THE WEAK CONTINUITY OF METRIC PROJECTIONS

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Let X be a Banach space, and let M be a closed subspace in X. Define P_M to be the metric projection (nearest-point operator, best-approximation operator) supported by M; that is, if x is an element of X, then

$$P_M(x) = \{y \in M \mid \|x - y\| = \inf_{z \in M} \|x - z\|\}.$$

M is said to be a Chebyshev subspace provided $P_M(x)$ is a singleton for each x in X.

There has been recent interest [2], [3], [7] in the continuity behavior of the metric projection P_M, especially when continuity is determined by topological conditions on the kernel $P_M^{-1}(\theta) = \{x \in X \mid P_M(x) = \theta\}$ [2]. The purpose of this paper is to establish sufficient conditions for the metric projection to be weakly continuous (that is, continuous as a mapping from the weak topology to the weak topology). The main result is Theorem 1. Theorem 2 and its corollaries are intended to simplify the hypotheses of Theorem 1. Theorem 3 is an extension of the result for the bw-topology. Two examples at the end of the paper establish the necessity of some of the hypotheses.

For the weak sequential topology, R. B. Holmes has recently proved a result [2, Theorem 11] analogous to Theorem 1.

THEOREM 1. *If* M *is a finite-dimensional Chevyshev subspace of* X *such that* $P_M^{-1}(\theta)$ *is weakly closed, then* P_M *is weakly continuous.*

Proof. Let $\{u_\alpha\}$ be a net converging weakly to u in X. We shall show that $\{P_M(u_\alpha)\}$ converges weakly to $P_M(u)$. We may assume $P_M(u) = \theta$. Let $S_M = \{x \in M \mid \|x\| = 1\}$ and $U_M = \{x \in M \mid \|x\| < 1\}$. Because S_M is weakly compact, $S_M + P_M^{-1}(\theta)$ is weakly closed. We claim that $V = P_M^{-1}(U_M)$ is weakly open. Supposing to the contrary that there is a net $\{y_\beta\}$ in $X \sim V$ that is convergent weakly to a point y in V, we have the inequality $\|P_M(y_\beta)\| \geq 1$ for each β, while $\|P_M(y)\| < 1$. Using the fact that P_M is norm-continuous (see for example [6, page 347]), we obtain for each β a number $t_\beta \in [0, 1]$ and a point $v_\beta = t_\beta y_\beta + (1 - t_\beta)y$ such that $\|P_M(v_\beta)\| = 1$, in other words, such that $\{v_\beta\} \subset S_M + P_M^{-1}(\theta) = P_M^{-1}(S_M)$. Because $\{v_\beta\}$ converges weakly to y and $S_M + P_M^{-1}(\theta)$ is weakly closed, y is an element of $X \sim V$, a contradiction. Thus V is weakly open, and since $u \in V$, we see that $\{u_\alpha\}$ is eventually in V. Hence $\{P_M(u_\alpha)\}$ is eventually in U_M, and therefore it has a norm cluster point, say z. Taking subnets if necessary, we may assume that $\{P_M(u_\alpha)\}$ converges in norm to z. For each α, let

$$d_\alpha = \inf\{\|u_\alpha - x\| \mid x \in P_M^{-1}(z)\}.$$

Then $\{d_\alpha\}$ converges to 0, since $u_\alpha + (z - P_M(u_\alpha))$ is in $P_M^{-1}(z)$ for each α, and $\{z - P_M(u_\alpha)\}$ converges in norm to θ. If for each α we choose $w_\alpha \in P_M^{-1}(z)$ so

Received October 17, 1969.

Michigan Math. J. 17 (1970).

that $\|u_\alpha - w_\alpha\| \le 2d_\alpha$, then $\{w_\alpha\}$ converges weakly to u. However, $P_M^{-1}(z) = z + P_M^{-1}(\theta)$ is weakly closed; therefore u is in $P_M^{-1}(z)$, and thus $z = \theta$. ∎

For points x and y of X, define

$$E(x, y) = \{z \in X \mid \|x - z\| = \|y - z\|\},$$

the equidistant set introduced by G. K. Kalisch and E. G. Straus [4]. In the following theorem and its corollaries, we continue the investigation begun by V. Klee [5] concerning the connection between the weak closure of equidistant sets and the weak continuity of metric projections. In the remainder of this paper, $[\{x_\alpha\}_{\alpha \in A}]$ denotes the closed linear span of the family $\{x_\alpha\}_{\alpha \in A}$.

LEMMA 1. *If* x *is an element of* X *such that* [x] *is a Chebyshev subspace, then* $P_{[x]}(E(-x, x)) \subset \{tx \mid -1 \le t \le 1\}$.

Proof. If y is an element of E(-x, x) but $P_{[x]}(y)$ is not an element of $\{tx \mid -1 \le t \le 1\}$, then the ball of radius $\|x - y\|$ centered at y would not be convex (x and -x are on its boundary, and $P_{[x]}(y)$ is in its interior). ∎

Actually, a much stronger form of Lemma 1 is true. In fact, it can be shown that E(-x, x) and $P_{[x]}^{-1}(\theta)$ are nearly parallel in the sense that if y is a point in one of the sets, there exists a point z in the other set such that $\|y - z\| \le \|x\|$ and $y - z \in [x]$.

THEOREM 2. *If* x *is a point in* X *such that* [x] *is a Chebyshev subspace and* E(-x, x) *is weakly closed, then* $P_{[x]}$ *is weakly continuous.*

Proof. Since E(-x, x) is weakly closed, the sets

$$E(\theta, 2x) = E(-x, x) + x \quad \text{and} \quad E(-2x, \theta) = E(-x, x) - x$$

are also weakly closed. Let $V = E(-x, x) + \{tx \mid -1 < t < 1\}$; that is, let V be the set between $E(\theta, 2x)$ and $E(-2x, \theta)$. An argument similar to that in the proof of Theorem 1 shows that $X \sim V$ is weakly closed, and thus V is weakly open. Furthermore, Lemma 1 shows that $P_{[x]}(V) \subset \{tx \mid -2 \le t \le 2\}$. The remainder of the proof is like that of Theorem 1. ∎

COROLLARY 1. *If* [x] *is a Chebyshev subspace and* E(-x, x) *is weakly closed, then* $P_{[x]}^{-1}(\theta)$ *is weakly closed.*

COROLLARY 2. *If* X *is smooth,* $M = [x_1, x_2, \cdots, x_n]$ *is a Chebyshev subspace,* $[x_i]$ *is a Chebyshev subspace, and* $P_{[x_i]}^{-1}(\theta)$ *is weakly closed for each* i $(i = 1, 2, \cdots, n)$, *then* P_M *is weakly continuous.*

Proof. Since X is smooth, we may apply a theorem of Holmes and Kripke [3, Proposition 4], which states that

$$P_{[\{x_\alpha\}_{\alpha \in A}]}^{-1}(\theta) = \bigcap_{\alpha \in A} P_{[x_\alpha]}^{-1}(\theta). \quad ∎$$

In the light of Corollaries 1 and 2, one may consider Theorem 1 as a strengthening of a result of Klee [5, Proposition 2.5] for smooth spaces. We shall show later (Example 2) that this theorem is strictly stronger than Klee's in some cases (in particular, when X is smooth and strictly convex); in the process, we shall also show that the converses of Theorem 2 and Corollary 1 are false.

If we replace the weak topology by the slightly stronger bounded weak (bw-) topology, we can demonstrate an improved version of Theorem 1, replacing finite dimensionality by reflexivity. We do not know whether the corresponding result is true in the weak topology. Recall [1, page 41] that a subset of a Banach space is bw-closed provided its intersection with each bounded set is weakly closed relative to the bounded set.

THEOREM 3. *Let* X *be a Banach space, and let* M *be a reflexive Chebyshev subspace such that* $P_M^{-1}(\theta)$ *is* bw-*closed. Then* P_M *is* bw-*continuous.*

Proof. Let $\{x_\alpha\}$ be a bounded net converging weakly to x. Since $\|P_M(y)\| \leq 2\|y\|$ for each y in X, it follows that $\{P_M(x_\alpha)\}$ is bounded and hence has a w(M, M*)-cluster point, say z. That is, θ is a w(M, M*)-cluster point of $\{z - P_M(x_\alpha)\}$, and thus x is a w(X, X*)-cluster point of $\{x_\alpha + (z - P_M(x_\alpha))\}$. But $x_\alpha + (z - P_M(x_\alpha))$ is in $P_M^{-1}(z)$ for each α, and $P_M^{-1}(z) = P_M^{-1}(\theta) + z$ is bw-closed. Thus $x \in P_M^{-1}(z)$. ∎

COROLLARY 3. *If* X *is smooth,* $M = [\{x_\alpha\}_{\alpha \in A}]$ *is a reflexive subspace, and for each* α, $[x_\alpha]$ *is a Chebyshev subspace and either* $P_{[x_\alpha]}^{-1}(\theta)$ *or* $E(-x_\alpha, x_\alpha)$ *is* bw-*closed, then* P_M *is* bw-*continuous.*

Proof. In view of the proof of Corollary 2, it suffices to show that if E(-x, x) is bw-closed and [x] is a Chebyshev subspace, then $P_{[x]}^{-1}(\theta)$ is bw-closed. This may be done either directly or in a manner parallel to the proof of Theorem 2 and Corollary 1. ∎

Example 1. *A one-dimensional Chebyshev subspace* M *such that* P_M *is not weakly* (*or* bw-) *continuous.* Let $X = c_0$, the space of all sequences of real numbers converging to 0 with the sup-norm topology. Define M to be the one-dimensional space spanned by x = (1, 1/2, 1/3, 1/4, ⋯). It is easy to verify that M is a Chebyshev subspace. Let $x_n = (0, \cdots, 0, 2 + 1/n, 0, \cdots)$ for n = 2, 3, ⋯, where the nonzero term occurs in the nth coordinate. Now $\|x - x_n\| = \|2x - x_n\| = 2$ for each n, and hence $\{x_n\} \subset E(x, 2x)$. Furthermore, $\{x_n\}$ converges weakly to θ; hence E(x, 2x) (and hence E(-x, x)) is not weakly closed. To see that P_M is not weak continuous, notice that Lemma 1 implies that $P_M(x_n)$ is in $\{tx \mid 1 \leq t \leq 2\}$ for each n, so that $\{P_M(x_n)\}$ does not converge weakly to θ.

Example 2. *A two-dimensional Chebyshev subspace* M *of a reflexive space* X, *for which* P_M *is weakly continuous but* E(-x, x) *is not weakly closed for some* $x \in M$. We begin by constructing a new norm on the separable Hilbert space ℓ_2, such that the unit ball is like the usual ball except for a bump at the north pole and at the south pole. This may be done as follows. Let B be the unit ball of two-dimensional Euclidean space, that is, let $B = \{(x, y) \mid (x^2 + y^2)^{1/2} \leq 1\}$. Let C be some set containing B and satisfying the following four conditions.

(i) C is closed, smooth, strictly convex, and symmetric with respect to the origin,

(ii) $(x, y) \in C$ if and only if $(x, -y) \in C$,

(iii) $B \cap D = C \cap D$, where $D = \{(x, y) \mid |y| \geq 1/2\}$, and

(iv) there exists a number $\varepsilon > 0$ such that $(1 + \varepsilon, 0)$ lies on the boundary of C.

Define $\|\cdot\|'$ to be the Minkowski functional induced on the two-dimensional space by C. Let $X = \ell_2$ with the norm $|||\cdot|||$ defined below. For $x = (\xi_1, \xi_2, \cdots) \in \ell_2$, define

$$|||x||| = \left\| \left(\xi_1, \left(\sum_{i=2}^{\infty} |\xi_i|^2 \right)^{1/2} \right) \right\|' .$$

Now if we choose $x_1 = (1 + \varepsilon, 0, 0, \cdots)$, $x_2 = (0, 1, 0, \cdots)$, and

$$x_3 = (\sqrt{2}/2, \sqrt{2}/2, 0, 0, \cdots),$$

we can easily verify, by symmetry, that $E(-x_1, x_1)$ and $E(-x_2, x_2)$ are weakly closed hyperplanes, and hence, if we let $M = [x_1, x_2]$, we see from Corollary 2 that P_M is weakly continuous. However, $E(-x_3, x_3)$ agrees with a hyperplane except on a bounded set, and therefore it is not weakly closed (not even sequentially weakly closed). We note in addition that $P_{[x_3]}^{-1}(\theta)$ is the weakly closed hyperplane $\{x = (\xi_1, \xi_2, \cdots) \mid \xi_1 + \xi_2 = 0\}$, and that the converses of Theorem 2 and Corollary 1 are therefore false.

The authors wish to thank Professor R. B. Holmes for helpful suggestions.

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