# SPLITTING LOCALLY COMPACT ABELIAN GROUPS

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The results of this paper constitute a part of a continuing investigation of the splitting problem in the category of locally compact abelian groups. More precisely, assume that $A \overset{\phi}{\rightarrowtail} B \overset{\theta}{\twoheadrightarrow} C$ is an extension of A by C in the category $\mathscr{L}$ of locally compact abelian groups (the morphisms of $\mathscr{L}$ are continuous homomorphisms). Under what conditions on A and C does the extension split? The papers [2] and [3] gave partial answers to the question, under various connectivity assumptions on A and C, respectively. In [1], the same problem was studied under the additional assumption that the extension be a pure exact sequence. The present paper is parallel to the development in [3]: we assume that G is a torsion-free group or a torsion group, rather than connected or totally disconnected (these analogues are suggested by Pontryagin's duality theory). Of course, this change of hypothesis greatly changes the problem; but our technique is similar to the technique used for the analogous problems in [1], [2], and [3], in that our investigation is basically homological.

More specifically, we determine the groups G in $\mathscr{L}$ for which the extension

$$G \rightarrowtail Y \twoheadrightarrow X \tag{1}$$

splits for each X in the class $\mathscr{C}$, where $\mathscr{C}$ may denote either the class of all locally compact, abelian torsion groups or the class of all locally compact, torsion-free abelian groups. We also determine the groups H in $\mathscr{L}$ for which the extension

$$X \rightarrowtail Y \twoheadrightarrow H \tag{2}$$

splits for each X in $\mathscr{C}$ with the same two choices of $\mathscr{C}$ as in the last statement.

In [2], we developed considerable homological apparatus for dealing with such problems. Suffice it to say that one may develop the extension functor in the category $\mathscr{L}$ along the lines used by S. MacLane [8] for R-modules. There are topological difficulties, but these are solved in [2]. One uses the usual definition of equivalence of extensions and the usual definition of Baer sum of extensions to make the class of all extensions $A \rightarrowtail B \twoheadrightarrow C$ into a (discrete) group Ext(C, A). The group Ext(C, A) has the usual functorial properties. One point should be mentioned: the definition of an extension is tailored to meet the requirement that if $A \overset{\phi}{\rightarrowtail} B \overset{\theta}{\twoheadrightarrow} C$ is an extension, then $\phi(A)$ is isomorphic to A *in the category* $\mathscr{L}$, and $\theta$ may be identified with the natural mapping $B \twoheadrightarrow B/\phi(A) = C$. In order to make such identifications, we *define* an extension $A \overset{\phi}{\rightarrowtail} B \twoheadrightarrow C$ in $\mathscr{L}$ to be an exact sequence, where $\phi$ is an injective morphism of $\mathscr{L}$, where $\theta$ is a surjective morphism of $\mathscr{L}$, and where both $\phi$ and $\theta$ are required to be open onto their respective images.

Once the group Ext has been constructed, we can rephrase the problems (1) and (2) above as follows: (1) determine the groups G of $\mathscr{L}$ for which Ext(X, G) = 0 for

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all X in $\mathscr{C}$, and (2) determine the groups H in $\mathscr{L}$ for which $\mathrm{Ext}(H, X) = 0$ for all X in $\mathscr{C}$.

Regarding notation, we follow [2] and [3], which in turn follow MacLane [8]. We use E. Hewitt and K. Ross [7] as our reference for facts and notation concerning topological groups. Briefly, if G is a group in $\mathscr{L}$, then $G_0$ denotes the identity component of G, and $G^\wedge$ denotes the Pontryagin dual of G. Also, R denotes the additive group of real numbers with its usual topology, and Z denotes the subgroup of integers of R.

THEOREM 1. *If* G *is a group in* $\mathscr{L}$, *then* $\mathrm{Ext}(T, G) = 0$ *for each torsion group* T *in* $\mathscr{L}$ *if and only if* G *is divisible.*

*Proof.* First assume that G is a group in $\mathscr{L}$ and that $\mathrm{Ext}(T, G) = 0$ for each torsion group T in $\mathscr{L}$. Then, for each positive integer m,

$$0 = \mathrm{Ext}(Z_m, G) = \mathrm{Ext}(Z_m, \mathscr{F}G) = G/mG,$$

where $\mathscr{F}G$ denotes G stripped of its topology. Thus G is divisible.

Conversely, assume G is divisible. Observe that $G^\wedge$ is torsion-free and thus, if C is any compact torsion group, then

$$\mathrm{Ext}(C, G) = \mathrm{Ext}(G^\wedge, C^\wedge) = \mathrm{Pext}(G^\wedge, C^\wedge).$$

Since $C^\wedge$ is bounded and reduced, it is discrete, reduced, and algebraically compact (see [4]). Thus $C^\wedge$ is a pure projective object of $\mathscr{L}$ (see [1, Proposition 8]). Thus $\mathrm{Pext}(G^\wedge, C^\wedge) = 0$ and $\mathrm{Ext}(C, G) = 0$. Note, however, that it follows from Theorem 24.30 of [7, page 389] that T contains a compact open subgroup C. Consider the exact sequence:

$$(*) \qquad \mathrm{Ext}(T/C, G) \longrightarrow \mathrm{Ext}(T, G) \twoheadrightarrow \mathrm{Ext}(C, G) = 0.$$

Since T/C is discrete and $\mathscr{F}G$ is divisible, $\mathrm{Ext}(T/C, G) = \mathrm{Ext}(T/C, \mathscr{F}G) = 0$ (see [2]). Thus it follows from (*) that $\mathrm{Ext}(T, G) = 0$. The theorem follows.

THEOREM 2. *If* G *is a group in* $\mathscr{L}$, *then* $\mathrm{Ext}(G, X) = 0$ *for each torsion-free group* X *in* $\mathscr{L}$ *if and only if* $G = R^m \oplus (\oplus_\sigma Z)$, *where* m *is a positive integer and* $\sigma$ *is a cardinal.*

*Proof.* First, because $R^m \oplus (\oplus_\sigma Z)$ is a projective object of $\mathscr{L}$ (see [9]), it is clear that $\mathrm{Ext}(R^m \oplus (\oplus_\sigma Z), X) = 0$ for each torsion-free group X in $\mathscr{L}$.

Conversely, assume that G is a group in $\mathscr{L}$ and that $\mathrm{Ext}(G, X) = 0$ for each torsion-free group X in $\mathscr{L}$. The sequence

$$\mathrm{Hom}(G_0, X) \longrightarrow \mathrm{Ext}(G/G_0, X) \longrightarrow \mathrm{Ext}(G, X) \longrightarrow \mathrm{Ext}(G_0, X)$$

is exact, and $\mathrm{Hom}(G_0, X) = 0$ for each totally disconnected group X. Thus $\mathrm{Ext}(G/G_0, X) = 0$ for each torsion-free, totally disconnected group X. We now wish to show that $H = (G/G_0)^\wedge$ is connected. Let $H_0$ denote the component of the identity of H, and observe that the sequence

$$\mathrm{Ext}(R/Z, H_0) \longrightarrow \mathrm{Ext}(R/Z, H) \twoheadrightarrow \mathrm{Ext}(R/Z, H/H_0)$$

is exact. Since $\mathrm{Ext}(R/Z, H) = \mathrm{Ext}(G/G_0, Z) = 0$, it follows from the latter exact sequence that $\mathrm{Ext}(R/Z, H/H_0) = 0$. We claim, however, that $H/H_0 = \mathrm{Ext}(R/Z, H/H_0)$, and thus that $H/H_0 = 0$. To see this, consider the exact sequence

$$0 = \mathrm{Hom}(R, H/H_0) \longrightarrow \mathrm{Hom}(Z, H/H_0) \longrightarrow \mathrm{Ext}(R/Z, H/H_0) \longrightarrow \mathrm{Ext}(R, H/H_0).$$

The needed result follows from the equalities

$$H/H_0 = \mathrm{Hom}(Z, H/H_0) \quad \text{and} \quad \mathrm{Ext}(R, H/H_0) = 0.$$

Thus $H = H_0$ and $H$ is connected. Since $(G/G_0)^\wedge$ cannot contain $R^m$, for $m > 0$, it follows from [7, page 389] that $(G/G_0)^\wedge$ is compact. Thus $G/G_0$ is discrete and torsion-free. By [7, page 395], $G = G_0 \oplus (G/G_0)$. We now compute the structure of $G_0$ and $G/G_0$. First write $G_0 = R^n \oplus C$, where $C$ is compact and connected. Since $Z$ is discrete and torsion-free, $\mathrm{Ext}(C, Z) = 0$. As above, $\mathrm{Ext}(R/Z, C^\wedge) = C^\wedge$. Since $\mathrm{Ext}(C, Z) = \mathrm{Ext}(R/Z, C^\wedge)$, we see that $C^\wedge = 0$. Thus $G_0 = R^n$. Finally, we compute $G/G_0$. Since $\mathrm{Ext}(G, X) = 0$ for each torsion-free group $X$, $\mathrm{Ext}(G/G_0, F) = 0$ for each discrete free group $F$. Since each discrete group $B$ is an epimorphic image of some free group, $\mathrm{Ext}(G/G_0, B) = 0$. Consequently, $G/G_0$ is a projective object in the category of discrete abelian groups, and therefore it is free. Thus $G/G_0 = \bigoplus_\sigma Z$ and $G = G_0 \oplus (G/G_0) = R^n \oplus (\bigoplus_\sigma Z)$. The theorem follows.

The following theorem is a consequence of the proof of one of the theorems in our joint paper with Griffith [2]. Since part of the proof of that theorem is rather brief, we elaborate that portion of it here for clarity. The remainder of the proof may be found in [2].

THEOREM 3. *If* $G$ *is a totally disconnected group in* $\mathscr{L}$ *such that* $\mathrm{Ext}(X, G) = 0$ *for each compact totally disconnected group* $X$, *then* $G$ *is discrete.*

*Proof.* Let $G$ denote a totally disconnected group of $\mathscr{L}$ such that $\mathrm{Ext}(X, G) = 0$ for each compact totally disconnected group $X$. Let $K$ denote any compact open subgroup of $G$. The sequence

$$\mathrm{Hom}(X, G/K) \longrightarrow \mathrm{Ext}(X, K) \longrightarrow \mathrm{Ext}(X, G) \longrightarrow \mathrm{Ext}(X, G/K)$$

is exact, and since $X$ is compact and $G/K$ is discrete, $\mathrm{Hom}(X, G/K)$ is a torsion group. Thus $\mathrm{Ext}(X, K)$ is a torsion group. Since it is also a cotorsion group [4, page 235], $\mathrm{Ext}(X, K)$ is the direct sum of a bounded group and a divisible group. Since $X^\wedge$ and $K^\wedge$ are torsion groups, it follows from [4, page 237] that the group $\mathrm{Ext}(X, K) = \mathrm{Ext}(K^\wedge, X^\wedge)$ is reduced. Thus $\mathrm{Ext}(X, K)$ is a bounded group for each compact, totally disconnected group $X$. Now let $B$ denote a basic subgroup of $K^\wedge$. Then $\mathrm{Ext}(B, J)$ is a bounded group, for each discrete torsion group $J$.

Let $B = \bigoplus_\lambda B_\lambda$, where $B_\lambda$ is cyclic of order $n_\lambda$, for each $\lambda$. Let $n$ denote a positive integer such that $n\,\mathrm{Ext}(B, B) = 0$. Then

$$n \prod_\lambda (B/n_\lambda B) = 0 \quad \text{and} \quad nB_\lambda \subseteq nB \cap B_\lambda \subseteq n_\lambda B \cap B_\lambda \subseteq n_\lambda B_\lambda = 0$$

for each $\lambda$. Thus $B$ is bounded, and $K^\wedge = B \oplus D$ for some divisible torsion group $D$. But the boundedness of $\mathrm{Ext}(K^\wedge, J)$ for a discrete torsion group $J$ implies that $\mathrm{Ext}(D, J)$ is bounded for each discrete torsion group $J$. If $D$ were not zero, it would follow that $D$ contains some $Z(p^\infty)$, and thus that the group

$$\mathrm{Ext}(\textstyle\bigoplus_\sigma Z(p^\infty), J) = \prod_\sigma \mathrm{Ext}(Z(p^\infty), J)$$

is bounded for each $J$. Since each $p$-group is contained in $\bigoplus_\sigma Z(p^\infty)$, for some $\sigma$, the group $\mathrm{Ext}(A, J)$ is bounded for each discrete $p$-group $A$ and each discrete torsion group J. That this is nonsense in case $A = J = \bigoplus_{i=1}^{\infty} Z(p^i)$ is clear from the computation

$$\operatorname{Ext}(A, J) = \prod_{i=1}^{\infty} \operatorname{Ext}(Z(p^i), J) = \prod_{i=1}^{\infty} [J/(p^i J)].$$

The group $\prod_{i=1}^{\infty} [J/(p^i J)]$ is clearly not bounded. Thus $D = 0$ and the group $K^\wedge$ is bounded. It follows that K is a bounded group, and thus that each compact open subgroup of G is bounded. One may now refer to the proof of Theorem 4.2 of [2] for the remainder of the proof that each compact open subgroup K of G is finite. The theorem follows.

Let $J_p$ denote the compact group of p-adic integers, and let Q denote the discrete group of rational numbers under addition. Then $Q^\wedge$ is the solenoidal group $\Sigma_a$ described by Hewitt and Ross [7, page 404].

THEOREM 4. *If* G *is a group in* $\mathscr{L}$, *then* $\operatorname{Ext}(G, T) = 0$ *for each torsion group* T *in* $\mathscr{L}$ *if and only if there exist a positive integer* m *and cardinals* $\lambda$, $\mu$, *and* $\sigma$ *such that*

$$G = R^m \oplus \left(\prod_{\lambda} J_p\right) \oplus \left(\prod_{\mu} \Sigma_a\right) \oplus (\oplus_\sigma Z).$$

*Proof.* Assume that G is a group in $\mathscr{L}$ such that $\operatorname{Ext}(G, T) = 0$ for each torsion group T in $\mathscr{L}$. We first show that $G = G_0 \oplus (G/G_0)$. The structure theorem for locally compact abelian groups implies that $G = R^m \oplus H$, where H contains a compact open subgroup (see [7, page 389]). Clearly, $H_0$ is compact, and thus $(H_0)^\wedge$ is discrete. Since $(H_0)^\wedge$ is discrete, $(H/H_0)^\wedge$ is an open subgroup of $H^\wedge$. Thus, if

$$(*) \qquad (H/H_0)^\wedge \rightarrowtail H^\wedge \twoheadrightarrow (H_0)^\wedge$$

splits algebraically, then it also splits topologically. Note, however, that for each torsion group T the sequence

$$\operatorname{Hom}(H_0, T) \longrightarrow \operatorname{Ext}(H/H_0, T) \longrightarrow \operatorname{Ext}(H, T) = 0$$

is exact, and that for a discrete group T, $\operatorname{Hom}(H_0, T) = 0$. Thus, for each discrete torsion group T, $\operatorname{Ext}(H/H_0, T) = 0$. For each n,

$$0 = \operatorname{Ext}(H/H_0, Z_n) = \operatorname{Ext}(Z_n, (H/H_0)^\wedge) = (H/H_0)^\wedge/n(H/H_0)^\wedge.$$

Thus $(H/H_0)^\wedge$ is a divisible group, and (*) splits. It follows that

$$G = R^m \oplus H_0 \oplus (H/H_0) = G_0 \oplus (G/G_0).$$

We now obtain the structure of $H_0$. Clearly, for each n,

$$0 = \operatorname{Ext}(H_0, Z_n) = (H_0)^\wedge/n(H_0)^\wedge,$$

and $(H_0)^\wedge$ is a divisible group. Thus $H_0$ is a compact, connected, and torsion-free group. It follows that $H_0 = \prod_{\mu} \Sigma_a$ for some cardinal $\mu$.

At this point, we need only compute the structure of $H/H_0$. Let $K = (H/H_0)^\wedge$, and observe that we have already shown that $K$ is divisible. For each compact, totally disconnected group $C$, $C^\wedge$ is a discrete torsion group, and there exists an exact sequence

$$\operatorname{Ext}(C, K_0) \longrightarrow \operatorname{Ext}(C, K) \twoheadrightarrow \operatorname{Ext}(C, K/K_0).$$

Since $\operatorname{Ext}(C, K) = \operatorname{Ext}(H/H_0, C^\wedge) = 0$, we see that $\operatorname{Ext}(C, K/K_0) = 0$. By Theorem 3, $K/K_0$ is discrete. Since $K_0$ is open and divisible, $K = K_0 \oplus (K/K_0)$. It follows that $\operatorname{Ext}((K_0)^\wedge, T) = 0$ for all torsion groups $T$. By Griffith's solution of the Baer problem [5], we see that $(K_0)^\wedge$ is free. Thus $H/H_0 = (K_0)^\wedge \oplus (K/K_0)^\wedge$, where $(K_0)^\wedge$ is free and $K/K_0$ is a discrete divisible group. Since $H/H_0$ is totally disconnected, it can contain no copy of $Q^\wedge$; thus $H/H_0 = \bigoplus_\sigma Z \oplus \prod_\lambda J_p$. Therefore

$$G = R^m \oplus \Big(\prod_\mu \Sigma_a\Big) \oplus (\textstyle\bigoplus_\sigma Z) \oplus \Big(\prod_\lambda J_p\Big).$$

Conversely, since $R^m \oplus (\bigoplus_\sigma Z)$ is a projective object in $\mathscr{L}$,

$$\operatorname{Ext}(R^m \oplus (\textstyle\bigoplus_\sigma Z), T) = 0$$

for each torsion group $T$. Thus the theorem will follow if we can show that $\operatorname{Ext}(D^\wedge, T) = 0$ for all discrete divisible groups $D$ and all torsion groups $T$. Since $\operatorname{Ext}(D^\wedge, T) = \operatorname{Ext}(T^\wedge, D)$, it suffices to show that $T^\wedge$ is totally disconnected (see [2] for a proof that discrete divisible groups are injective among totally disconnected groups). To show that $T^\wedge$ is totally disconnected, we show that its identity component $(T^\wedge)_0$ is trivial. Now $(T^\wedge)_0 = R^m \oplus C$ for some compact connected group $C$ (see [7, page 389]). There exists an epimorphism $T \twoheadrightarrow ((T^\wedge)_0)^\wedge$; thus $R^m$ and $C^\wedge$ are homomorphic images of $T$. Since $T$ is a torsion group, $R^m = 0$. Since $C$ is compact and connected, $C^\wedge$ is discrete and torsion-free. Thus $C^\wedge = 0$ and $(T^\wedge)_0 = 0$, as was to be proved. The theorem follows.

This brings us to the final stage of our program. It is the purpose of this section of the paper to investigate the groups that we shall call $\mathscr{L}$-cotorsion groups. Recall that a discrete group $G$ is a *cotorsion group* if and only if $\operatorname{Ext}(X, G) = 0$ for all discrete torsion-free groups $X$. The theory of cotorsion groups was developed by D. K. Harrison [6], and an account of this theory may be found in the book [4] by Fuchs.

A locally compact abelian group $G$ will be called an *$\mathscr{L}$-cotorsion group* if and only if $\operatorname{Ext}(X, G) = 0$ for each torsion-free group $X$ in $\mathscr{L}$.

THEOREM 5. *A group $G$ in $\mathscr{L}$ is an $\mathscr{L}$-cotorsion group if and only if $G/G_0$ is an $\mathscr{L}$-cotorsion group.*

*Proof.* Let $G$ and $X$ denote groups in $\mathscr{L}$, and let $X$ denote a torsion-free group. First observe that the sequence

$$\operatorname{Ext}(X, G_0) \longrightarrow \operatorname{Ext}(X, G) \twoheadrightarrow \operatorname{Ext}(X, G/G_0)$$

is exact; thus, if we can show that $\operatorname{Ext}(X, G_0) = 0$, then it will follow that $\operatorname{Ext}(X, G) = \operatorname{Ext}(X, G/G_0)$. The theorem is a clear consequence of the last equation. Now $X = R^n \oplus Y$, where $Y$ contains some compact open subgroup $K$. Thus we have an exact sequence

$$(*)\qquad \operatorname{Ext}(Y/K, G_0) \longrightarrow \operatorname{Ext}(Y, G_0) \twoheadrightarrow \operatorname{Ext}(K, G_0).$$

Also, $G_0 = R^m \oplus C$ for some compact connected group $C$; thus

$$\operatorname{Ext}(K, G_0) = \operatorname{Ext}(K, C) = \operatorname{Ext}(C^\wedge, K^\wedge) = 0$$

and

$$\operatorname{Ext}(Y/K, G_0) = \operatorname{Ext}(Y/K, \mathscr{F} G_0) = 0 ,$$

where $\mathscr{F} G_0$ denotes $G_0$ stripped of its topology (recall that $G_0$, being connected, is divisible). It follows from the sequence (*) that $\operatorname{Ext}(Y, G_0) = 0$. The theorem follows.

COROLLARY 6. *Each connected group in $\mathscr{L}$ is an $\mathscr{L}$-cotorsion group.*

THEOREM 7. *If $G$ is an $\mathscr{L}$-cotorsion group, then $(G/G_0)/K$ is a discrete divisible torsion group for each compact open subgroup $K$ of $G/G_0$.*

*Proof.* By Theorem 5, it suffices to prove the theorem for totally disconnected $\mathscr{L}$-cotorsion groups. Let $G$ denote such a group and $K$ any compact open subgroup of $G$. First we shall show that $G/K$ is a torsion group. We do this by showing that the identity component $H_0$ of the group $H = (G/K)^\wedge$ is trivial. Consider the exact sequence

$$(*)\qquad \operatorname{Ext}(H/H_0, D) \longrightarrow \operatorname{Ext}(H, D) \twoheadrightarrow \operatorname{Ext}(H_0, D)$$

for discrete divisible $D$. Since $D^\wedge$ is a compact and torsion-free group,

$$\operatorname{Ext}(H, D) = \operatorname{Ext}(D^\wedge, H^\wedge) = \operatorname{Ext}(D^\wedge, G/K) = 0$$

(we use the fact that $G/K$ is an $\mathscr{L}$-cotorsion group). It follows from (*) that $\operatorname{Ext}(H_0, D) = 0$. But if $B$ is any discrete group, then $B \subseteq D$ for some $D$, and there exists an exact sequence

$$\operatorname{Hom}(H_0, D/B) \longrightarrow \operatorname{Ext}(H_0, B) \longrightarrow \operatorname{Ext}(H_0, D) = 0 .$$

Since $H_0$ is connected and $D/B$ is discrete, $\operatorname{Hom}(H_0, D/B) = 0$. Thus $\operatorname{Ext}(H_0, B) = 0$ for each discrete group $B$. It follows that

$$(H_0)^\wedge = \operatorname{Ext}(R/Z, (H_0)^\wedge) = \operatorname{Ext}(H_0, Z) = 0$$

and $H_0 = 0$. This implies that $G/K$ is a torsion group.

We now show that $G/K$ is divisible. Since $G$ is a cotorsion group, so is $G/K$. Recall that if a group is both a torsion group and a cotorsion group, it is the direct sum of a bounded group and a divisible group (see Fuchs [4]). Let $G/K = B \oplus D$, where $B$ is a bounded group and $D$ is a divisible group. We show that $B = 0$. If $B$ is not zero, then $B = \bigoplus_{i=1}^m B_i$, where $B_i = \bigoplus_{\sigma_i} Z(n_i)$ for some positive integer $n_i$ and some nonzero cardinal $\sigma_i$. Clearly, $\operatorname{Ext}(X, B_i) = 0$ for each $i$ and for each torsion-free group $X$. Thus $X^\wedge/n_i X^\wedge = 0$ for each $i$ and each torsion-free group $X$. By a modification of a construction by Hewitt and Ross [7, page 393], one obtains a locally compact, abelian, torsion-free group $X$ for which $X^\wedge$ is not $n_i$-divisible (actually, Hewitt and Ross show that such an $X$ exists for which $X^\wedge$ is not 2-divisible). This contradiction shows that $B = 0$ and that $G/K = D$ is a divisible group. The theorem follows.

COROLLARY 8. *If* G *is an* $\mathscr{L}$*-cotorsion group, then* $G/G_0$ *is a projective limit of discrete, divisible torsion groups.*

*Proof.* Each totally disconnected group G in $\mathscr{L}$ is a projective limit of groups of the form $G/K$, where K denotes a compact open subgroup of G. The corollary follows from this and the theorem.

COROLLARY 9. *A compact abelian group is an* $\mathscr{L}$*-cotorsion group if and only if it is connected.*

*Proof.* Let C be a compact abelian $\mathscr{L}$-cotorsion group. We show that C is connected, by showing that $C/C_0$ is trivial. By the theorem, $(C/C_0)/K$ is divisible for each compact open subgroup K of $C/C_0$. But if K is a compact open subgroup of $C/C_0$, then $(C/C_0)/K$ is finite. Thus $K = C/C_0$ is the only compact open subgroup of $C/C_0$. But $C/C_0$, being totally disconnected, has a neighborhood basis of compact open subgroups at 0. Thus $C/C_0$ is trivial and C is connected. The converse is clear from Corollary 6.

COROLLARY 10. *A discrete abelian group is an* $\mathscr{L}$*-cotorsion group if and only if it is a divisible torsion group.*

*Proof.* It is an immediate consequence of the theorem that each discrete $\mathscr{L}$-cotorsion group is a divisible torsion group. Conversely, assume that A is any divisible group and that X is a torsion-free member of $\mathscr{L}$. Consider the exact sequence

$$(*) \qquad \operatorname{Ext}(X/X_0, A) \longrightarrow \operatorname{Ext}(X, A) \twoheadrightarrow \operatorname{Ext}(X_0, A).$$

Since $X/X_0$ is totally disconnected and A is discrete and divisible, $\operatorname{Ext}(X/X_0, A) = 0$. Let $X_0 = R^n \oplus C$, for some compact, connected group C. Since $A^\wedge$ is totally disconnected and $C^\wedge$ is a discrete, divisible group,

$$\operatorname{Ext}(X_0, A) = \operatorname{Ext}(C, A) = \operatorname{Ext}(A^\wedge, C^\wedge) = 0.$$

It follows immediately from (*) that $\operatorname{Ext}(X, A) = 0$. This completes the proof of the corollary.

THEOREM 11. *A group* G *in* $\mathscr{L}$ *is an* $\mathscr{L}$*-cotorsion group if and only if the totally disconnected group* $G/G_0$ *is an* $\mathscr{L}$*-cotorsion group. A totally disconnected group* H *of* $\mathscr{L}$ *is an* $\mathscr{L}$*-cotorsion group if and only if*

(1) *as a discrete abelian group,* H *is a cotorsion group,*

(2) *whenever* K *is a compact open subgroup of* H, *then* $H/K$ *is a discrete, divisible torsion group, and*

(3) *whenever* K *is a compact open subgroup of* H *and* E *is the divisible hull of a product of* p*-adic groups, then the connecting homomorphism*

$$\partial_E\colon \operatorname{Hom}(E, H/K) \longrightarrow \operatorname{Ext}(E, K)$$

*is surjective.*

*Proof.* The first statement in the theorem is a restatement of Theorem 5. It follows from Theorem 7 that if H is an $\mathscr{L}$-cotorsion group and is disconnected, then conditions (1) and (2) of Theorem 11 are satisfied. To see that (3) holds, recall that the divisible hull of a torsion-free group is torsion-free, and that thus $\operatorname{Ext}(E, H) = 0$ (see Hewitt and Ross [7, page 419] for a description of the topology carried by the divisible hull of a group). It follows from the exact sequence

$$\mathrm{Hom}(E, H/K) \xrightarrow{\partial} \mathrm{Ext}(E, K) \twoheadrightarrow \mathrm{Ext}(E, H)$$

that the connecting homomorphism $\partial$ is surjective.

Conversely, assume that (1), (2), and (3) hold. We show that H is an $\mathscr{L}$-cotorsion group. Let K denote any compact open subgroup of H, and let $D = H/K$. Then D is a discrete, divisible torsion group, and hence, by Corollary 10, $\mathrm{Ext}(X, D) = 0$ for each torsion-free group X in $\mathscr{L}$. Thus, in the exact sequence

$$\mathrm{Hom}(X, H/K) \xrightarrow{\partial_X} \mathrm{Ext}(X, K) \xrightarrow{\sigma} \mathrm{Ext}(X, H) \twoheadrightarrow \mathrm{Ext}(X, D),$$

$\sigma$ is surjective. It follows that $\mathrm{Ext}(X, H)$ is zero in case the group $\mathrm{Ker}\,\sigma = \mathrm{Im}\,\partial_X$ is all of $\mathrm{Ext}(X, K)$. Thus it suffices to show that $\partial_X$ is surjective for each torsion-free group X. Let X denote any torsion-free member of $\mathscr{L}$. By a remark of Hewitt and Ross [7, page 418], $X = R^n \oplus C \oplus T$, where C is a compact, connected group, and where T is a totally disconnected group. We show that $\mathrm{Ext}(R^n \oplus C, H) = 0$. From this it will follow that $\partial_X$ is surjective if $\partial_T$ is surjective. Thus we show that $\mathrm{Ext}(C, H) = 0$. Consider the exact sequence

$$\mathrm{Ext}(C, K) \longrightarrow \mathrm{Ext}(C, H) \twoheadrightarrow \mathrm{Ext}(C, D). \tag{*}$$

Since $C^\wedge$ is a discrete divisible group and $K^\wedge$ is a discrete group,

$$\mathrm{Ext}(C, K) = \mathrm{Ext}(K^\wedge, C^\wedge) = 0.$$

Since $D^\wedge$ is a totally disconnected group and $C^\wedge$ is a discrete and divisible group, $\mathrm{Ext}(C, D) = \mathrm{Ext}(D^\wedge, C^\wedge) = 0$. By (*), $\mathrm{Ext}(C, H) = 0$. We have shown that it suffices to prove that the connecting homomorphism

$$\partial_T: \mathrm{Hom}(T, K/K) \longrightarrow \mathrm{Ext}(T, K)$$

is surjective. Now let $E^*$ denote the minimal divisible hull of T, topologized as in Hewitt and Ross [7, page 419]. Since T is a totally disconnected group, $E^*$ is also totally disconnected. Moreover, $E^*/T$ is discrete. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(E^*, H/K) \arrow[r] \arrow[d, "\partial_{E^*}"'] & \mathrm{Hom}(T, H/K) \arrow[r] \arrow[d, "\partial_T"'] & \mathrm{Ext}(E^*/T, H/K) \arrow[r] \arrow[d] & \mathrm{Ext}(E^*, H/K) \arrow[d] \\
\mathrm{Ext}(E^*, K) \arrow[r] & \mathrm{Ext}(T, K) \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}.$$

Since $H/K$ is a discrete, divisible group and both $E^*$ and $E^*/T$ are totally disconnected groups, it follows from [2] that $\mathrm{Ext}(E^*/T, H/K) = 0$ and $\mathrm{Ext}(E^*, H/K) = 0$. If $\partial_{E^*}$ were surjective, it would follow from the five-lemma that $\partial_T$ is surjective. By [7, page 421], $E^* = \oplus Q \oplus E$, where E is the divisible hull of a group that is a product of p-adic groups. Since H is a cotorsion group,

$$\mathrm{Ext}(\oplus_\sigma Q, H) = \mathrm{Ext}(\oplus_\sigma Q, \mathscr{F}H) = 0,$$

where $\mathscr{F}H$ denotes H stripped of its topology. It follows that $\partial_{E^*}$ is surjective whenever $\partial_E$ is surjective. But by (3), $\partial_E$ is surjective. Thus $\partial_X$ is surjective and the theorem follows.

*Remark.* Although Theorem 11 gives a characterization of $\mathscr{L}$-cotorsion groups, this characterization is not completely satisfactory. We feel that conditions (1) and (2) of the theorem are reasonable conditions. In particular, the condition (1) is a purely algebraic condition, and much is known about cotorsion groups. Actually this structure is almost as well known as is the structure of discrete torsion groups (see, for example, Fuchs [4]). The condition (3) of Theorem 11 needs some illumination. In the presence of our other conditions, it is clearly equivalent to the condition that $\operatorname{Ext}(E, G) = 0$ whenever $E$ is the minimal divisible hull of a product of $p$-adic groups. We do not know the structure of these groups. Actually, it may well be that conditions (1) and (2) imply (3).

## REFERENCES

1. R. Fulp, *Homological study of purity in locally compact groups.* Proc. London Math. Soc. (3) 21 (1970), 502-512.

2. R. Fulp and P. Griffith, *Extensions of locally compact abelian groups.* Trans. Amer. Math. Soc. (to appear).

3. ———, *Extensions of locally compact abelian groups II.* Trans. Amer. Math. Soc. (to appear).

4. L. Fuchs, *Infinite abelian groups I.* Academic Press, New York and London, 1970.

5. P. Griffith, *A solution to the splitting mixed group problem of Baer* (to appear).

6. D. K. Harrison, *Infinite abelian groups and homological methods.* Ann. of Math. (2) 69 (1959), 366-391.

7. E. Hewitt and K. Ross, *Abstract harmonic analysis.* I. Die Grundlehren der mathematischen Wissenschaften, Bd. 115. Springer-Verlag, Berlin, 1963.

8. S. MacLane, *Homology.* Die Grundlehren der mathematischen Wissenschaften, Bd. 114. Springer-Verlag, New York, 1967.

9. M. Moskowitz, *Homological algebra in locally compact abelian groups.* Trans. Amer. Math. Soc. 127 (1967), 361-404.

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