A NOTE ON MAPPING CYLINDERS

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THEOREM. *Let* X *and* Y *be topological spaces, and let* $f: X \to Y$ *be a homotopy equivalence. Then* $X \times \{0\}$ *is a strong deformation retract of the mapping cylinder* $M_f = X \times I \cup_f Y$.

This theorem is known in homotopy theory, but the proof normally comes out of the context of cofibration (see [1, Corollary 3.7], for example). I feel that a direct proof is desirable, since it is simple and since in special cases, for example, for fiber-homotopy equivalences or for equivariant-homotopy equivalences, one can verify directly that a corresponding theorem holds. Thus one is not forced to reappraise a whole theory.

The first theorem in this direction was proved by R. Fox in 1943 (see [2, Theorem 3.7]).

Proof. Let $g: Y \to X$ be a homotopy inverse of f, and let $K: X \times I \to X$ be such that $K(x, 0) = x$ and $K(x, 1) = g \circ f(x)$. Consider the set

$$\{S \mid S: X \times I \to X \text{ with } S(x, 0) = x,\ S(x, 1) = g \circ f(x)\}.$$

We denote the set of homotopy classes of these maps by $\mathscr{P}(X \times I, X; 1_X, g \circ f)$. Also, we form the set of homotopy classes $\mathscr{P}(Y \times I, Y; 1_Y, f \circ g)$. The map $f: X \to Y$ induces the maps

$$\mathscr{P}(X \times I, X; 1_X, g \circ f) \xrightarrow[f_*]{} \mathscr{P}(X \times I, Y; f, f \circ g \circ f)$$

and

$$\mathscr{P}(Y \times I, Y; 1_Y, f \circ g) \xrightarrow[f^*]{} \mathscr{P}(X \times I, Y; f, f \circ g \circ f);$$

f^* and f_* are equivalences of sets, since f is a homotopy equivalence (for the proof, one needs the proposition that canonical change of "base points" induces equivalences of sets). Therefore there must exist a map $L: Y \times I \to Y$ such that $L(y, 0) = y$ and $L(y, 1) = f \circ g(y)$, and such that $f^*[L] = f_*[K]$. Consequently, there exists a homotopy $M: X \times I \times I \to Y$ such that

$$M(x, t, 0) = f \circ K(x, t), \qquad M(x, 0, s) = f(x)$$

and

$$M(x, t, 1) = L(f(x), t), \qquad M(x, 1, s) = f \circ g \circ f(x)$$

for $x \in X$ and $t, s \in I$.

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The identity map of M_f is homotopic to the retraction $r\colon M_f \to Y$; therefore it is homotopic to the mapping

$$\begin{cases} (x, \tau) \\ (x, \tau) \\ (x, \tau) \\ y \end{cases} \to \begin{cases} (x, 3\tau) & (0 \leq \tau \leq 1/3), \\ f(x) & (1/3 \leq \tau \leq 2/3), \\ f(x) & (2/3 \leq \tau \leq 1), \\ y\,. \end{cases}$$

Using L and M, we can deform this map to

$$\begin{cases} (x, \tau) \\ (x, \tau) \\ (x, \tau) \\ y \end{cases} \to \begin{cases} (x, 3\tau) & (0 \leq \tau \leq 1/3), \\ L \circ (f(x), 3\tau - 1) & (1/3 \leq \tau \leq 2/3), \\ f \circ g \circ f(x) & (2/3 \leq \tau \leq 1), \\ f \circ g(y), \end{cases}$$

and then to

$$\begin{cases} (x, \tau) \\ (x, \tau) \\ (x, \tau) \\ y \end{cases} \to \begin{cases} (x, 3\tau) & (0 \leq \tau \leq 1/3), \\ f \circ K(x, 3\tau - 1) & (1/3 \leq \tau \leq 2/3), \\ f \circ g \circ f(x) & (2/3 \leq \tau \leq 1), \\ f \circ g(y)\,. \end{cases}$$

Again, because r is homotopic to the identity of M_f, we can deform this map to

$$\begin{cases} (x, \tau) \\ (x, \tau) \\ (x, \tau) \\ y \end{cases} \to \begin{cases} (x, 0) & (0 \leq \tau \leq 1/3), \\ (K(x, 3\tau - 1), 0) & (1/3 \leq \tau \leq 2/3), \\ (g \circ f(x), 0) & (2/3 \leq \tau \leq 1), \\ (g(y), 0)\,. \end{cases}$$

Hence the last map is a strong deformation retraction, as we claimed.

REFERENCES

1. A. Dold, *Halbexakte Homotopiefunktoren.* Lecture Notes in Mathematics, 12. Springer-Verlag, Berlin-New York, 1966.

2. R. H. Fox, *On homotopy type and deformation retracts.* Ann. of Math. (2) 44 (1943), 40-50.

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