THE MARX CONJECTURE FOR STARLIKE FUNCTIONS

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1. INTRODUCTION

In 1932, A. Marx [4] conjectured that for each fixed z_0 in the unit disc, the set of possible values of $f'(z_0)$ for all f in the class of starlike functions is contained in the set of all values of $k'(z)$ for $|z| \leq |z_0|$, where $k(z) = z/(1-z)^2$ is the Koebe function. Marx showed that this must be true for $|z_0| \leq \sin \pi/8 = 0.382 \cdots$. R. M. Robinson [5], [6] improved the results of Marx, and more recently, P. L. Duren [1] proved that the conjecture holds for $|z_0| \leq 0.736 \cdots$.

In this paper we present a counterexample, which shows that the conjecture is in fact false. We also prove that every point on the boundary of the set of possible values is given by a function having at most two slits; this simplifies the problem considerably.

First, we must state precisely the form of the problem as we shall investigate it. Following Robinson [5], [6], we replace Marx's original formulation of the problem by the investigation of the domain of variability of $\log f'(z_0)$. This has the advantage of making the mappings involved univalent, at the expense of requiring care in keeping track of the proper branch of the logarithm.

2. THE MARX REGION

Let U denote the unit disc $\{z: |z| < 1\}$. Let $\mathcal{S}^*$ denote the class of *starlike functions*, that is, the class of all functions that are regular and univalent in U with $f(0) = 0$, $f'(0) = 1$, and that map U onto a domain starshaped with respect to the origin. For $z_0 \in U$, we define the *Marx region* for z_0 as

$$M(z_0) = \{w: w = \log f'(z_0),\ f \in \mathcal{S}^*\}. \tag{2.1}$$

The determination of the branch of the logarithm is fixed by the specification that $0 \in M(z_0)$ and that $f_t(z) = \frac{1}{t} f(tz)$ $(0 < t \leq 1)$. Then $f_1(z) = f(z)$, and since $f_t(z) = z + a_2 tz^2 + \cdots$, we may let $f_0(z) = z$. Each f_t is in $\mathcal{S}^*$; hence $\log f_t'(z_0)$ $(0 \leq t \leq 1)$ defines a path in $M(z_0)$ joining 0 to $\log f'(z_0)$.

The set $M(z_0)$ is bounded, because of the well-known bounds on $f'(z)$ for $f \in \mathcal{S}^*$. Since $\mathcal{S}^*$ is a normal family, $M(z_0)$ is closed. It is symmetric about the real axis; for if $f(z) = \sum a_n z^n$ is in $\mathcal{S}^*$, then $\sum \bar{a}_n z^n$ is in $\mathcal{S}^*$.

The set $M(z_0)$ depends only on $r = |z_0|$. To see this, note that $f \in \mathcal{S}^*$ implies $e^{-i\alpha} f(e^{i\alpha} z) \in \mathcal{S}^*$ for every real α. From this one easily shows $M(z_0) = M(|z_0|)$.

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For each r $(0 < r < 1)$, define

$$K(r) = \{w\colon w = \log k'(z),\ |z| \leq r\}, \tag{2.2}$$

where $k(z) = z/(1 - z)^2$ is the Koebe function. Thus, $k'(z) = (1 + z)/(1 - z)^3$. Again we must specify the branch of the logarithm. We do so by requiring that 0 is in the set and letting the path $\log k'(tz)$ $(0 \leq t \leq 1)$ continuously define the branch.

Duren [1] observes that the mapping $\log k'(z)$ is starlike, and he proves that it is convex for $|z| < 0.886 \cdots$. It is difficult to prove either of these facts, but one easily sees that the mapping is univalent, by considering the image of the unit circle $z = e^{i\theta}$ under the mapping

$$w = \log k'(z) = \log[1 + z)/(1 - z)] - 2 \log(1 - z).$$

As θ varies from 0 to π, $(1 + z)/(1 - z)$ varies decreasingly along the imaginary axis from ∞ to 0, while $1 - z$ moves from 0 to 2 along the lower half of the circle with center 1 and radius 1. It follows that the real part of w decreases from $+\infty$ to $-\infty$, while the imaginary part of w decreases from $3\pi/2$ to $\pi/2$.

The Marx conjecture is equivalent to the statement that

$$M(r) = K(r) \quad (0 < r < 1). \tag{2.3}$$

3. THE BOUNDARY OF THE MARX REGION

THEOREM. *Choose* r $(0 < r < 1)$, *and let* w_0 *be a boundary point of* $M(r)$. *Then* $w_0 = \log f_0'(r)$, *where* f_0 *is some function in* $\mathscr{S}^*$ *mapping* U *onto the exterior of at most two radial slits. That is,*

$$f_0(z) = z/(1 - e^{i\alpha_1} z)^{2s} (1 - e^{i\alpha_2} z)^{2-2s},$$

where α_1, α_2, *and* s *are real and* $0 \leq s \leq 1$.

Remark. The author has been told that this result has also been obtained by Z. Lewandowski and A. Wesołowski.

Proof. Since $M(r)$ is closed, there exists at least one function $f_0 \in \mathscr{S}^*$ that will produce the boundary value $w_0 = \log f_0'(r)$. Set

$$f_0(r) = A, \quad f_0'(r) = B,$$

where $B = \exp w_0$. Define

$$D(r, A) = \{w\colon w = f'(r),\ f(r) = A,\ f \in \mathscr{S}^*\}.$$

The point B is in this set, so that the set is not empty. Moreover, $D(r, A)$ is clearly closed and bounded.

If f_1 and f_2 are two functions in $\mathscr{S}^*$, then $h = f_1^{(1-t)} f_2^t$ is in $\mathscr{S}^*$ for each real t $(0 \leq t \leq 1)$, if we choose the proper branches so as to obtain the correct normalization at 0. This follows from the relation

$$\frac{z\,h'(z)}{h(z)} = (1 - t)\,\frac{z\,f_1'(z)}{f_1(z)} + t\,\frac{z\,f_2'(z)}{f_2(z)} \tag{3.1}$$

and the fact that a normalized f is in $\mathscr{S}^*$ if and only if $z f'(z)/f(z)$ has a positive real part in U.

If $f_1(r) = f_2(r) = A$, then $h(r) = A$ also, and (3.1) thus shows that $D(r, A)$ is a convex set.

Next we show that B is a boundary point of $D(r, A)$. Suppose to the contrary that the open set $V = \{w: |w - B| < \rho\}$ is contained in $D(r, A)$. For each $\omega_1 \in V$, there must exist an $f_1 \in \mathscr{S}^*$ with $f_1'(r) = \omega_1$ and $f_1(r) = A$. Let $h_s = f_0^{(1-s)} f_1^s$ $(0 \leq s \leq 1)$. Then the derivatives $h_s'(r)$ determine a path from B to ω_1 in V, and $\log h_s'(r)$ will thus vary continuously from w_0 to $\log \omega_1$. It follows that the mapping $w = \log \omega$ carries V onto a neighborhood of w_0. This contradicts the assumption that w_0 is a boundary point of $M(r)$.

Since B is a boundary point of the compact, convex region $D(r, A)$, it must lie on a supporting line. That is, there must exist some $\lambda = e^{i\alpha}$ (α real) such that

$$\Re\,\lambda B = \max\{\Re\lambda\omega: \omega \in D(r, A)\}.$$

This is equivalent to the assertion that f_0 is a solution of the extremal problem

$$\max\{\Re\,\lambda f'(r): f(r) = A,\ f \in \mathscr{S}^*\}. \tag{3.2}$$

We observe that f_0 might not be the only solution; but this causes no difficulty.

Note that (3.2) is an extremal problem with a constraint. The method of Lagrange multipliers can be applied to show that there exists a complex number λ_1 such that f_0 is also a (local) solution to the extremal problem

$$\max\{\Re[\lambda f'(r) + \lambda_1 f(r)]: f \in \mathscr{S}^*\}. \tag{3.3}$$

(The applicability of the method of Lagrange multipliers will be discussed in Section 5, at the end of this paper.)

Theorem 3 of [3] then applies and shows that f_0, the solution of the extremal problem (3.3) must be precisely of the form specified in the conclusion of the theorem. This proves the theorem.

4. THE COUNTEREXAMPLE

The boundary of the region $K(r)$ of (2.2) is the image of $|z| = r$ by the mapping

$$b(z) = \log\frac{1 + z}{(1 - z)^3}. \tag{4.1}$$

Let $z = re^{i\theta}$; then the inward normal to the boundary is given by $-\partial b(z)/\partial r$. Except for the factor $2/r$, which is real and positive, this is

$$n(z) = -\frac{z(2 + z)}{(1 - z^2)}. \tag{4.2}$$

Notice that since $n(z) \neq 0$, the boundary of $K(r)$ $(0 < r < 1)$ is an analytic curve.

Every two-slit function $f(z)$ of the type described in the theorem of Section 3 can be written as a combination of two Koebe functions:

$$f(z) = \left(\frac{z}{(1 - e^{i\alpha_1} z)^2}\right)^{1-s} \left(\frac{z}{(1 - e^{i\alpha_2} z)^2}\right)^{s}$$

As s varies from 0 to 1, we obtain a path that joins the two Koebe functions in the space of starlike functions. Therefore $\log f'(r)$ also follows a path in $M(r)$ joining two points on the boundary of $K(r)$. If we set

$$z_1 = re^{i\alpha_1} \quad \text{and} \quad z_2 = re^{i\alpha_2},$$

we easily compute that

$$(4.3)\quad \log f'(r) = \log \frac{1 + z_1}{(1 - z_1)^3} + 2s \log\left(\frac{1 - z_1}{1 - z_2}\right) + \log\left[1 + 2s\,\frac{(z_2 - z_1)}{(1 + z_1)(1 - z_2)}\right].$$

The proper branch of the logarithms must be chosen in a continuous manner. We accomplish this in (4.1) and in the first term of the right-hand side of (4.3) by choosing

$$|\arg(1 + z)| < \pi/2 \quad \text{and} \quad |\arg(1 - z)| < \pi/2$$

and letting

$$\Im b(z) = \arg(1 + z) - 3\arg(1 - z).$$

In the second term of (4.3), we similarly choose $|\arg(1 - z_1)| < \pi/2$ and $|\arg(1 - z_2)| < \pi/2$. The third term is more difficult. However, we see that if we fix $z_1 \in U$, then for each $z_2 \in U$ the value of the expression inside the brackets lies in the half-plane not containing 0 and bounded by the line through $1 - s$ making an angle

$$\beta = \arg\left\{\frac{1}{i}\,\frac{(z_1 - 1)}{(z_1 + 1)}\right\}$$

with the real axis. We see that $0 < \beta_1 < \pi$, and hence we may fix the branch of the third term in (4.3) to have an imaginary part between $-\pi$ and $+\pi$. This is exactly the principal value of the logarithm.

The tangent vector to the path defined by (4.3) at $s = 0$ is given by the expression

$$(4.4)\qquad 2t(z_1, z_2) = \frac{2(z_2 - z_1)}{(1 + z_1)(1 - z_2)} + 2\log\left(\frac{1 - z_1}{1 - z_2}\right).$$

In order to determine the angle at which this path leaves the boundary of $K(r)$, we compute the ratio

$$(4.5)\qquad R(z_1, z_2) = \frac{t(z_1, z_2)}{n(z_1)} = \frac{(z_1 - z_2)(1 - z_1)}{z_1(2 + z_1)(1 - z_2)} - \frac{(1 - z_1^2)}{z_1(2 + z_1)} \log\left(\frac{1 - z_1}{1 - z_2}\right).$$

If $R(z_1, z_2)$ has a positive real part, the path defined by (4.3) departs from $b(z_1)$ in a direction leading into the interior of $K(r)$. If $\Re R(z_1, z_2) < 0$, then the path must go outside the Koebe region. That is, if there exist an r $(0 < r < 1)$ and two complex numbers z_1 and z_2 with $|z_1| = |z_2| = r$ such that $\Re R(z_1, z_2) < 0$, then there exists a two-slit function f(z) such that $\log f'(r)$ is outside of the region K(r). This would be a counterexample to the Marx conjecture.

With the help of a computer, we have investigated the function $R(z_1, z_2)$ for various values of r. When we found values that made the real part negative, we searched further to try to find the best values. That is, since

$$T(z_1, z_2) = \frac{\Re\{R(z_1, z_2)\}}{|\Im\{R(z_1, z_2)\}|}$$

is the cotangent of the angle the path makes with the inward normal, we adjusted the values $\alpha_1 = \arg z_1$ and $\alpha_2 = \arg z_2$ to obtain the minimum T. Some representative results of the computations are listed in Table 1.

r	Min T	α_1	α_2
0.99	-0.303965	2.346	6.177
0.96	-0.084978	2.548	6.006
0.95	-0.040465	2.596	5.944
0.94	-0.002789	2.641	5.874
0.93	+0.029330	2.688	5.787

Table 1.

The computations suggest that a counterexample of the specified form exists for each $r \geq 0.94$. Since the proof of the accuracy of numerical computations is tedious at best, we offer here an example that can be checked by hand.

Using the machine computations as a guide, we set

$$z_1 = -\frac{20}{29} + i\,\frac{21}{29}, \qquad z_2 = \frac{840}{841} - i\,\frac{41}{841}.$$

These two points are both on the unit circle. From (4.5) we compute the expression

$$R(z_1, z_2) = \left(\frac{-390{,}224 + 48{,}778\,i}{317{,}057}\right)\left(\frac{812 + 2009\,i}{58}\right) + \left(\frac{882 + 1596\,i}{1885}\right)\left(\frac{1}{2}\log\frac{1{,}173{,}845}{841} - i\theta\right),$$

where $\theta = \pi - \operatorname{Arctan}\frac{2009}{812}$. This gives the result

$$R(z_1, z_2) = -19.2629\cdots - (38.759\cdots)\,i,$$

and there is no question that the real part is negative. The continuity of $R(z_1, z_2)$ then shows that there must exist z_1 and z_2 with $|z_1| = |z_2| = r < 1$ such that $\Re R(z_1, z_2) < 0$. Hence the Marx conjecture is false.

Figure 1 shows a sketch of the region K(0.99). The lack of convexity is evident. The dotted curve labeled γ_1 shows the path defined by (4.3) from $b(z_1)$ to $b(z_2)$, where

$$|z_1| = |z_2| = 0.99, \qquad \arg z_1 = \alpha_1 = 2.346, \qquad \arg z_2 = \alpha_2 = 6.177.$$

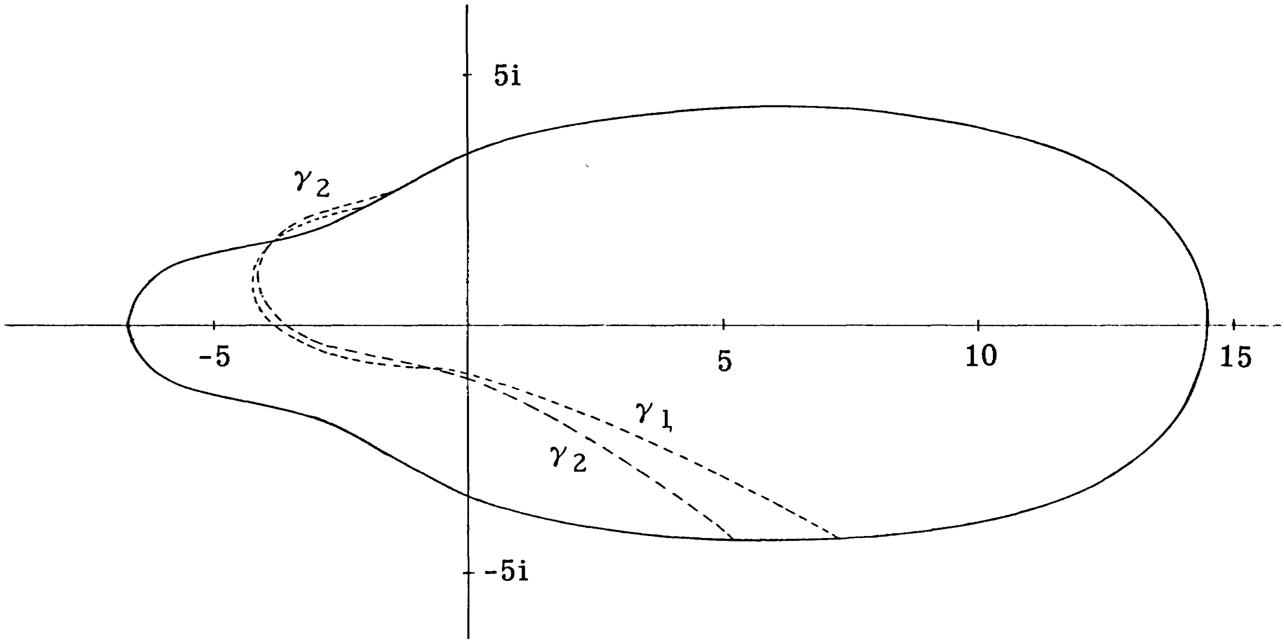

Figure 1.

A second curve, labeled γ_2, is similar, except that the endpoints are defined by $\alpha_1 = 1.9889$ and $\alpha_2 = 6.0650$. This curve appears to be the one that goes farthest outside K(0.99). It is clear from the figure that all points on the two curves lie within the convex hull of M(r). We have computed other curves for various endpoints α_1 and α_2 and have found none that goes outside this convex hull. Thus, one is led to the conjecture that M(r) is contained in the convex hull of K(r).

However, a careful search of examples for $r = 0.99$ found no two-slit functions that come closer to filling out the convex hull of K(r) than the curve γ_2. It therefore seems unlikely that M(r) is actually the convex hull of K(r).

5. REMARKS CONCERNING LAGRANGE MULTIPLIERS

The method of Lagrange multipliers is not commonly applied to extremal problems involving complex quantities as in Section 3. There are also some complications in the use of the function class $\mathscr{S}^*$, since this is not a linear space. In this section we give some of the details of the proof that a function f_0 extremal with respect to (3.2) must also be extremal with respect to (3.3).

To be more precise, we shall show that corresponding to every solution f_0 of the extremal problem (3.2), there exists a complex number λ_1 such that if $f = f_0 + \varepsilon\phi + o(\varepsilon)$ (where ϕ denotes an "allowable" variation of the type used in the proof of Theorem 1 in [3]) is in $\mathscr{S}^*$, then

$$\Re\{\lambda f'(r) + \lambda_1 f(r)\} = \Re\{\lambda f_0'(r) + \lambda_1 f_0(r)\} + o(\varepsilon).$$

This is all we need in the proof of Theorem 3 of [3] to establish that f_0 has the desired form.

In its usual form, the method of Lagrange multipliers applies only to linear spaces; but the set of functions in $\mathscr{S}^*$ does not form a linear space. However, an inspection shows that the proof of Theorem 3 in [3] requires only the knowledge that f_0 is extremal with respect to a set of allowable variations. In [2] and [3], it is shown that if f_0 is a function in $\mathcal{S}^*$, ζ is a point of U, and θ is a real number, then there exists a function $\delta f(z;\, \theta,\, \zeta)$, such that to each positive ε there corresponds a function $f \in \mathcal{S}^*$ of the form

$$f(z) = f_0(z) + \varepsilon\,\delta f(z;\, \theta,\, \zeta) + o(\varepsilon)\,. \tag{5.1}$$

The δf used here is that defined in [2], except that here we include neither the ε nor the $o(\varepsilon)$. In our case, the $o(\varepsilon)$-term is uniform for all θ and ζ and for all z in a compact subset of U. While this is not specifically shown in [2], it is easily deduced from the fact that the Schiffer variation has the same property.

Let $\mathcal{V}$ be the real linear space whose elements are finite linear combinations of these $\delta f(z;\, \theta,\, \zeta)$, with real coefficients. Again, the proof in [2] shows that every element of $\mathcal{V}$ is an allowable variation.

Next, we turn $\mathcal{V}$ into an inner-product space. This can be done in many ways, but the following is convenient. For each $\phi \in \mathcal{V}$, set

$$\Phi(z) = \phi\left(\frac{z+r}{1+rz}\right) = \sum_{n=0}^{\infty} c_n(\phi)\, z^n\,.$$

We see that $c_0(\phi) = \phi(r)$ and $c_1(\phi) = (1 - r^2)\,\phi'(r)$. For $n = 0, 1, 2, \cdots$, define

$$\alpha_{2n}(\phi) = \Re\, c_n(\phi)\,,$$

$$\alpha_{2n+1}(\phi) = \Im\, c_n(\phi)\,.$$

With each ϕ in $\mathcal{V}$ we have now associated a real sequence $\{\alpha_n\}_{n=0}^{\infty}$. Fix R $(0 < R < 1)$. Since Φ is analytic in U, $\sum \alpha_n^2 R^n < \infty$, and

$$(\phi,\, \psi) = \sum_{n=0}^{\infty} R^n\, \alpha_n(\phi)\, \alpha_n(\psi)$$

is an inner product on $\mathcal{V}$.

Let $\mathcal{V}_1$ be the completion of $\mathcal{V}$ with respect to the norm $\|\phi\| = (\phi,\, \phi)^{1/2}$. The elements of $\mathcal{V}_1$ may no longer be analytic in all of U; but they are analytic in some neighborhood of r whose size depends only on R. The space $\mathcal{V}_1$ is now a Hilbert space.

Let $\lambda = e^{i\alpha}$ be the complex number used in (3.2). Set $\lambda = \sigma_2 + i\sigma_3$, where σ_2 and σ_3 are real. Then

$$J(\phi) = \Re\,\lambda\,\phi'(r) = \frac{1}{(1 - r^2)}\,[\sigma_2\,\alpha_2(\phi) - \sigma_3\,\alpha_3(\phi)]$$

is a bounded linear functional on $\mathcal{V}_1$. Since $\mathcal{V}_1$ is a Hilbert space, there must exist a $\phi_2 \in \mathcal{V}_1$ such that $J(\phi) = (\phi,\, \phi_2)$ for all $\phi \in \mathcal{V}_1$. (This ϕ_2 need not be the function $[R\sigma_2 - i\sigma_3](z - r)/R^3(1 - r^2)$, since this function may not be in $\mathcal{V}_1$.)

Similarly, there must also exist ϕ_0 and ϕ_1 in $\mathcal{V}_1$ such that for each $\phi \in \mathcal{V}_1$,

$$\alpha_0(\phi) = (\phi,\, \phi_0) \quad \text{and} \quad \alpha_1(\phi) = (\phi,\, \phi_1)\,.$$

Notice that $\phi(r) = (\phi, \phi_0) + i(\phi, \phi_1)$.

Next, let

$$\phi_4 = \phi_2 + \sigma_0 \phi_0 + \sigma_1 \phi_1 , \tag{5.2}$$

where σ_0 and σ_1 are real constants chosen so that $(\phi_4 , \phi_0) = (\phi_4 , \phi_1) = 0$ (this can be done by the Gram-Schmidt process, for example). Then

$$\|\phi_4\|^2 = (\phi_4 , \phi_2 + \sigma_0 \phi_0 + \sigma_1 \phi_1) = (\phi_4 , \phi_2) = J(\phi_4) = \Re \lambda \phi_4'(r) .$$

We shall show that the extremal property assumed for f_0 implies that $\Re \lambda \phi_4'(r) \leq 0$. It follows that $\phi_4 = 0$, and hence, for each $\phi \in \mathscr{V}$,

$$\begin{aligned}(\phi, \phi_4) &= (\phi, \phi_2 + \sigma_0 \phi_0 + \sigma_1 \phi_1) = J(\phi) + \sigma_0 \alpha_0(\phi) + \sigma_1 \alpha_1(\phi) \\ &= \Re \{ \lambda \phi'(r) + \lambda_1 \phi(r) \} = 0 ,\end{aligned}$$

where $\lambda_1 = \sigma_0 - i\sigma_1$. This implies the desired result, as we explained in the second paragraph of this section.

It remains to show that $\Re \lambda \phi_4'(r) \leq 0$. Assume to the contrary that $\phi_4'(r) = \beta$, with $\Re \{ \lambda\beta \} > 0$. We shall show that this leads to a contradiction of the fact that f_0 maximizes $\Re \lambda f'(r)$ for all $f \in \mathscr{S}^*$ with $f(r) = A$. To do this we must approximate $\phi_4 \in \mathscr{V}_1$ by some $\phi \in \mathscr{V}$, then approximate $f = f_0 + \varepsilon\phi + o(\varepsilon)$ by some $h \in \mathscr{S}^*$ with $h(r) = A$, and show that these approximations can be made so that $\Re \lambda h'(r) > \Re \lambda f_0'(r)$.

Suppose $0 < \rho_0 < 1$. In the steps that follow, we shall let ρ_n $(n = 1, 2, \cdots)$ denote a complex quantity such that $|\rho_n| < n\rho_0$. This convention allows us room to make successive approximations at each stage. At the end, we choose ρ_0 small enough to obtain the contradiction.

Clearly, $\phi_4'(r) = \beta$, and by the choice of σ_0 and σ_1 in (5.2), $\phi_4(r) = 0$. Since $\mathscr{V}_1$ is the completion of $\mathscr{V}$, there must exist a $\phi \in \mathscr{V}$ such that

$$\phi(r) = \rho_1 \qquad \text{and} \qquad \phi'(r) = \beta + \rho_2 .$$

Since ϕ is an allowable variation, (5.1) implies that for each $\varepsilon > 0$ there exists a function $f(z) = f_0(z) + \varepsilon\phi(z) + o(\varepsilon)$ in $\mathscr{S}^*$. We can choose ε_0 sufficiently small so that if $0 < \varepsilon < \varepsilon_0$, then

$$f(r) = A' = A + \varepsilon\rho_3 \qquad \text{and} \qquad f'(r) = B' = B + \varepsilon\beta + \varepsilon\rho_4 .$$

In order to use the extremal property of f_0, we must have functions with value A at $z = r$. We must therefore approximate this f by another function h such that $h(r) = A$.

We do this by letting $h = k_1^s f^{(1-s)}$, where k_1 is a suitably chosen Koebe function and s is a real number between 0 and 1. In this case,

$$\log h(r) = s \log k_1(r) + (1 - s) \log f(r) .$$

That is, $\log h(r)$ lies on the line joining $\log A'$ and $\log k_1(r)$.

It is well known that $G_r = \{\log f(r) : f \in \mathscr{S}^*\}$ is a convex set whose boundary is defined by $\log[r/(1 - e^{i\theta} r)^2]$ $(0 \leq \theta \leq 2\pi)$. The value of $\log A$ is an interior point of G_r, unless $f_0(z) = z/(1 - e^{i\theta} z)^2$ for some θ. In the latter case, f_0 is a one-slit function and the theorem is proved.

Since f is in $\mathscr{S}^*$, $\log A'$ is also in G_r. Draw the straight line from $\log A'$ through $\log A$ until it meets the boundary of G_r at $\log k_1(r)$, where $k_1(z) = z/(1 - e^{i\theta_1} z)^2$. This determines k_1, and

$$s = \frac{\log A/A'}{\log k_1(r)/A'}$$

is a real number $(0 < s < 1)$ such that $h = k_1^s f^{(1-s)}$ is in $\mathscr{S}^*$ with $h(r) = A$.

Next we show that $h'(r)$ is close to $f'(r)$. Let d be the distance from $\log A$ to ∂G_r. This d depends only on A and r. There then exists an $\varepsilon_1 \leq \varepsilon_0$ such that the condition $0 < \varepsilon < \varepsilon_1$ implies $|\log A - \log A'| < d/2$ and hence

$$|\log k_1(r) - \log A'| \geq d/2 .$$

Furthermore, since $\log A/A' = -\log(1 + \varepsilon\rho_3/A)$, there exists an $\varepsilon_2 \leq \varepsilon_1$ such that $0 < \varepsilon < \varepsilon_2$ implies

$$s = \varepsilon\rho_5 M,$$

where M is a positive constant depending only on A and r.

Now

$$\begin{aligned} h'(r) &= (1 - s)\,\frac{f'(r)\,h(r)}{f(r)} + s\,\frac{k_1'(r)\,h(r)}{k(r)} \\ &= \frac{B'A}{A'} + s\left[\frac{r\,k_1'(r)}{k_1(r)}\,\frac{h(r)}{r} - \frac{r\,f'(r)}{f(r)}\,\frac{h(r)}{r}\right] . \end{aligned} \tag{5.3}$$

The inequalities $|r\,g'(r)/g(r)| \leq (1 + r)/(1 - r)$ and $|g(r)/r| \leq 1/(1 - r)^2$ hold for each function in $\mathscr{S}$. Let C be the expression in brackets in (5.3). Then $|C| \leq 2(1 + r)/(1 - r)^3$, and there must exist an $\varepsilon_3 \leq \varepsilon_2$ such that if $0 < \varepsilon < \varepsilon_3$, then

$$h'(r) = \frac{B + \varepsilon(\beta + \rho_4)}{1 + \varepsilon\rho_3/A} + \varepsilon\rho_5 MC = B + \varepsilon\beta + \varepsilon\rho_0 K .$$

Here one easily verifies that the complex number K satisfies the condition $|K| < 3|B/A| + 5M|C| + 1$; since $|B| = |f_0'(r)| \leq (1 + r)/(1 - r)^3$, this gives a bound depending only on A and r.

The function h is in $\mathscr{S}^*$, and $h(r) = A$. Hence the extremal property of f_0 implies that

$$\Re\,\lambda h'(r) \leq \Re\,\lambda f_0'(r) = \Re\,\lambda B,$$

or, equivalently, that

$$\varepsilon\,\Re\,\{\lambda\beta + \lambda\rho_0 K\} \leq 0$$

for each ε $(0 < \varepsilon < \varepsilon_3)$. We have assumed that $\Re\{\lambda\beta\} > 0$. This results in a contradiction if we choose ρ_0 sufficiently small so that $|\rho_0| < |\beta/2K|$. Therefore, the assumption must have been false. This shows that $\Re\{\lambda\phi_4'(r)\} \leq 0$, and the proof is complete.

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