AN EXAMPLE OF TORRIGIANI RELATED TO MULTIPLE FOURIER SERIES

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A function of n variables is in W_p^1 if it is absolutely continuous in each variable for almost all values of the other variables and if both it and its partial derivatives are in L_p; it is in V_p if it is of bounded variation in each variable for almost all values of the other variables and if the n variation functions are in L_p, each as a function of the other $n - 1$ variables. For each $p \geq 1$, $V_p \supset W_p^1$.

For the case $n = 2$, L. Cesari [1, Theorem 3, p. 290] showed that if $f \in V_1$, then the rectangular sums of its double Fourier series converge almost everywhere. Subsequently, L. Tonelli [4, p. 325] suggested a new proof of this. For the case $n = 3$, Cesari [2] also showed that if $p > 1$ and $f \in V_p$, then the rectangular sums of the triple Fourier series of f converge almost everywhere. The issue whether Cesari's result holds for $p = 1$ and $n = 3$ remains unresolved. Evidence toward a negative answer is furnished by an example of G. Torrigiani [5], for $n = 3$, of a function $f \in V_1$ possessing a certain property nowhere. For $n = 2$, each $f \in V_1$ has this property almost everywhere. It is used by Tonelli [4] to show that for $n = 2$ the double Fourier series of each $f \in V_1$ converges almost everywhere. The example given by Torrigiani is long and involved. In view of the revived interest in this topic, we feel that it is worthwhile to give a short, simple discussion, which, however, is based on Torrigiani's idea.

Let Q_3 be the unit cube, and Q_2 the unit square. For each interval $[a, b] \subset [0, 1]$, each function f, and each point $(x, y) \in Q_2$, let $V_z(f; x, y; [a, b])$ denote the variation of f as a function of z on the interval $[a, b]$ with x and y fixed. Since we shall be dealing only with absolutely continuous functions, we need not worry about jumps at the endpoints. Tonelli's condition for functions of two variables at a point (x_0, y_0) is that for each $\varepsilon > 0$ there exists a $\lambda > 0$ such that $0 < \delta < \lambda$ implies

$$\frac{1}{2\delta} \int_{y_0-\delta}^{y_0+\delta} V_x(f; y_0; [x_0, x_0 + \lambda])\, dy < \varepsilon .$$

He showed that if $f \in V_1$, the condition is satisfied almost everywhere, and this implies that the Fourier series of f converges almost everywhere.

A function g is an *S-function* (after S. Saks, see [3]) if it is summable and

$$\limsup_{h,k \to 0} \frac{1}{4hk} \int_{x-h}^{x+h} \int_{y-k}^{y+k} g(u, v)\, du\, dv = +\infty ,$$

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for each $(x, y) \in Q_2$. Saks showed that there exists a sequence $\{g_n\}$ of piecewise linear (continuous) functions on Q_2 that are 0 on ∂Q_2 and such that $g = \sum_{n=1}^{\infty} g_n$ is an S-function.

Define f on Q_3 in terms of the functions g_n as follows. Let

$$M_n = \max[g_n(x, y): (x, y) \in Q_2];$$

let r_n be such that the x-axis can be partitioned into r_n intervals on each of which g_n is linear in x for every fixed y, and similarly in the other direction; let $k_n = 2(n+1)r_n M_n$; and let

$$z_{nj} = 1 - \frac{1}{n} + \frac{j}{n(n+1)k_n} \qquad (j = 0, 1, \cdots, k_n;\ n = 1, 2, \cdots).$$

Define

$$f(x, y, z_{nj}) = \begin{cases} \frac{1}{k_n} g_n(x, y) & (j = 1, 3, \cdots, k_n - 1), \\ 0 & (j = 2, 4, \cdots, k_n), \end{cases}$$

let $f(x_0, y_0, z)$ be linear on each segment $x = x_0$, $y = y_0$, $z_{nj} \leq z \leq z_{n,j+1}$, and let $f(x, y, 1) = 0$. Then f is continuous and takes the value 0 on ∂Q_3. Moreover,

$$V_y(f; x_0, z_0; [0, 1]) \leq 1 \qquad \text{for each } (x_0, z_0) \in Q_2$$

and

$$V_x(f; y_0, z_0; [0, 1]) \leq 1 \qquad \text{for each } (y_0, z_0) \in Q_2.$$

Finally, $V_z(f; x, y; [0, 1]) = g(x, y)$, so that V_z is an S-function. It is easy to show that $f \in W_1^1$.

We use the function f to obtain a more poorly behaved function F. For each $J = [a, b] \subset [0, 1]$, let f_J be defined in the domain $Q_2 \times J$ by the formula

$$f_J(x, y, z) = f(x, y, a + (b - a)z).$$

Let $f_1 = f$. Assume that $f_1, \cdots, f_{n-1}$ have been defined, and divide $[0, 1]$ into n equal intervals $I_{n1}, \cdots, I_{nn}$. For each pair of indices n and i, there exists a $J_{ni} \subset I_{ni}$ such that for all $(x, y) \in Q_2$, each of the functions $f_1, \cdots, f_{n-1}$ is monotonic in z on J_{ni}. We define f_n as $2^{-n} f_{J_{ni}}$ whenever $z \in J_{ni}$, and as 0 whenever $z \notin \bigcup_{i=1}^{n} J_{ni}$. It follows that each function $V_z(f_n; x, y; J_{ni})$ is an S-function. Let $F = \sum_{n=1}^{\infty} f_n$. For each $[a, b] \subset [0, 1]$, there is a $J_{ni} \subset [a, b]$, and

$$V_z(F; x, y; [a, b]) \geq V_z(f_n; x, y; J_{ni})$$

for all $(x, y) \in Q_2$. Therefore $V_z(F; x, y; [a, b])$ is an S-function.

The series $\sum_{n=1}^{\infty} f_n$ converges in the W_1^1-norm, so that $F \in W_1^1$. Torrigiani's example is in V_1. Our discussion yields the following slight extension of his result.

THEOREM. *There exists a function* F *in* W_1^1 *on* Q_3 *that is* 0 *on* ∂Q_3 *and such that for every point* $(x, y, z) \in Q_3$ *and every* $\lambda > 0$,

$$\limsup_{h,k \to 0} \frac{1}{4hk} \int_{x-h}^{x+h} \int_{y-k}^{y+k} V_z(F; u, v; [z, z+\lambda])\,du\,dv = +\infty .$$

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