A NOTE ON INVARIANT SUBSPACES

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P. R. Halmos and L. J. Wallen have asked whether on Hilbert space, there exists an algebraic linear transformation (not necessarily continuous) having no proper closed invariant subspace. In this note, we show that such transformations do exist. We also show, using a result of H. H. Schaefer [5], that every continuous linear transformation on the Fréchet space (s) of all sequences has a proper closed invariant subspace.

The author benefitted from discussions with Halmos and Wallen in the Diamond Head Circle Seminar.

THEOREM 1. *Let* H *denote a separable, infinite-dimensional, complex Hilbert space. Then there exists an algebraic linear transformation* T *of* H *into itself having no proper closed invariant subspace.*

Proof. Let c denote the cardinal number of the continuum, and let ω_c denote the first transfinite ordinal number for which the set of all smaller ordinal numbers has cardinality c. The set of all infinite-dimensional proper closed subspaces of H has cardinality c; well-order this set in a minimal well-ordering. That is, assign to each ordinal number α $(1 \leq \alpha < \omega_c)$ an infinite-dimensional proper closed subspace M_α such that each subspace occurs exactly once.

Using transfinite induction, we shall assign two vectors f_α, g_α to each ordinal number $\alpha < \omega_c$ such that

i) $f_\alpha \in M_\alpha$,

ii) $g_\alpha \notin M_\alpha$,

iii) the set of all vectors f and g is linearly independent.

Choose f_1 and g_1 to satisfy i), ii), iii) above for $\alpha = 1$. Now assume that $\alpha < \omega_c$ is given and that f_β, g_β have been chosen for all $\beta < \alpha$. Then the vector subspace V_α determined by all these f_β and g_β cannot contain M_α, since the cardinality of a Hamel basis for M_α is c. For f_α, select some vector that is in M_α but not in V_α. The vector subspace W_α determined by V_α and f_α is not all of H, since it has a Hamel basis of cardinality less than c; therefore it cannot contain any nonvoid open subset of H. Hence the set-theoretic union of M_α and W_α is not all of H. For g_α, choose some vector not in this union. Then the set of all vectors f_β, g_β $(\beta \leq \alpha)$ is linearly independent, and the induction is complete.

Extend the set of all f_α, g_α $(\alpha < \omega_c)$ to a Hamel basis for H by the adjunction of a set $\{h_\gamma\}$. Define the linear transformation T by requiring that

$$Tf_\alpha = g_\alpha, \quad Tg_\alpha = f_{\alpha+1} \qquad (\alpha < \omega_c).$$

Received November 25, 1969.

This research was done while the author was a visiting professor at the University of Hawaii; it was supported in part by the National Science Foundation.

Michigan Math. J. 17 (1970).

The definition of Th_γ requires the consideration of two cases. If the set $\{h_\gamma\}$ is infinite, let T be a permutation of this set with no finite orbit. If this set is finite, say

$$\{h_j\} = \{h_1, \cdots, h_n\},$$

put $Th_i = h_{i+1}$ $(i < n)$ and $Th_n = f_1$. Now that T has been defined on a Hamel basis, extend the definition to all of H by linearity. Then no proper closed infinite-dimensional subspace can be invariant under T, since each such subspace is an M_α for some α, and $Tf_\alpha = g_\alpha$ is not in M_α. Also, T can have no eigenvectors (every vector is a finite linear combination of vectors f, g, and h; the image under T of such a linear combination involves at least one new basis vector). Hence T can have no finite-dimensional invariant subspace, which completes the proof.

In 1963, H. H. Schaefer [5] proved that every algebraic linear transformation on an infinite-dimensional vector space has a proper invariant vector subspace. This leads to the following result. (This result may have been known to D. A. Raikov, though neither the statement nor the proof seems to have been published.)

Let (s) denote the space of all complex sequences f, with the seminorms

$$\|f\|_n = \max_{k \leq n} |f(k)|.$$

The set of all continuous linear functionals on (s) may be identified with the set (P) of all sequences that are finitely nonzero, by means of the pairing

$$(f, g) = \sum_{n=1}^{\infty} f(n)g(n) \qquad (f \in (s),\ g \in (P)).$$

Under this pairing, (s) may be identified with the algebraic dual of (P), the set of all algebraic linear functionals on the vector space (P) (no topology, no continuity). Hence, if V is a vector subspace of (P) and if g is a vector not in V, then there exists an element $f \in (s)$ such that

$$f \perp V \quad \text{and} \quad (f, g) = 1.$$

The set of all f $(f \perp V)$ is easily seen to be a closed subspace of (s).

THEOREM 2. *Every continuous linear transformation of* (s) *into itself has a proper closed invariant subspace.*

Proof. Let T be a continuous linear transformation on (s), and let T^* be the adjoint transformation on (P). By Schaefer's results referred to above, there exists a proper vector subspace V of (P) that is mapped into itself by T^*. The subspace $V^\perp$ of all $f \in (s)$ such that $f \perp V$ is then a proper closed subspace of (s), invariant under T.

In conclusion, we state two problems, the first of which arises in connection with W. Arveson's work [1] on transitive operator algebras.

1. Let H denote a separable, infinite-dimensional, complex Hilbert space. Let A be a bounded linear transformation on H such that neither A nor A^* has any eigenvectors. Does there exist an algebraic linear transformation T of H into itself, commuting with A and having no proper closed invariant subspace?

2. Does every commuting family of continuous linear transformations of the space (s) into itself have a common proper closed invariant subspace?

Added July 1, 1970. Theorem 2 occurs in a paper of K.-H. Körber [3, Satz 3].

In his dissertation [4], Alan Lambert has described a class of bounded operators with the property that each densely defined commuting operator (not necessarily closed or closable) must actually be bounded. The adjoints of his operators always have eigenvalues.

Thanks to recent work of B. E. Johnson and A. M. Sinclair [2], we have complete knowledge of when a compact or quasi-nilpotent operator A on a Banach space has discontinuous operators that are everywhere defined and commute with A.

REFERENCES

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