A NEW APPROACH TO GELFAND-MAZUR THEORY AND THE EXTENSION THEOREM

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To state the theorem on which our approach is based, we need the following definitions: A function N from a ring A to the real numbers $\mathbb{R}$ is a (ring) *seminorm* if for all $x, y \in A$, $N(x) \geq 0$, $N(-x) = N(x)$, $N(xy) \leq N(x)N(y)$, and $N(x+y) \leq N(x) + N(y)$. The null space $N^{-1}(0)$ of a seminorm N is an ideal; N is a *norm* if $N^{-1}(0) = (0)$. The *core* of a seminorm N on A is the set C(N), defined by

$$C(N) = \{c \in A: N(c) \neq 0, \text{ and } N(cx) = N(c)N(x) = N(xc) \text{ for all } x \in A\}.$$

A function V from a ring A with identity to $\mathbb{R}$ is an *absolute semivalue* if V is a seminorm satisfying $V(1) = 1$ and $V(xy) = V(x)V(y)$ for all $x, y \in A$. The null space $V^{-1}(0)$ of an absolute semivalue V is a prime ideal; V is an *absolute value* if $V^{-1}(0) = (0)$.

If $|\,.\,.\,|$ is an absolute value on a field K and if A is a K-algebra, a ring norm N on A is an *algebra norm* if $N(\lambda . x) = |\lambda| N(x)$ for all $\lambda \in K$, $x \in A$.

Let N be a seminorm on a commutative ring A. As is well known, $\lim_{n\to\infty} N(x^n)^{1/n}$ exists for each $x \in A$, and $N_s: x \mapsto \lim_{n\to\infty} N(x^n)^{1/n}$ is a seminorm on A, called the *spectral seminorm* associated to N. A seminorm N on A is *spectral* if $N = N_s$, or equivalently, if $N(x^n) = N(x)^n$ for all $x \in A$ and all $n \geq 1$. If N is any seminorm on A, N_s is a spectral seminorm.

Our discussion is based on the following theorem of Aurora [3, Theorem 1]:

THEOREM 1. *If* N *is a nonzero spectral seminorm on a commutative ring* A *with identity and if* $J = N^{-1}(0)$, *there is a family* $(V_c)_{c \in A \setminus J}$ *of absolute semivalues on* A *such that for each* $c \in A \setminus J$, $V_c \leq N$, $C(V_c) \supseteq C(N) \cup \{c\}$, $V_c(x) = N(x)$ *for all* $x \in C(N) \cup \{c\}$, *and therefore* $N = \sup_{c \in A \setminus J} V_c$.

We have stated somewhat more than appears in [3], but a slight modification of the proof yields our statement. Earlier, Cohn [8, Theorem 13.3] had shown that a spectral norm on a field was the supremum of a family of absolute values, but the further properties mentioned in Theorem 1, which are crucial for applications, are not apparently derivable from his proof. Aurora's theorem has subsequently been rediscovered, in whole by Bergman [4], and in part by Szpiro [17] and Kiyek [11].

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1. THE EXTENSION THEOREM

An absolute value $|..|$ on a field K is *proper* if the topology defined by $|..|$ is not the discrete topology, that is, if there exists $x \in K$ such that $0 < |x| < 1$. The basic extension theorem is the following:

THEOREM 2. *Let* $|..|$ *be a proper, complete absolute value on a field* K, *and let* E *be a field extension of* K *of finite degree. There is a unique absolute value* V *on* E *extending* $|..|$.

Proof. Let $\{e_1, \ldots, e_n\}$ be a basis of the K-vector space E, where $e_1 = 1$. By [7, Theorem 2, p. 18] there is a unique topology on E making E into a Hausdorff topological vector space, and that topology is therefore given by the norm N on the K-vector space E, defined by $N\left(\sum_{i=1}^{n} \lambda_i e_i\right) = \sup_{1 \leq i \leq n} |\lambda_i|$. Multiplication is a K-bilinear mapping from $E \times E$ to E and hence is continuous (*cf.* [7, Corollary 2, p. 19]), so there is an equivalent algebra norm N_1 on the K-algebra E (defined by $N_1(x) = |\alpha|^{-2} N(x)$, where $\alpha \in K^*$ is such that for all $x, y \in E$, $N(x) \leq |\alpha|$ and $N(y) \leq |\alpha|$ imply $N(xy) \leq 1$). By a familiar technique, we may replace N_1 by an equivalent algebra norm M satisfying $M(1) = 1$ by defining

$$M(x) = \sup \{N_1(xy) N_1(y)^{-1} : y \neq 0\}.$$

Then $C(M) \supseteq K$, and M is a norm extending $|..|$. As E is a field, the associated spectral seminorm M_s is a norm; furthermore, $C(M_s) \supseteq C(M) \supseteq K$, and M_s agrees with M on $C(M)$, so in particular, M_s is an extension of $|..|$. By Theorem 1, there is an absolute semivalue V on E agreeing with M_s on $C(M_s)$ and, in particular, on K. As E is a field, V is an absolute value, which is thus the desired extension of $|..|$. For uniqueness, see [6, Lemma 2; p. 132].

THEOREM 3. *If* $|..|$ *is a proper absolute value on a field* K *and if* E *is a field extension of* K *of finite degree, there is an absolute value on* E *extending* $|..|$.

Proof. Let $|..|^\wedge$ be the absolute value of the completion $K^\wedge$ of K. There is a K-isomorphism σ from E onto a subfield of the algebraic closure Ω of $K^\wedge$, and $[\sigma(E)^\wedge : K^\wedge] \leq [\sigma(E) : K] < +\infty$. By Theorem 2, there is an absolute value V on $\sigma(E)^\wedge$ extending $|..|^\wedge$, so $x \mapsto V(\sigma(x))$ is the desired extension of $|..|$ to E.

Kürschak's original proof [12] of the Extension Theorem depended on theorems of Hadamard concerning the radius of convergence of the product of a power series and a polynomial. Ostrowski [13] observed that Hensel's Lemma permitted a much simpler proof for nonarchimedean absolute values, and established the theorem for archimedean absolute values by proving the theorem that bears his name. Essentially, this method of proof of the Extension Theorem has remained current since 1918 (*e.g.,* [2, Ch. 2]), although some presentations replace the appeal to Hensel's Lemma by one to Krull's extension theorem for valuations (*e.g.,* [6, Proposition 9, p. 151]).

2. THE GELFAND-MAZUR THEOREM

Besides Theorem 1, we shall need the following two theorems:

THEOREM 4. (*Frobenius*) *If* D *is a real division algebra such that every commutative division subalgebra has dimension at most 2, then* D *is isomorphic to* $\mathbb{R}$, $\mathbb{C}$, *or the division algebra* $\mathbb{H}$ *of quaternions.*

A proof is given in [14] and in [6, Exercise 2, p. 186].

Henceforth we shall use $|\,.\,.\,|$ to denote the usual absolute value on division subrings of $\mathbb{H}$.

THEOREM 5. (*Ostrowski*) *If a field* K *is a proper extension of* $\mathbb{C}$, *the absolute value* $|\,.\,.\,|^p$ *on* $\mathbb{C}$, *where* $0 < p \leq 1$, *admits no extension to an absolute value on* K.

This theorem is crucial to the proof of Ostrowski's description of archimedean absolute values. The proof [13, pp. 281–282] uses the fact that $\mathbb{C}$ is locally compact and contains roots of unity of all orders.

Gelfand's proof of the Gelfand-Mazur theorem depended on the Hahn-Banach Theorem and Liouville's Theorem concerning bounded entire functions. Subsequent elementary proofs ([18], [16], [10], [15]) used instead the same properties of $\mathbb{C}$ that Ostrowski used in proving Theorem 5. Żelazko [19] established more explicitly the fact the convexity played no role by showing that $|\,.\,.\,|$ on the scalar field $\mathbb{C}$ could be replaced by $|\,.\,.\,|^p$ where $0 < p \leq 1$. All presentations heretofore have proved the complex Gelfand-Mazur theorem first and derived from it the real version, first established by Arens [1]. We may prove the real version directly, however, from which the complex version follows.

THEOREM 6. (*Gelfand-Mazur*) *If* D *is a normed division algebra over* $\mathbb{R}$, *equipped with the absolute value* $|\,.\,.\,|^p$ *where* $0 < p \leq 1$, *there is a topological isomorphism from* D *onto one of the* $\mathbb{R}$*-algebras* $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$.

Proof. As in the proof of Theorem 2, there is an algebra norm N on D that is equivalent to the given one and satisfies $N(1) = 1$. Consequently,

$$N(\lambda.1) = |\lambda|^p N(1) = |\lambda|^p$$

for all $\lambda \in \mathbb{R}$. We identify $\mathbb{R}$ with $\mathbb{R}.1$; thus N is a norm on D that extends $|\,.\,.\,|^p$ and contains $\mathbb{R}$ in its core. To apply Theorem 4, let K be a commutative division subalgebra of D, N' the restriction of N to K. The corresponding spectral seminorm N'_s on K agrees with $|\,.\,.\,|^p$ on $\mathbb{R}$ and contains $\mathbb{R}$ in its core, and N'_s is a norm since K is a field. By Theorem 1, there is an absolute semivalue V on K that agrees with N'_s and hence $|\,.\,.\,|^p$ on $\mathbb{R}$, and again, V is an absolute value as K is a field. By Theorem 3, there is an absolute value V' extending V on K(i), the field obtained by adjoining a root i of $X^2 + 1$ to K. But $K(i) \supseteq \mathbb{R}(i) = \mathbb{C}$, so as $\mathbb{R}$ is complete for $|\,.\,.\,|^p$, $V'(x) = |x|^p$ for all $x \in \mathbb{R}(i)$ by Theorem 2. Therefore $K(i) = \mathbb{C}$ by Theorem 5, so the dimension of K does not exceed 2. Consequently by Theorem 4 and [7, Corollary 2, p. 19], the assertion follows.

3. THE SPECTRAL RADIUS THEOREM

Gelfand [9] used complex variable methods in establishing the spectral radius formula. Rickart [15] has given an elementary proof of it. In a complex, commutative Banach algebra with identity, the spectrum of an element x coincides, of course, with the set of all the numbers u(x) where u is a nonzero homomorphism from A to $\mathbb{C}$. Consequently, the spectral radius theorem for complex Banach algebras is contained in the following theorem:

THEOREM 7. *(Spectral Radius Theorem) Let* K *be either* $\mathbb{R}$ *or* $\mathbb{C}$, *equipped with the absolute value* $|\cdot\cdot|^p$ *where* $0 < p \leq 1$. *Let* N *be the norm of a commutative Banach algebra* A *with identity over* K, *and let* Δ_A *be the set of all nonzero homomorphisms from* A *into the* K*-algebra* $\mathbb{C}$. *For each* $x \in A$,

$$\sup_{u \in \Delta_A} |u(x)|^p = N_s(x).$$

Proof. As is well known, every maximal ideal of A is closed, so as the kernel of each $u \in \Delta_A$ is a maximal ideal of finite codimension (for its range must be either K or $\mathbb{C}$), each $u \in \Delta_A$ is continuous and therefore has norm 1. Consequently, by a familiar argument, $|u(x)|^p \leq N_s(x)$ for every $u \in \Delta_A$.

To prove the converse, let $c \in A$ be such that $N_s(c) \neq 0$. By Theorem 1, there is an absolute semivalue V_c on A such that $V_c(c) = N_s(c)$ and V_c agrees with N_s on its core. In particular, $V_c(\lambda.1) = N_s(\lambda.1) = |\lambda|^p$ for all $\lambda \in K$, so $V_c(\lambda.x) = |\lambda|^p V_c(x)$ for all $\lambda \in K$, $x \in V$. Therefore the prime ideal $V_c^{-1}(0)$ is an algebra ideal; let $P = V_c^{-1}(0)$, let B be the K-algebra A/P, and let $\bar{V}_c$ be the absolute value on B satisfying $\bar{V}_c(x + P) = V_c(x)$ for all $x \in A$. Let V_c' be the absolute value on the quotient field L of B induced by $\bar{V}_c$. We make L into a K-algebra by defining $\lambda.(s/t) = (\lambda.s)/t$ for all $\lambda \in K$, $s, t \in B$, $t \neq 0$. Clearly $V_c'(\lambda.(s/t) = |\lambda|^p V_c'(s/t)$ for all $\lambda \in K$ and all $s, t \in B$, $t \neq 0$. Thus V_c' is an algebra norm on L. By Theorem 6, there is a K-isomorphism w from L onto either K or $\mathbb{C}$. As w(B) is a K-subalgebra, w(B) is a field and hence B is also. Thus B = L, and $V_c' = \bar{V}_c$. In particular, the dimension of B does not exceed 2. As $s \mapsto |w(s)|^p$ and $\bar{V}_c$ are absolute values on B that agree on K.(1 + P) and as K.(1 + P) is complete, by Theorem 2 they coincide on all of B. Let $v: x \mapsto w(x + P)$, $x \in A$. Then $v \in \Delta_A$, and $|v(x)|^p = V_c(x)$ for all $x \in A$; in particular, $|v(c)|^p = V_c(c) = N_s(c)$.

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