cap f^{-1}(W))$$

$$N(W,r) = \int_0^r n(W,t)\, d\log t$$

(Here we assume $f(0) \neq W$, otherwise one must modify the counting function N(W,r).)

$$T_1(r) = \int_0^r \left\{ \int_{|z|\leq t} f^*\omega \wedge dd^c\tau \right\} d\log t$$

$$T_2(r) = \int_0^r \left\{ \int_{|z|\leq t} f^*\omega \wedge f^*\omega \right\} d\log t$$

$$S(W,r) = \int_{|z|\leq r} f^*\Lambda_W \wedge dd^c\tau.$$

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One has the equation of currents $dd^c\Lambda_W = \omega \wedge \omega - \delta_W$. Applying f^* to this equation and integrating twice gives the First Main Theorem of Nevanlinna.

2. LOCAL REPRESENTATION OF $f^*\Lambda_W$

Choose coordinates in $\mathbb{C}P^2$ such that $W = [1:0:0]$. Let $(w_1, w_2) \leftrightarrow [1:w_1:w_2]$ be local coordinates near W. Near W one has

$$\Lambda_W = \log[(1 + |w_1|^2 + |w_2|^2)/(|w_1|^2 + |w_2|^2)](\omega + \omega_o),$$
$$\omega = dd^c \log(1 + |w_1|^2 + |w_2|^2), \quad \omega_o = dd^c \log(|w_1|^2 + |w_2|^2).$$

The most singular term of Λ_W is

$$\Omega = \log[1/(|w_1|^2 + |w_2|^2)]\, dd^c \log(|w_1|^2 + |w_2|^2).$$

For a certain class of functions $f: \mathbb{C}^2 \to \mathbb{C}P^2$ we first estimate $f^*\Omega$.

3. THE CLASS E

Let $f: \mathbb{C}^2 \to \mathbb{C}P^2$ be a holomorphic map having the representation $[1:f_1:f_2]$ near $W = [1:0:0]$. We say that f belongs to the class E if there exists some positive $\rho << 1$ such that the following conditions hold:

(i) There exists ρ_1 with $0 < \rho_1 < \rho$ such that for each line $\mathbb{C}_\xi$ through the origin in $\mathbb{C}^2$ (a) or (b) holds

(a) $\mathbb{C}_\xi \cap \{|f_1(z)| \le \rho_1\} = \amalg K_\nu$ (disjoint union)

(b) $\mathbb{C}_\xi \cap \{|f_2(z)| \le \rho_1\} = \amalg K_\nu$

where K_ν are compact in $\mathbb{C}_\xi$ and in each K_ν the number of points p^i_ν such that $f(p^i_\nu) = W$ is less than or equal to N (N independent of ξ and ν).

(ii) For each line $\mathbb{C}_\xi$ through the origin in $\mathbb{C}^2$ there exists a holomorphic function F^2_ξ on $\mathbb{C}_\xi$ which vanishes (with multiplicity) exactly on

$$\{z \in \mathbb{C}_\xi : |f_1(z)|^2 + |f_2(z)|^2 = 0\}$$

and satisfies the estimate $|F_\xi(z)|^2 \le M(r)(|f_1(z)|^2 + |f_2(z)|^2)$ if $|z| \le r$ and where $M(r)$ is independent of ξ.

4. ESTIMATE FOR S(W,r)

Assume $W = [1:0:0]$ and $f = [1:f_1:f_2]$ near W. Let $N_\rho = \{w_1, w_2): |w_i| < \rho\}$ be a neighborhood of W in $\mathbb{C}P^2$. Put

$$I_1(r) = \int_{\substack{|z|\le r \\ z\in f^{-1}(N_\rho)}} f^*\Lambda_W \wedge dd^c\tau \quad \text{and} \quad I_2(r) = \int_{\substack{|z|\le r \\ z\in \mathbb{C}^2 - f^{-1}(N_\rho)}} f^*\Lambda_W \wedge dd^c\tau.$$

Then $S(W,r) = I_1(r) + I_2(r)$. The main contribution to $S(W,r)$ comes from $I_1(r)$. Put

$$J_\xi(r) = \int_{\substack{|z|\le r \\ z\in \mathbb{C}_\xi \cap f^{-1}(N_\rho)}} f_\xi^* \Lambda_W.$$

Then $I_1(r) = \int_{\xi\in \mathbb{C}P^1} J_\xi(r)\, d\xi$, where $d\xi$ is the usual normalized volume on $\mathbb{C}P^1$. The main contribution to $J_\xi(r)$ comes from integration of $f^*\Omega\,|\,\mathbb{C}_\xi$. Let

$$\begin{aligned}\tilde{J}_\xi(r) &= \int_{\substack{|z|\le r \\ z\in \mathbb{C}_\xi \cap f^{-1}(N_\rho)}} f_\xi^*\Omega. \\ &= \int_{\substack{|z|\le r \\ z\in \mathbb{C}_\xi \cap f^{-1}(N_\rho)}} \log\,[1/(|f_1|^2 + |f_2|^2)] \\ &\quad \cdot [|f_1' f_2 - f_2' f_1|^2/(|f_1|^2 + |f_2|^2)]\, \frac{\sqrt{-1}}{2}\, dz \wedge d\bar{z}.\end{aligned}$$

Let $M_\xi(r) = \sup\limits_{\substack{|z|\le r \\ z\in \mathbb{C}_\xi}} |f_1'(z) f_2(z) - f_2'(z) f_1(z)|^2/(|f_1(z)|^2 + |f_2(z)|^2)^2$.

Remark. A simple calculation shows this exists. Then

$$\tilde{J}_\xi(r) \le M_\xi(r) \int_{\substack{|z|\le r \\ z\in \mathbb{C}_\xi \cap f^{-1}(N_\rho)}} \log\,[1/(|f_1|^2 + |f_2|^2)]\, \frac{\sqrt{-1}}{2}\, dz \wedge d\bar{z}.$$

Now suppose $f \in E$ and condition (ia) holds. Then we have

$$\begin{aligned}&\int_{\substack{|z|\le r \\ z\in \mathbb{C}_\xi \cap f^{-1}(N_\rho)}} \log\,[1/(|f_1|^2 + |f_2|^2)]\, \frac{\sqrt{-1}}{2}\, dz \wedge d\bar{z} \\ &\le \int_{\substack{|z|\le r \\ z\in \mathbb{C}_\xi \cap f^{-1}(N_\rho)}} \log\,[\rho_1^2/|f_1|^2]\, \frac{\sqrt{-1}}{2}\, dz \wedge d\bar{z} + Cr^2\end{aligned}$$

where C is a constant which depends only on ρ_1 and ρ. Using condition (ia) $\mathbb{C}_\xi \cap f^{-1}N_{\rho_1}) = \coprod_\nu K_\nu$ where each K_ν is compact and $K_\nu \cap K_\mu = \emptyset$ if $\nu \ne \mu$. Let

$p_\nu^i \in K_\nu$ such that $f_1(p_\nu^i) = 0$. Let the Green's function of K_ν with pole at p_ν^i be $g(z, p_\nu^i)$. Then on K_ν

$$\log [\rho_1/|f_1(z)|] = \sum_{\substack{i=1,m_\nu \\ m_\nu \leq N}} g(z, p_\nu^i).$$

Therefore

$$\int_{\substack{|z|\leq r \\ z\in \mathbb{C}_\xi \cap \{z:|f_1(z)|\leq \rho}} 2\log [\rho_1/|f_1(z)|] \frac{\sqrt{-1}}{2} dz \wedge d\bar{z}$$

$$= \sum_\nu \int_{\substack{|z|\leq r \\ z\in K_\nu}} \log [\rho_1/|f_1(z)|] \sqrt{-1}\, dz \wedge dz$$

$$= \int_{t=0}^{r} \left(\sum_{\substack{\nu,j \\ j\leq N}} \int_{\substack{0 \\ te^{i\theta}\in K_\nu}}^{2\pi} g(te^{i\theta}, p_\nu^j)\, d\theta \right) t\, dt.$$

Applying Selberg's lemma on Green's function (see Tsuji [3]) gives for $t \geq r_o$

$$\sum_j \int_{\substack{0 \\ te^{i\theta}\in K_\nu}}^{2\pi} g(te^{i\theta}, p_\nu^j)\, d\theta \leq N(2\pi^2 + \log(t/r_o))$$

where $r_o = \text{dist.}(0, \amalg K_\nu) > 0$ is independent of $\mathbb{C}_\xi$. Hence

$$\int_{t=0}^{r} \left(\sum_{\nu,j} \int_{\substack{\theta=0 \\ te^{i\theta}\in K_\nu}}^{2\pi} g(te^{i\theta}, p_\nu^i)\, d\theta \right) t\, dt \leq N \int_{t=0}^{r} [2\pi^2 + \log(t/r_o)]\, t\, dt.$$

Putting $L(r) = N \int_{t=0}^{r} [2\pi^2 + \log(t/r_o)]\, t\, dt$ one has

$$\tilde{J}_\xi(r) \leq M_\xi(r)[L(r) + c \cdot r^2].$$

Now since $\log[(1 + |f_1|^2 + |f_2|^2)/(|f_1|^2 + |f_2|^2)] \leq 2\log[1/(|f_1|^2 + |f_2|^2)]$ on N_ρ for $\rho << 1$ we have the estimate $J_\xi(r) \leq c\tilde{J}_\xi(r)$. If we obtain an estimate for $M_\xi(r)$ independent on ξ we will have an estimate for $I_1(r)$. Using condition (ii) we have:

$$M_\xi(r) \leq M^2(r) \sup_{\substack{|z|\leq r \\ z\in \mathbb{C}_\xi}} |(f_1'(z)f_2(z) - f_2'(z)f_1(z))/F_\xi^2(z)|^2.$$

Since F_ξ^2 has the same zeros as $|f_1|^2 + |f_2|^2$ this exists. Let $G_\xi = (f_1'f_2 - f_2'f_1)/F_\xi$. Then G_ξ is holomorphic and therefore letting $K_{G_\xi}(r) = \sup_{|z|\leq r} |G_\xi(z)|^2$ and using the estimates

$$T(r) \leq \log K_{G_\xi}(r) + O(1) \leq 3T(2r) + O(1) \tag{4.1}$$

where $T(r)$ is the order function of an entire function, we have

$$M_\xi(r) \leq M^2(r)\, K^2_{G_\xi}(r) \leq M^2(r) \cdot c\,\{K^3_g(2r) \cdot M^3(2r)(K^2_{f_1}(2r) + K^2_{f_2}(2r))^3\}$$

where $g = f_1' f_2 - f_2' f_1$. $M_\xi(r) \leq c \cdot M^5(2r) \cdot P(r)$ where $P(r)$ depends only on the growth of f. We now have $J_\xi(r) \leq c_1 M^5(2r) \cdot P(r)[L(r) + c_2 r^2]$ which is independent of ξ. Therefore $I_1(r) \leq c_1 M^5(2r) \cdot P(r)[L(r) + c_2 r^2]$. Now estimate $I_2(r)$. Since $\omega_o \leq \text{const.}\, \omega$ on $\mathbb{C}^2 - f^{-1}(N_\rho)$ and $\log[1/(|f_1|^2 + |f_2|^2)]$ is bounded on $\mathbb{C}^2 - f^{-1}(N_\rho)$ we have the estimate

$$I_2(r) \leq c_3 \int_{|z|\leq r} f^*\omega \wedge dd^c\tau = c_3 \cdot r \cdot T_1'(r) + c_4.$$

Since $S(W,r) = I_1(r) + I_2(r)$ we have

$$S(W,r) \leq c_1 M^5(2r) \cdot P(r)[L(r) + c_2 r^2] + c_3 r T_1'(r) + c_4.$$

Hence the growth of the error term depends on the function $M(r)$ and functions of r which depend only on the growth of f for functions in the class E.

5. SOME EXAMPLES

We will now give some examples which will show that condition (i) says something about the clustering of common zeros of f_1 and f_2 along radial lines while condition (ii) says something about the proximity to zero of $|f_1|^2 + |f_2|^2$ along radial lines. We remark that all polynomial mappings are obviously in the class E.

Example 1. Our first example will deal with the "exponential surfaces" $f = (f_1, f_2)$ with $f_1(z,w) = e^{\alpha_1 z + \beta_1 w} - 1$ and $f_2(z,w) = e^{\alpha_2 z + \beta_2 w} - 1$, where $(\alpha_i, \beta_i) \neq (0,0)$ $i = 1,2$. In this example we will use a slightly stronger version of the main result than we have proved. Note that in section 4 $I_1(r) = \int_{\xi \in \mathbb{C}P^1} J_\xi(r)\, d\xi$ where $d\xi$ is the usual normalized volume on $\mathbb{C}P^1$. Suppose D is a set of measure zero in $\mathbb{C}P^1$. Then we have $I_1(r) = \int_{\xi \in \mathbb{C}P^1 - D} J_\xi(r)\, d\xi$. Condition (ii) was used to obtain a uniform (*i.e.*, independent of ξ) estimate for $J_\xi(r)$. It will suffice to have condition:

(ii)′ For each line $\mathbb{C}_\xi$ through the origin in $\mathbb{C}^2$ with $\xi \in \mathbb{C}P^1 - D$ there exists a holomorphic function F^2_ξ on $\mathbb{C}_\xi$ which vanishes exactly on

$$\{z \in \mathbb{C}_\xi : |f_1(z)|^2 + |f_2(z)|^2 = 0\}$$

and satisfies the estimate $|F_\xi^2(z)| \le M(r)(|f_1(z)|^2 + |f_2(z)|^2)$ if $|z| \le r$ and where $M(r)$ is independent of ξ.

Condition (ii)$'$ can be used exactly as before to obtain the estimate

$$J_\xi(r) \le c_1 M^5(2r) \cdot P(r)[L(r) + c_2 r^2] \qquad \text{for all } \xi \in \mathbb{C}P^1 - D.$$

Integrating $J_\xi(r)$ over $\mathbb{C}P^1 - D$ gives the estimate for $I_1(r)$. Let E' be the set E with condition (ii) replaced by condition (ii)$'$. We now show that the "exponential surfaces" above are in E'.

First verify condition (i). Let $\zeta_0 z + \zeta_1 w = 0$ be a line through the origin in $\mathbb{C}^2$. For $\zeta_1 \neq 0$ this may be written $w = \xi z$. The restriction of f to $w = \xi z$ is

$$f(z) = (e^{\mu_1 z} - 1, e^{\mu_2 z} - 1)$$

where $\mu_i = \alpha_i + \xi\beta_i$. Let $\mu = \mu_1$ if $\mu_1 \neq 0$, otherwise let $\mu = \mu_2$. Let $N \subset \mathbb{C}$ be a small neighborhood of μ; $N = \{\mu' : |\mu' - \mu| < \varepsilon << 1\}$. We show there exists $\rho > 0$ and $\varepsilon > 0$ such that $\{z: |e^{\mu' z} - 1| \le \rho\} = \amalg K_\nu$ with K_ν disjoint compact and $\mu' \in N$. (The K_ν of course depends upon μ' but for each μ' we get a collection of K_ν with the stated property.)

This shows that for any given $\xi \in \mathbb{C}P^1$ there exists a neighborhood $M \subset \mathbb{C}P^1$ of ξ such that the given ρ works for all $\xi \in \mathbb{C}P^1$. Cover $\mathbb{C}P^1$ by such neighborhoods. Since $\mathbb{C}P^1$ is a compact topological space this will imply that there exists a ρ which works for all lines $\zeta_0 z + \zeta_1 w = 0$ through the origin. By rotating the z-plane we may assume μ real. Then $|e^{\mu z} - 1| = 2(\cosh \mu x - \cos \mu y)$. Let $\tilde{N} \subset \mathbb{R}$ be the set $\{e^{i\theta}\mu' : \mu' \in N \text{ and } e^{i\theta}\mu' \in \mathbb{R}\} = \{t \in \mathbb{R} : |t - \mu| < \varepsilon\}$. By the continuity of cosh and cos we may choose ρ and ε sufficiently small so that

$$\{(x,y): 2(\cosh tx - \cos ty) \le \rho\} = \amalg K_\nu \qquad \text{for each } t \in \tilde{N}.$$

This implies $\{z: |e^{\mu' z} - 1| \le \rho\} = \amalg K_\nu$ for each $\mu' \in N$. Condition (i) has therefore been verified for f. Next we verify condition (ii)$'$. Again the restriction of f to $w = \xi z$ is $(e^{\mu_1 z} - 1, e^{\mu_2 z} - 1)$. We may assume $\mu_1 \neq 0$ and $\mu_2 \neq 0$ since this occurs for only a finite number of ξ. There are two possible situations.

(a) The zeros of $e^{\mu_1 z} - 1$ and $e^{\mu_2 z} - 1$ lie on the same real line through the origin in $\mathbb{C}$. This occurs when $\mu_1/\mu_2 \in \mathbb{R}$; hence when

$$(\alpha_1 + \beta_1\xi)/(\alpha_2 + \beta_2\xi) \in \mathbb{R} \quad \text{or} \quad \xi = (\alpha_1 - t\alpha_2)/(t\beta_2 - \beta_1), \quad t \in \mathbb{R}.$$

$\{\xi \in \mathbb{C}P^1 : \xi = (\alpha_1 - t\alpha_2)/(t\beta_2 - \beta_1); t \in \mathbb{R}\}$ is a set of measure zero with respect to $d\xi$, the normalized volume on $\mathbb{C}P^1$. Hence we may ignore this set of ξ in verifying condition (ii)$'$.

(b) The zeros of $e^{\mu_1 z} - 1$ and $e^{\mu_2 z} - 1$ lie on different real lines through the origin in $\mathbb{C}$. From the discussion on the verification of (i), given $\varepsilon > 0$ we may choose $\rho_i > 0$ $(i = 1,2)$ so small that $\{z: |e^{\mu_i z} - 1| \le \rho_i\} = \amalg K_\nu^i$ with the K_ν^i disjoint compact and K_ν^i contained in a ball of radius ε about the zero p_ν^i of $e^{\mu_i z} - 1$. Let $\rho = \inf(\rho_1, \rho_2)$. Choose R so large that if $|z| \ge R$ then

$$\{z: |e^{\mu_1 z} - 1| \leq \rho\} \cap \{z: |e^{\mu_2 z} - 1| \leq \rho\} = \emptyset.$$

Let $c = \inf(|e^{\mu_1 z} - 1|^2 + |e^{\mu_2 z} - 1|^2)/|z|^2$. The constants ρ, R, and c all depend upon ξ. We have also if $|z| \geq R$ then $|e^{\mu_1 z} - 1|^2 + |e^{\mu_2 z} - 1|^2 \geq \rho^2$. Hence if $|z| \geq R$ then $\rho^2|z|^2 \leq r^2(|e^{\mu_1 z} - 1|^2 + |e^{\mu_2 z} - 1|^2)$. We may assume, without changing the respective inequalities, that c and ρ are less than or equal to 1. Therefore $|\rho \cdot cz|^2 \leq (1 + r^2)(|e^{\mu_1 z} - 1|^2 + |e^{\mu_2 z} - 1|^2)$. Condition (ii)$'$ is therefore satisfied if we take $M(r) = 1 + r^2$. Therefore the "exponential surface"

$$f(z,w) = (e^{\alpha_1 z + \beta_1 w} - 1, e^{\alpha_2 z + \beta_2 w} - 1)$$

is in the class E$'$.

Remark. The lack of restriction on M(r) in condition (ii) in fact makes E a rather large class of functions. For application one might want to restrict E by restricting the functions M(r) allowed. We do this in our next example.

Example 2. In this example we will examine the Cornalba-Shiffman counterexample to the transcendental Bezout problem in the context of our main result. They construct two holomorphic functions g(z,w) and f(z,w) on $\mathbb{C}^2$ as follows. (See [1].)

Put $g(z,w) = g(z) = \prod_{h=1,\infty} (1 - z2^{-h})$ and $g_k(z) = g(z)(1 - z2^{-k})^{-1}$ if $k \geq 1$.

Observe that for any $\varepsilon > 0$, $|g(z)| \leq K_\varepsilon e^{r^\varepsilon}$ for $|z| \leq r$ and

$$|g_k(z)| \leq K'_\varepsilon e^{r^\varepsilon} \qquad \text{for } |z| \leq r.$$

Now put $P_h(w) = \prod_{j=1,c_h} (w - 1/j)$ where $c_h = 2^{2^h}$. Define

$$f(z,w) = \sum_{h=1,\infty} 2^{-c_h^2} g_h(z) P_h(w).$$

Then $|f(z,w)| \leq K_\varepsilon e^{|z|^\varepsilon + |w+1|^\varepsilon}$. We now restrict the class E to a smaller class of functions as follows. Suppose $k \in \mathbb{Z}$. Define the class E_k by replacing condition (ii) in the definition of E by (ii)$'$:

(ii)$'$ For each line $\mathbb{C}_\zeta$ through the origin in $\mathbb{C}^2$ there exists a holomorphic function F_ζ^2 on $\mathbb{C}_\zeta$ which vanishes (with multiplicity) exactly on

$$\{z \in \mathbb{C}_\zeta: |f_1(z)|^2 + |f_2(z)|^2 = 0\}$$

and satisfies the estimate $|F_\zeta(z)|^2 \leq r^k [|f_1(z)|^2 + |f_2(z)|^2]$ for all z with $|z| \leq r$.

We will show that the Cornalba-Shiffman example is not in the class E_k for any $k \in \mathbb{Z}$. The set of common zeros of f and g is

$$\{(2^h, 1/j): h = 1, \ldots, n, \ldots; j = 1, \ldots, c_h\}.$$

We restrict (g,f) to the line $w = \zeta z$ where ζ is chosen such that $f|\mathbb{C}_\zeta$ and $g|\mathbb{C}_\zeta$ have no common zeros and such that $|\zeta| \leq 1$. We remark that almost all ζ with $|\zeta| \leq 1$ have this property. Suppose there exist an entire function $F_\zeta(z)$, referred to from now on as $F(z)$, such that $|F^2(z)| \leq r^k [|g(z)|^2 + |f(z,\zeta z)|^2]$ for $|z| \leq r$ and $F(z)$ is nonvanishing (since there are no z such that $|g(z)|^2 + |f(z,\zeta z)|^2 = 0$). Write $F(z) = e^{H(z)}$. Then we would have

$$(2.1) \qquad |e^{H(z)}|^2 \leq r^k [|g(z)|^2 + |f(z,\zeta z)|^2]; \qquad |z| \leq r.$$

Now $|f(z,\zeta z)| \leq K_\varepsilon e^{|z|^\varepsilon + |\zeta z + 1|^\varepsilon}$; hence with $|\zeta| \leq 1$ we have

$$|f(z,\zeta z)| \leq K_\varepsilon e^{r^\varepsilon + (r+1)^\varepsilon} \qquad \text{for } |z| \leq r.$$

Therefore

$$|e^{H(z)}|^2 \leq r^k (K_\varepsilon e^{2r^\varepsilon} + K'_\varepsilon e^{2r^\varepsilon + 2(r+1)^\varepsilon}) \leq r^k \tilde{K}_\varepsilon e^{4(r+1)^\varepsilon}, \qquad |z| \leq r;$$

where we have collected the various constants which gives:

$$(2.2) \qquad \max_{|z| \leq r} |e^{H(z)}|^2 \leq r^k \tilde{K}_\varepsilon e^{4(r+1)^\varepsilon}.$$

Now consider the RHS of (2.1) at the point $z = 2^\ell, \ell \in \mathbb{Z}$. Since $g(2^\ell) = 0$ we have RHS $= r^k \cdot |f(2^\ell, \zeta 2^\ell)|$. Now

$$f(2^\ell, \zeta 2^\ell) = \sum_{h=1,\infty} 2^{-c_h^2} \cdot g_h(2^\ell) \cdot P_h(\zeta 2^\ell).$$

Since $g_h(2^\ell) = 0$ if $h \neq \ell$; $f(2^\ell, \zeta 2^\ell) = 2^{-c_\ell^2} \cdot g_\ell(2^\ell) \cdot P_\ell(\zeta 2^\ell)$ where

$$g_\ell(2^\ell) = \prod_{h \neq \ell} (1 - 2^\ell \cdot 2^{-h}) \quad \text{and} \quad P_\ell(\zeta 2^\ell) = \prod_{j=1,c_\ell} (\zeta 2^\ell - 1/j).$$

Therefore $|g_\ell(2^\ell)| \leq 2^{\ell^2}$ and $|P_\ell(\zeta 2^\ell)| \leq (2^\ell)^{c_\ell}$ where we have used $|\zeta| \leq 1$. Finally we get

$$|f(2^\ell, \zeta 2^\ell)| \leq 2^{-c_\ell^2} \cdot 2^{\ell c_\ell} \cdot 2^{\ell^2}$$

so

$$\text{RHS} \leq 2^{\ell k} \cdot 2^{-2c_\ell^2} \cdot 2^{2\ell c_\ell} \cdot 2^{2\ell^2} \qquad \text{for } z = 2^\ell.$$

This implies

$$(2.3) \qquad \min_{|z| \leq 2^\ell} |e^{H(z)}| \leq 2^{-c_\ell} \qquad \text{for } \ell \text{ sufficiently large and } |z| \leq 2^\ell.$$

Therefore

$$(2.4) \qquad \max_{|z| \leq 2^\ell} |e^{-H(z)}|^2 \geq 2^{c_\ell} \text{ for } \ell \text{ sufficiently large.}$$

From the classical Nevanlinna theory, for any meromorphic function f, $T_f(r) = T_{1/f}(r)$. If f is holomorphic for all z then we have the relationship between $T_f(r)$ and $\max_{|z|\leq r} |f(z)|^2$ given in inequality (4.1). Applying this to the entire functions $e^{H(z)}$ and $e^{-H(z)}$ and using the inequalities (2.2) and (2.4) we get:

$$2^{c_\ell} \leq K_\varepsilon 2^{3\cdot\ell\cdot k} e^{12(2^\ell+1)^\varepsilon}$$

for some K_ε depending upon ε and for all ℓ.

But with $c_\ell = 2^{2\ell}$ this gives a contradiction; hence no such function H(z) exists and the Cornalba-Shiffman example is not an element of E_k for any k.

Remark. Notice that in this second example the "clustering" of zeros on *radial lines* is nonexistent. Although the set $\{(2^h, 1/j): h = 1, \ldots, n, \ldots; j = 1, \ldots, c_h\}$ clusters its intersection with a line, $\zeta_0 z + \zeta_1 w = 0$ does not. We will give a final example which will show that it is not really the distribution of common zeros of f_1 and f_2 along radial lines but proximity to zero along radial lines which determines elements of E. Of course near a common zero of f_1 and f_2 on a radial line the functions must be near zero.

Example 3. Let $f(z,w) = (e^{P(z,w)}, e^{Q(z,w)})$ where P and Q are non-constant first order polynomials. Consider any line $w = \xi z$ such that $P(z, \xi z)$ and $Q(z, \xi z)$ are non-constant polynomials. Then no matter how small we take $\rho > 0$ the set $\{z: |e^{P(z,\xi z)}| \leq \rho\}$ cannot be written as the disjoint union of compact sets. The same remark applies to $\{z: |e^{Q(z,\xi z)}| \leq \rho\}$. Hence $f \notin E$ because condition (i) is not satisfied.

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