SOME QUOTIENT VARIETIES HAVE RATIONAL SINGULARITIES

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Let $\mathbb{V}$ be a representation of a reductive algebraic group G over a field k of characteristic zero. The quotient variety $G\backslash\mathbb{V}$ is the affine variety, whose regular functions are the regular functions on $\mathbb{V}$, which are invariant under G.

The famous theorem of Hochster-Roberts [7] asserts that $G\backslash\mathbb{V}$ is a Cohen-Macaulay variety. In this note, I will demonstrate a generalization of the following

THEOREM 0. *The variety* $G\backslash\mathbb{V}$ *has rational singularities if the connected component of* G *is a semi-simple group.*

The essential case for the proof is when G is semi-simple (*cf.* Lemma 8). One may also prove the same result when the connected component of G is a torus. (See [6] and [9]). Of course, one would like to have the same result for an arbitrary reductive group G acting on an affine variety with (as weak as possible) conditions on its singularities.

1. PROPERTIES OF DIFFERENTIALS

In this paper, we will be working in the category of k-schemes of finite type over a field k, which is algebraically closed. All points are k-points. A variety is a k-scheme, which is separated, reduced and irreducible. Thus, the set X_{sing} of singular points of a variety X forms a closed subset of X, which is not all of X.

A variety X is normal, by definition, if and only if all its local rings $\mathcal{O}_{X,x}$ are integrally closed. Recall that, if X is a normal variety, the codimension of X_{sing} in X is at least two. In fact, the depth of $\mathcal{O}_X$ along X_{sing} is at least two.

If X is a normal variety, let $i: U \equiv X - X_{sing} \hookrightarrow X$ be the open immersion of the smooth part U of X. The sheaf ω_X of regular dualizing differentials is defined to be the sheaf $i_*(\Omega_U^{\dim U})$ on X. If X is smooth, then ω_X equals the sheaf of differential forms $\Omega_X^{\dim U}$ of highest degree. In general, a rational dualizing differential (*i.e.*, a rational section of ω_X) on a normal variety is regular (*i.e.*, a section of ω_X) if and only if it has no poles. For further information, the reader should consult [3].

Let $f: X \to Y$ be a birational morphism between normal varieties. We may identify a rational dualizing differential on X with one on Y. If ω is a regular dualizing differential on Y, the $f^*\omega$ may only have poles along divisors D in X such that f(D) is contained in the singularities of Y. Conversely, if ω is a regular differential on X, then $f_*\omega$ has no pole along any divisor E on Y such that $E \cap f(X)$ is dense in E.

Let $f: X' \to X$ be a proper birational morphism between normal varities. By the last remark, $f_*\omega_{X'} \subset \omega_X$. Define the sheaf K_X of absolutely regular dualizing

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differentials to be the intersection $K_{X'} \equiv \bigcap f_* \omega_{X'}$, where we take the intersection for all such morphisms f. Clearly, in the above situation, we always have the equality $f_* K_{X'} = K_X$.

If X is smooth, then $K_X = \omega_X = \Omega_X^{\dim X}$. So, if $f: X' \to X$ is a proper birational morphism, where X' is smooth and X is normal, then $K_X = f_* \omega_{X'}$, which is a coherent sheaf on X.

By Hironaka's theorem [5] in characteristic zero, we are assured that such a resolution f of the normal variety X exists. Thus, we have the following criterion for a rational dualizing differential to be absolutely regular.

LEMMA 1 (Characteristic zero). *Let ω be a rational section of ω_X on a normal variety* X. *Then, ω is absolutely regular (i.e., $\omega \in K_X$) if and only if, for any birational morphism $f: S \to X$, where* S *is a smooth affine variety, $f^* \omega$ is a regular section of ω_S.*

The reader should also consult Serre's discussion in [13].

Next, I want to recall the definition of rational singularities [8], [9]. A variety X in characteristic zero is said to have rational singularities if X is a normal Cohen-Macaulay variety such that $\omega_X = K_X$. (That is to say that any regular dualizing differential is absolutely regular.)

Lastly, we come to the main technical tool in this paper which does not require invariant theory.

LEMMA 2. *Let $f: X \to Y$ be a surjective morphism between two varieties in characteristic zero. Assume that* Y *is smooth and* X *is normal. Let ω be a rational differential on* Y *(i.e., a rational section of the sheaf Ω_Y^i for some* i*). Then, ω is regular if and only if the pull-backed rational differential $f^* \omega$ has no poles on* X.

Proof. The forward implication is evident. Contrapositively, if ω is not regular, then we will see that $f^*\omega$ must have a pole. As Y is smooth variety and ω is not regular, there must be an irreducible divisor F on Y along which ω has a pole. Furthermore, locally we may find a regular function t on Y such that t generates the ideal of F.

Let E be the inverse image $f^{-1}(F)$. As E is locally defined by the equation $f^*t = 0$ and is not all of X, E has codimension one in X. The mapping $f: E \to F$ is surjective. Thus, we may find an irreducible component D of E such that the image of D contains an open dense subset of F. We intend to show that $f^*\omega$ has a pole along D.

For the rest of the proof, we will fix a point d of D, which is general enough to satisfy the following conditions:

a) D and X are smooth near d,

b) the ideal of D is generated near d by a function s on X, which is regular at d,

c) $f^*t = v \cdot s^r$, where v is a unit near d and r is a positive integer,

d) F is smooth near f(d), and

e) if $t, y_1, \cdots, y_k$ are regular parameter functions on Y at the point f(d), then $x_i = f^*y_i$ have independent differentials on D near d.

The existence of such a point d is fairly easy. The hardest part is e), which follows from Sard's lemma applied to the almost surjective morphism $D \to F$. I leave the details to the reader.

As a direct consequence of the above, we may find other functions $z_1, \cdots, z_e$ on X such that $s, x_1, \cdots, x_k, z_1, \cdots, z_e$ are regular parameters on X at our general point d. To prove that $f^*\omega$ has a pole along D, we will begin with some special cases.

Assume that ω has either of the following forms:

$$\frac{1}{t^n}\, dt \wedge dy_1 \wedge \cdots \wedge dy_{i-1} \quad \text{or} \quad \frac{1}{t^n}\, dy_1 \wedge \cdots \wedge dy_i .$$

In the first case, $f^*\omega$ is $\frac{r \cdot v \cdot s^{r-1}}{(v \cdot s^r)^n}\, ds \wedge dx_1 \wedge \cdots \wedge dx_{i-1}$ plus a term with a pole along D of order less than or equal to $r(n-1)$. In the second case, $f^*\omega$ is $\frac{1}{(v \cdot s^r)^n}\, dx_1 \wedge \cdots \wedge dx_i$. Therefore, in general, if n is the order of the pole of ω along F, then the order of the pole of $f^*\omega$ along D is either $r \cdot n$ or $r(n-1)+1$. As these orders are positive, we are done.

2. INVARIANT THEORY

I will begin by recalling some basic notions. Let G be a reductive algebraic group over a field $k = \bar{k}$ of characteristic zero. Let X be an algebraic k-scheme. A group action of G on X is a morphism $G \times X \to X$ satisfying the usual rules for a group action.

If X is an affine scheme with a G-action, the quotient scheme $G\backslash X$ is the affine scheme where $\Gamma(G\backslash X, \mathcal{O}_{G\backslash X})$ is the ring of G-invariants ${}^G\Gamma(X, \mathcal{O}_X)$ in $\Gamma(X, \mathcal{O}_X)$. We have an obvious quotient morphism $\pi: X \to G\backslash X$ corresponding to the inclusion between the two rings.

The quotient morphism π has many pleasant properties. We will next recall

LEMMA 3. a) *π is surjective.*

b) *π takes G-invariant closed subsets of X onto closed subsets of $G\backslash X$.*

c) *If X is a normal variety, then $G\backslash X$ is a normal variety.*

d) *Let* $\begin{array}{ccc} T & \to & X \\ \downarrow & & \downarrow \\ S & \to & G\backslash X \end{array}$ *be a cartesian square of affine k-schemes. Then, S is naturally isomorphic to the quotient $G\backslash T$, where T is given the G-action induced by the G-action on X.*

For the proof, see Theorem 1.1 in Mumford's book [11]. We will use the above results in the form of

COROLLARY 4. *Given a G-action on a normal affine variety X and a birational morphism $S \to G\backslash X$ from an affine variety S. We may find a commutative diagram,*

$$\begin{array}{ccc} R & \longrightarrow & X \\ \downarrow & & \downarrow \\ S & \longrightarrow & G\backslash X \end{array}$$

where R *is a normal affine variety, which is birational to* X *and maps surjectively onto* S.

Proof. Let $R' = S \times_{G\backslash X} X$. Let R equal the normalization of the component of R', which is mapped birationally to X. R exists because S is birational to G\X and X dominates G\X. Furthermore, the image of R in S is dense.

On the other hand, the image of R in R' is a closed G-invariant subset. Hence, by the lemma, the image of R in S is closed. Thus, the morphism $R \to S$ is surjective.

Remark. If S is normal, S = G\R, where R is given the induced G-action.

The next lemma deals with codimension one behavior. Most of it is well-known [2], [10].

LEMMA 5. *Let* X *be an affine* k-*scheme such that* $\Gamma(X, \mathcal{O}_X)$ *is a unique factorization domain and has only constants as units. Assume that* (1) *we are given a* G-*action on* X *and* (2) *all characters of* G *are constant on the connected components of* G.

a) *Let* D *be an effective* G-*invariant divisor on* X. *Then,* πD *is the support of an effective principal divisor* E *on* G\X *such that* D *is the set-theoretic inverse image of* E.

b) *If* G *is connected, in the above situation, we may find a unique divisor* E *such that* $\pi^{-1}E = D$ *as divisors: hence,* $\Gamma(G\backslash X, \mathcal{O}_{G\backslash X})$ *is a unique factorization domain.*

c) *For any such* G, *if* ω *is a rational differential form on* G\X, *which has no poles, then* $\pi^*\omega$ *has no poles.*

Proof. (a) As $\Gamma(X, \mathcal{O}_X)$ is factorial, we may find a regular function f on X such that D is its divisor. f is determined up to constant multiple by the assumption on the units. Thus, as D is G-invariant, $f(gx) = \chi(g)f(x)$, where χ is a character of G. By the assumption on G, the values of χ are m-th roots of unity, where m is some number which divides the number of connected components of the variety G. Thus, f^m is a G-invariant function on X. The divisor E of f^m regarded as a regular function on G\X has the required properties.

b) In the above notation, we may take m = 1. This gives the first statement. Let g be a regular function on G\X. As any factor of π^*g corresponds to G-invariant divisor on X, the first statements show that π^*g is irreducible if and only if g is irreducible. This implies the second statement.

c) $\pi^*\omega$ is a G-invariant differential form on X. If it has any poles, they must be situated along a G-invariant divisor D. By (a), πD is a divisor on the normal variety G\X and $D = \pi^{-1}\pi D$. As ω has no poles along πD, it cannot have any poles along $D = \pi^{-1}\pi D$. Thus, $\pi^*\omega$ has no poles.

Remark. For the proof of (c), the above assumptions are a rather brutal way to insure that π(G-invariant divisor on X) is a divisor on the quotient G\X.

The next results bring together the above information.

PROPOSITION 6. *Let* X *be a smooth affine variety with* G-*action. Assume that*

1) $\Gamma(X, \mathcal{O}_X)$ *is a unique factorization ring with only constants as units; and*

2) G *has no radical. Then, any regular dualizing differential on the normal quotient variety* G\X *is absolutely regular.*

Proof. Let ω be a regular dualizing differential on G\X. By Lemma 5c, we know that $\pi^*\omega$ has no poles on X. Hence, $\pi^*\omega$ is regular differential form on the smooth variety X. We will use the criterion of Lemma 1 to check that ω is absolutely regular.

Let $S \to G\backslash X$ be any birational morphism from a smooth affine variety S. We will use the diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d, "\alpha"] & X \arrow[d, "\pi"] \\ S \arrow[r] & G\backslash X \end{tikzcd}$$

of Corollary 4, where you recall that R is normal and α is surjective. Thus, the pull-back of ω to R via X is regular by the above remarks. Hence, the pull-back ω' to S remains regular under α^*. By Lemma 2, this means that ω' is regular. This is what we need.

We can now state the

THEOREM 7. *With some assumptions as Proposition* 6, *the quotient variety* G\X *has rational singularities.*

Proof. G\X is automatically normal by Lemma 3.c. We have just seen that any regular dualizing differential on G\X is absolutely regular. The hardest part of the result is the Hochster-Roberts theorem [7] which gives in particular: any quotient G\X of a smooth affine variety by a reductive group is a Cohen-Macaulay variety (Char. 0). Hence, G\X has all three of the properties in the definition of rational singularities.

I want to remind the reader that if G is connected (*i.e.* semisimple), one may add the properties:

a) G\X is Gorenstein and

b) $\Gamma(G\backslash X, \mathcal{O}_{G\backslash X})$ is a UFD.

This result was already noted by Hochster-Roberts, but it is easily understood in terms of the methods of this paper. In fact, (b) follows from Lemma 5.6. As we know that G\X is a factorial Cohen-Macaulay variety, Murthy has proven that these properties for a variety imply that it is Gorenstein (*i.e.*, $\omega_{G\backslash X} \approx \cdot \mathcal{O}_{G\backslash X}$). Murthy's point is that $\omega_{G\backslash X}$ is a reflexive rank-one coherent sheaf on a factorial variety and, hence, the freeness follows from the factorial property [12].

Next, I want to point out a rather well-known result (see [1] for instance) when G is a finite group.

LEMMA 8. *Let* G *be a finite group of order not divisible by the characteristic of the ground field. If* G *acts on an affine scheme* Y, *then*

a) *the projection* $Y \to G\backslash Y$ *is a finite pure morphism,*

b) $G\backslash Y$ *is Cohen-Macaulay if* Y *is, and*

c) $G\backslash Y$ *has rational singularities if* Y *does.*

This is trivial compared to the last theorem. I leave the proof to the reader.

I will make a remark which might be helpful to researchers of questions related to this paper. The results of Luna [10] on the behavior of a group acting along a closed orbit give a better picture of Hochster-Roberts' "reduction to the graded case".

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