STRONG INERTIAL COEFFICIENT RINGS

William C. Brown

INTRODUCTION

A well-known theorem due to J. H. M. Wedderburn and A. Malcev states that if A is a finite-dimensional algebra over a field F and if A/N is separable over F, then there exists a subalgebra S of A such that

$$S + N = A \quad \text{and} \quad S \cap N = 0.$$

Here N denotes the radical of A. The theorem further states that S is unique up to an inner automorphism G of A. The automorphism G is of the form

$$G(X) = (1 - n)X(1 - n)^{-1},$$

for some n in N. Many authors have attempted to generalize this result by removing the restriction that F be a field. In particular, G. Azumaya [3] has extended the Wedderburn-Malcev theorem to the case where F is a Hensel ring. Azumaya proved the following result: Let A be an algebra over a Hensel ring R. If A is finitely generated as an R-module and if A/N is separable over R/p, then A contains a subalgebra S that is separable over R and has the property that $S + N = A$. Here N and p are the Jacobson radicals of A and R, respectively. Azumaya further proved that S is unique up to an inner automorphism G of A, where G is as in the original classical theorem. If R is a field, then $S \cap N = 0$, and we retrieve the original theorem. The Wedderburn-Malcev theorem yields an F-algebra isomorphism of A/N into A. Since $S \cap N \neq 0$ in general, we lose this isomorphism in Azumaya's generalization.

In [6], E. Ingraham has studied a class of commutative rings, called inertial coefficient rings, that permit a generalization of the Wedderburn-Malcev theorem along the lines of Azumaya's result. Specifically, a commutative ring R with identity is called an *inertial coefficient ring* if it has the following property: If A is an R-algebra that is finitely generated as an R-module and has the property that A/N is separable over R, then there exists an R-separable subalgebra S of A with $S + N = A$. If S is unique up to an inner automorphism of A generated by 1 plus an element of N, then R is said to have the *uniqueness property*. In these terms, Azumaya's result says that every Hensel ring is an inertial coefficient ring with the uniqueness property.

If A is an algebra over an inertial coefficient ring R and A satisfies the usual hypotheses, then there need not exist an algebra isomorphism of A/N into A that splits the sequence

$$0 \to N \to A \to A/N \to 0.$$

Received May 23, 1969.

This paper is based on my doctoral thesis, which was written at Northwestern University while I was a National Science Foundation Fellow. I wish to thank Professor Daniel Zelinsky for his guidance in preparing this work.

Michigan Math. J. 17 (1970).

However, if there exists a ring homomorphism $j: R/p \to R$ that splits

$$0 \to p \to R \to R/p \to 0,$$

then there may exist an (R/p)-algebra isomorphism $\varepsilon: A/N \to A$ that splits $0 \to N \to A \to A/N \to 0$. The purpose of this paper is to study those pairs (R, j) that permit $0 \to N \to A \to A/N \to 0$ to be split by some (R/p)-algebra isomorphism ε. We call these pairs strong inertial coefficient rings. We shall show that they can be identified with a proper subclass of inertial coefficient rings and enjoy many of the properties of inertial coefficient rings. Our main result will show that if R is a Hensel ring for which a splitting map j exists, then (R, j) is a strong inertial coefficient ring. We shall also obtain a partial converse to this theorem in the case where R is a local domain.

Strong inertial coefficient rings are of interest, because they retrieve the isomorphism that was lost in Azumaya's result.

PRELIMINARIES

Throughout this paper, we assume that all rings have an identity. All subrings contain the identity of the overring and all ring homomorphisms map the identity to the identity. By R, we always denote a commutative ring and by A, an R-algebra; that is, A is a ring, together with a ring homomorphism θ of R into the center of A. If θ is a monomorphism, we say A is a faithful R-algebra. We say A is a finitely generated or projective R-algebra if A is finitely generated or projective as an R-module. We say A is separable over R if A is projective as an $(A \otimes_R A^o)$-module. Here A^o of course denotes the opposite ring of A.

We shall let N denote the Jacobson radical of A and p the Jacobson radical of R. Since all rings are assumed to contain an identity, both N and p can be taken to be the intersection of all maximal right ideals of A and R, respectively. We state the following lemma concerning p and N.

LEMMA. *Let* A *be a finitely generated* R-*algebra, and let* $\bigcap mA$ *denote the intersection of the ideals* mA, *as* m *runs through all maximal ideals of* R. *Then*

a) $pA \subseteq N$,

b) *there exists a positive integer* n *such that* $N^n \subseteq \bigcap mA$,

c) *if* A *is projective over* R, *then* $pA = \bigcap mA$, *and*

d) *if* A *is separable over* R, *then* $N = \bigcap mA$.

For a proof, see [6, Lemma 1.1].

The lemma implies that if A is a finitely generated R-algebra, then A/N is a finitely generated (R/p)-algebra. The algebra A/N is separable over R if and only if A/N is separable over R/p. More generally, if I is an ideal of R contained in the annihilator of A, then A is naturally an (R/I)-algebra. The algebra A is separable over R if and only if A is separable over R/I. We shall make this shift between coefficient rings frequently without further comment.

Throughout this paper, π and π_0 denote the canonical projections of A and R onto A/N and R/p, respectively. We say R splits as a ring if there exists a ring homomorphism $j: R/p \to R$ such that $\pi_0 j$ is the identity on R/p. We shall indicate that R is split by j by writing the pair (R, j). If we have a pair (R, j) and a finitely generated R-algebra A, then A can be regarded as an (R/p)-algebra via the mapping j. The sequence

$$0 \to N \to A \to A/N \to 0$$

is exact both as R-algebras and as (R/p)-algebras. We say A splits as an (R/p)-algebra if there exists an (R/p)-algebra homomorphism ζ: A/N → A such that $\pi\zeta$ is the identity on A/N.

Finally, we said in the introduction that an inertial coefficient ring R has the uniqueness property if S is unique in A up to a special type of inner automorphism. This means if S' is another separable subalgebra of A with S' + N = A, then there exists an element n in N such that $(1 - n)S(1 - n)^{-1} = S'$.

1. STRONG INERTIAL COEFFICIENT RINGS AND THEIR ELEMENTARY PROPERTIES

Definition. A pair (R, j) is called a *strong inertial coefficient ring* if every finitely generated R-algebra A splits as an (R/p)-algebra, provided A/N is separable over R. A strong inertial coefficient ring (R, j) is said to have the *uniqueness property* if for each pair of (R/p)-algebra homomorphisms ζ and ζ^1 splitting A, there exists an element n in N such that

$$(1 - n)\zeta(x)(1 - n)^{-1} = \zeta^1(x) \qquad \text{for all } x \text{ in } A/N.$$

In terms of subalgebras of A, the pair (R, j) is a strong inertial coefficient ring if and only if there exists an (R/p)-subalgebra S of A such that S + N = A and S ∩ N = 0. The uniqueness property means that for every two such subalgebras S and S', there exists an element n in N such that $(1 - n)S(1 - n)^{-1} = S'$.

PROPOSITION 1. *If* (R, j) *is a strong inertial coefficient ring, then* R *is an inertial coefficient ring.*

Proof. Let A be a finitely generated R-algebra such that A/N is separable over R. Since (R, j) is a strong inertial coefficient ring, there exists an (R/p)-subalgebra S of A such that $S \oplus N = A$. The algebras S and A/N are isomorphic as (R/p)-algebras, and hence S is separable over R/p. Consider RS ⊂ A. The algebra RS is an R-subalgebra of A, and RS + N = A. The fact that S is separable over R/p implies that $R \otimes S$ is separable over R. Here the tensor product is taken over R/p. Since RS is a homomorphic image of $R \otimes S$, it follows that RS is separable over R. ■

Proposition 1 implies that the collection of strong inertial coefficient rings can be identified with a subclass of inertial coefficient rings. In general, if an inertial coefficient ring R is split by some ring map j, the pair (R, j) need not be a strong inertial coefficient ring. It is known [6, Theorem 3.16] that every Dedekind domain with radical 0 is an inertial coefficient ring. Hence the integers Z are an example of an inertial coefficient ring that is split as a ring by the identity mapping 1. It is easy to see that (Z, 1) is not a strong inertial coefficient ring by considering, for example, Z/4Z as our algebra A. More generally, if R is a commutative ring with identity having radical 0 and a proper maximal ideal I such that $I \neq I^2 \neq 0$, then (R, 1) cannot be a strong inertial coefficient ring. For if we let $A = R/I^2$, then A is finitely generated over R and A/N = R/I is separable over R; but R/I^2 cannot split as an R-algebra. Thus the mapping that sends (R, j) to R maps the class of strong inertial coefficient rings onto a proper subclass of inertial coefficient rings.

PROPOSITION 2. *Let* (R, j) *be a strong inertial coefficient ring with the uniqueness property. Let* $\sigma\colon R \to S$ *be a ring homomorphism of* R *onto the commutative ring* S. *Then there exists a ring mapping* j' *that splits* S *and has the property that* (S, j') *is a strong inertial coefficient ring with the uniqueness property.*

Proof. Let p' denote the Jacobson radical of S. Then $\sigma(p) \subset p'$. Hence S/p' is a ring-homomorphic image of R/p ($R/p \to S/\sigma(p) \to S/p'$). Therefore, S/p' is separable over R/p and hence also over R. Since (R, j) is a strong inertial coefficient ring, there exists an (R/p)-algebra homomorphism j': $S/p' \to S$ splitting S. Now let A be a finitely generated S-algebra such that A/N is separable over S. Then, via σ, we can regard A as an algebra over R. Clearly, A is finitely generated over R. Moreover, A/N is separable over S/p' and hence also over R/p. Therefore A/N is separable over R. Since (R, j) is a strong inertial coefficient ring, there exists an (R/p)-subalgebra T of A such that $T \oplus N = A$. The algebra T is also an (S/p')-subalgebra of A, and hence A splits as an (S/p')-algebra. Hence (S, j') is a strong inertial coefficient ring. If (R, j) has the uniqueness property, then (S, j') has the uniqueness property, because every (S/p')-subalgebra of A is also an (R/p)-subalgebra. ∎

PROPOSITION 3. *Let* (R_i, j_i) $(i = 1, \cdots, n)$ *denote a finite number of pairs. Then* $(\oplus R_i, \oplus j_i)$ *is a strong inertial coefficient ring with the uniqueness property if and only if each* (R_i, j_i) *is a strong inertial coefficient ring with the uniqueness property.*

Proof. Suppose $(\oplus R_i, \oplus j_i)$ is a strong inertial coefficient ring with the uniqueness property. Since each (R_i, j_i) is a homomorphic image of $(\oplus R_i, \oplus j_i)$, we may use Proposition 2 (with j_i in place of j') to conclude that each (R_i, j_i) is also a strong inertial coefficient ring with the uniqueness property. To prove the other direction of Proposition 3, it suffices, by induction, to consider only the case $n = 2$. Let A be a finitely generated algebra over $R_1 \oplus R_2$ such that A/N is separable over $R_1/p_1 \oplus R_2/p_2$. Then A decomposes into an orthogonal direct sum of an R_1-subalgebra A_1 and an R_2-subalgebra A_2. Moreover, N decomposes into a direct sum of N_1 and N_2, where N_i is the radical of A_i. Each A_i is finitely generated over R_i. The fact that A/N is separable over $R_1 \oplus R_2$ implies that A_i/N_i is separable over R_i. Since each (R_i, j_i) is a strong inertial coefficient ring, each A_i splits as (R_i/p_i)-algebras via, say, ζ_i. Then $\zeta_1 \oplus \zeta_2$ splits A as an $(R_1/p_1 \oplus R_2/p_2)$-algebra. Hence $(R_1 \oplus R_2, j_1 \oplus j_2)$ is a strong inertial coefficient ring. To show it has the uniqueness property, we proceed in a similar manner. If S and T are two $(R_1/p_1 \oplus R_2/p_2)$-subalgebras of A such that $S \oplus N = A$ and $T \oplus N = A$, we break S and T into components S_i and T_i in A_i. Since each (R_i, j_i) has the uniqueness property, there exists an element n_i in N_i such that

$$(1_i - n_i) S_i (1_i - n_i)^{-1} = T_i .$$

Here 1_i denotes the identity of A_i, and we have the relations $1_1 + 1_2 = 1$ and $1_1 1_2 = 0$. Since $A_1 A_2 = 0$, a direct calculation shows that if $n = n_1 + n_2$, then $(1 - n) S (1 - n)^{-1} = T$. ∎

Propositions 2 and 3 show that the class of strong inertial coefficient rings is closed under the formation of direct sums and homomorphic images.

If R is an inertial coefficient ring and p a nonzero prime ideal of R, then R_p, the localization of R at p, need not be another inertial coefficient ring. We may again cite the integers Z as an example. It is known [6, Theorem 3.11] that every noetherian local domain with perfect residue class field is an inertial coefficient ring if and only if it is a Hensel ring. (The reader may consult Section 2 of this paper for the definition of a Hensel ring.) Since Z_p is not a Hensel ring for any $p \neq 0$, Z_p cannot be an inertial coefficient ring.

Strong inertial coefficient rings are not closed under localizations either. To see this, we need the following proposition, which is due to P. Samuel.

PROPOSITION 4. *Let* R *be a noetherian local domain and* p *a nonzero prime ideal of* R. *If* R_p *is a Hensel ring, then* p *is the maximal ideal of* R.

Proof. Let m denote the maximal ideal of R. By a quadratic transformation, there exists a noetherian local domain R_1 of altitude 1 that dominates R [1, page 15, 1.33]. The integral closure $\overline{R}_1$ in its quotient field is a Krull ring of altitude 1. Hence, if we localize $\overline{R}_1$ at some maximal ideal, we obtain a discrete valuation ring dominating R. Since R_1 lies in the quotient field of R, there exists a discrete valuation ν (with values in Z) of the quotient field of R that dominates R. Then $\nu(R)$ is a subsemigroup of Z that generates Z. Hence $\nu(R)$ contains all large integers. Let $n > 2$ be an integer prime to the characteristic of R/m (and of R/p, also). Suppose $p \neq m$. There exists an a in p $(a \neq 0)$ such that $\nu(a)$ is prime to n. Let s' be an element of $m - p$. Then a suitable power s of s' satisfies the conditions

$$n \mid \nu(s) \quad \text{and} \quad \nu(a) < \nu(s).$$

Consider now the polynomial $x^n - (1 + a/s)$ over R_p. Its reduced polynomial (modulo pR_p) is $x^n - 1$ and admits 1 as a simple root. Since R_p is Henselian, $x^n - (1 + a/s)$ has a root in R_p. Call this root χ. Then $1 + a/s = \chi^n$. Hence $a + s = s\chi^n$; but $\nu(a + s) = \nu(a)$ (since $\nu(a) < \nu(s)$). It follows that $n \mid \nu(s)$ and $n \mid \nu(s\chi^n)$, and thus $n \mid \nu(a)$, a contradiction. ∎

Using Proposition 4, we can show that localizations of strong inertial coefficient rings need not be strong inertial coefficient rings, even when some splitting map is available. Let $F[[x, y]]$ denote the power series ring in two indeterminates x and y over an arbitrary field F. Since $F[[x, y]]$ is a Hensel ring that is split as a ring via 1, Theorem 1 in this paper implies that $(F[[x, y]], 1)$ is a strong inertial coefficient ring. If we localize $F[[x, y]]$ at the prime ideal generated by x, we obtain a noetherian, integrally closed, local domain that is split as a ring by 1. Call this ring $F[[x, y]]_{(x)}$. By Proposition 4, $F[[x, y]]_{(x)}$ is not a Hensel ring. Now it is known that an integrally closed local domain is a Hensel ring if and only if it is an inertial coefficient ring [6, Theorem 3.11]. Using Proposition 1, we conclude that $(F[[x, y]]_{(x)}, 1)$ is not a strong inertial coefficient ring.

Finally, we shall show that the class of strong inertial coefficient rings is closed under some types of extensions.

Definitions. A pair (R', j') is said to be an *extension* of the pair (R, j) if

(a) R' is a finitely generated algebra over R (say via $\theta: R \to R'$) and

(b) $\theta j = j' \overline{\theta}$, where $\overline{\theta}$ is the induced mapping on the rings modulo their radicals.

An extension (R', j') of (R, j) is said to be *residually separable* if R'/p' is separable over R/p. Here p' is the Jacobson radical of R'.

E. Ingraham suggested the next proposition to me.

PROPOSITION 5. *Let* S *be a separable, commutative* R-*algebra, and let* A *be an* S-*algebra. If* A *splits as an* R-*algebra, then* A *splits as an* S-*algebra.*

Proof. The set $\mathrm{Hom}_R(A/N, N)$ is an $S \otimes_R S$-module in the following way: let

$$\{(s \otimes s')f\}(a) = sf(s'a) \qquad (f \in \mathrm{Hom}_R(A/N, N),\ a \in A/N).$$

Since S is separable over R, there exists an element $e = \sum x_i \otimes y_i$ in $S \otimes_R S$ such that $\sum x_i y_i = 1$ and

$$\sum sx_i \otimes y_i = \sum x_i \otimes y_i s,$$

for all s in S. Now let Ψ be an R-algebra homomorphism splitting

$$0 \to N \to A \to A/N \to 0.$$

Then Ψ is an element of $\mathrm{Hom}_R(A/N, N)$, and thus $e \cdot \Psi$ is well defined. One can directly verify that $e \cdot \Psi$ is an S-module homomorphism splitting

$$0 \to N \to A \to A/N \to 0.$$

Since Ψ is a ring homomorphism, $e \cdot \Psi$ is also a ring homomorphism. Hence $e \cdot \Psi$ is an S-algebra homomorphism splitting A. ■

COROLLARY. *If* (R', j') *is a residually separable extension of the strong inertial coefficient ring* (R, j), *then* (R', j') *is a strong inertial coefficient ring. If* (R, j) *has the uniqueness property, then* (R', j') *has the uniqueness property.*

Proof. Since R'/p' is a commutative, separable extension of R/p, the result follows directly from Proposition 5. Every (R'/p')-subalgebra of an R'-algebra A is also an (R/p)-subalgebra. Hence the uniqueness statement follows. ■

2. SPLIT HENSEL RINGS

As usual, let R denote a commutative ring with identity.

Definition. R is called a *quasi-local ring* if the nonunits of R form an ideal.

If R is a quasi-local ring, it has a unique maximal ideal p, which is its Jacobson radical. Let R[t] denote the ring of polynomials in an indeterminate t over R. Then we have a natural ring homomorphism λ of R[t] onto (R/p)[t], defined as follows: if $g(t) = \sum r_i t^i$ is in R[t], then

$$\lambda(g(t)) = \sum \pi_0(r_i)t^i.$$

We shall usually denote $\lambda(g(t))$ in (R/p)[t] by $\bar{g}(t)$ or just by $\bar{g}$.

Definition. A quasi-local ring R is said to be a *Hensel ring* if every monic polynomial f(t) in R[t] satisfies the following condition. If there exist two relatively prime polynomials $g_1(t)$ and $g_2(t)$ in (R/p)[t] such that $\bar{f} = g_1 g_2$ and $g_1(t)$ is monic, then there exist two polynomials h_1 and h_2 in R[t] such that $h_1 h_2 = f$, $\bar{h}_1 = g_1$, $\bar{h}_2 = g_2$, and h_1 is monic.

With these definitions, we proceed to the main results of this paper.

THEOREM 1. *Let* R *be a Hensel ring that is split as a ring by a ring homomorphism* j. *Then* (R, j) *is a strong inertial coefficient ring with the uniqueness property.*

Proof. Let A be a finitely generated R-algebra such that A/N is separable over R. First we prove the theorem in the following special case: Assume A is central-separable over R. In this case, we have that N = pA, and $0 \to N \to A \to A/N \to 0$ becomes $0 \to pA \to A \to A/pA \to 0$. It is known that the Brauer group $\beta(R)$ of R and the Brauer group $\beta(R/p)$ of R/p are isomorphic under the mapping that takes an element (A) in $\beta(R)$ to the element (A/pA) in $\beta(R/p)$ [3, Theorem 31]. Consider the R-algebra $A' = A/pA \otimes R$, where the tensor product is taken over R/p and R is regarded as an (R/p)-algebra via j. Then A' is a central separable R-algebra having A/pA as its corresponding residue class algebra. It is known that there exists (up to an R-isomorphism) only one central separable R-algebra having a prescribed central simple (R/p)-algebra as residue class algebra [3, Theorem 32]. Since A and A' have the same residue class algebra A/pA, there exists an R-algebra isomorphism Ψ mapping $A/pA \otimes R$ onto A. If we denote the composite map

$$A/pA \to A/pA \otimes R \to A$$

by σ, then σ is an (R/p)-algebra homomorphism of A/pA into A. One can verify that $\pi(\sigma(A/pA)) = A/pA$. Hence

$$\sigma(A/pA) + pA = A.$$

Since $\sigma(A/pA)$ is simple, we have that $pA \cap \sigma(A/pA) = 0$. Hence $\sigma(A/pA)$ determines an (R/p)-algebra homomorphism of A/pA into A that splits

$$0 \to pA \to A \to A/pA \to 0.$$

Therefore we proved the theorem for central separable R-algebras.

We now proceed to the general case. By [3, Theorem 33], A contains an inertial subalgebra S that is separable over R. It is clear that if S splits as an (R/p)-algebra, then A splits as an (R/p)-algebra. Hence we may assume without loss of generality that A is separable over R. Let C(A) denote the center of A. Then C(A) is separable and finitely generated over R [2, Theorems 2.1 and 2.3]. Since C(A) contains a homomorphic image of R (the image being another split Hensel ring with the same residue class field as R), it follows from [8, 43.15 and 43.16] that C(A) is a finite direct sum of Hensel rings. Let us write

$$C(A) = R_1 \oplus \cdots \oplus R_n.$$

Each R_i is a Hensel ring and a finitely generated, separable R-algebra. We now follow the same procedure as in the special case to show that each R_i is split as a ring via an (R/p)-algebra homomorphism. Fix i ($1 \leq i \leq n$). Suppose p_i is the Jacobson radical (that is, maximal ideal) of R_i. Then

$$0 \to p_i \to R_i \to R_i/p_i \to 0$$

is exact, and R_i/p_i is a separable field extension of R/p. Let θ_i denote the ring homomorphism giving R_i the structure of an R-algebra. Then R_i contains $\theta_i(R)$, and $\theta_i(R)$ is a Hensel ring having R/p as its residue class field. The ring $\theta_i(R)$ is also split as a ring via $\theta_i\, j$. Now let

$$Z = R_i/p_i \otimes \theta_i(R),$$

where the tensor product is taken over R/p, as usual. Then Z is a quasi-local, unramified, regular extension of $\theta_i(R)$. The residue class field of Z is isomorphic to R_i/p_i. It now follows from [3, Lemma 5] that there exists a unique $\theta_i(R)$-algebra homomorphism ϕ: $Z \to R_i$ that is an isomorphism of R_i/p_i, modulo pZ. Then ϕ is also an R-algebra homomorphism. Let us denote the composite map

$$R_i/p_i \to R_i/p_i \otimes \theta_i(R) = Z \xrightarrow{\phi} R_i$$

by ψ. Then ψ is an (R/p)-algebra homomorphism of R_i/p_i into R_i. One can readily verify that $\psi(R_i/p_i) \oplus p_i = R_i$. Hence R_i is split as an (R/p)-algebra.

Now A is central separable over $C(A) = R_1 \oplus \cdots \oplus R_n$. Hence A can be written as $A_1 \oplus \cdots \oplus A_n$, where each A_i is a central separable R_i-algebra. Since each R_i is a Hensel ring that is split as a ring, it follows from the special case that A_i is split as an (R_i/p_i)-algebra. In particular, each A_i is split as an (R/p)-algebra. Hence A is split as an (R/p)-algebra.

We have now proved that (R, j) is a strong inertial coefficient ring. To show that (R, j) has the uniqueness property, we can proceed as in the classical case. Suppose A is split by two (R/p)-algebra homomorphisms ζ_1 and ζ_2. Then N can be regarded as an (A/N)-bimodule in the following way: let

$$n\bar{a} = n\zeta_2(\bar{a}) \quad \text{and} \quad \bar{a}n = \zeta_1(\bar{a})n \qquad (\bar{a} \in A/N,\ n \in N).$$

Let f: $A/N \to N$ be defined by $f(\bar{a}) = \zeta_1(\bar{a}) - \zeta_2(\bar{a})$. Then f is a crossed homomorphism of A/N into N and hence determines an element of Hochschild's first cohomology group $H^1(A/N, N)$. Since A/N is separable over R/p, it follows that $H^1(A/N, N) = 0$. Thus f is a principal homomorphism, in other words, there exists an element m in N such that $f(\bar{a}) = \bar{a}m - m\bar{a}$. Hence

$$\zeta_1(\bar{a}) - \zeta_2(\bar{a}) = \zeta_1(\bar{a})m - m\zeta_2(\bar{a})$$

or, equivalently, $(1 - m)\zeta_2(\bar{a})(1 - m)^{-1} = \zeta_1(\bar{a})$, for all $\bar{a}$ in A/N. ∎

Theorem 1 is false if R is not a Hensel ring. Consider the following example: Let Q denote the rational numbers and $Q[x]_{(x)}$ the localization of the polynomial ring $Q[x]$ at the prime ideal generated by x. It is easy to see that $Q[x]_{(x)}$ is not a Hensel ring. The polynomial

$$y^2 + y + x$$

in $\{Q[x]_{(x)}\}[y]$, for example, does not satisfy Hensel's lemma. The fact that $(Q[x]_{(x)}, 1)$ is not a strong inertial coefficient ring follows directly from Proposition 1 and [6, Theorem 3.11]. We can even exhibit a central separable $(Q[x]_{(x)})$-algebra that fails to split as a Q-algebra.

Let Λ be the generalized quaternion algebra over $Q[x]_{(x)}$ with basis 1, α, β, and $\alpha\beta$, where

$$\alpha^2 = \beta^2 = x - 2 \quad \text{and} \quad \alpha\beta = -\beta\alpha .$$

Then Λ is a free $Q[x]_{(x)}$-algebra with basis 1, α, β, and $\alpha\beta$. Since $Q[x]_{(x)}$ is noetherian, Λ is separable over $Q[x]_{(x)}$ if and only if $\Lambda/x\Lambda$ is separable over Q

[2, Theorem 4.7]. But $\Lambda/x\Lambda$ is the quaternion algebra over Q generated by 1, $\bar{\alpha}$, $\bar{\beta}$, $\bar{\alpha}\bar{\beta}$ with relations

$$\bar{\alpha}^2 = -2 = \bar{\beta}^2 \quad \text{and} \quad \bar{\alpha}\bar{\beta} = -\bar{\beta}\bar{\alpha}.$$

It follows from [5, page 67, Theorem 1] that $\Lambda/x\Lambda$ is a division algebra over Q. Since $\Lambda/x\Lambda$ has center Q, $\Lambda/x\Lambda$ is central-separable over Q. It follows that Λ is central-separable over $Q[x]_{(x)}$. The Jacobson radical of Λ is $x\Lambda$. We shall show that Λ cannot split as a Q-algebra. For if a splitting map ζ existed, then Λ would have an element whose square is -2. Hence there exist constants a_0, a_1, a_2, and a_3 in $Q[x]_{(x)}$ such that

$$a_0^2 + (x-2)a_1^2 + (x-2)a_2^2 - (x-2)^2 a_3^2 = -2$$

and $a_0 a_i = 0$ $(i = 1, 2, 3)$. Clearly $a_0 = 0$. The resulting equation is equivalent to the equation

$$(x-2)A_1^2 + (x-2)A_2^2 - (x-2)^2 A_3^2 = -2A_4^2$$

in $Q[x]$. If this equation had a solution in $Q[x]$, we could assume that the A_i have no common factor, and by repeatedly letting $x = 2$, we could proceed to the contradiction that $x - 2$ divides all the A_i. Hence Λ cannot have an element whose square is -2, and thus no such splitting map ζ exists.

We say a quasi-local ring R is local if it is noetherian. If R is a local integral domain, we can obtain a converse to Theorem 1. We proceed by proving some new results about inertial coefficient rings.

Definition. If R is an integral domain with quotient field K, then the *derived normal ring* of R is the integral closure R' of R in K.

THEOREM 2. *Let* R *be a local domain with maximal ideal* p *and derived normal ring* R'. *Let* N *denote the Jacobson radical of* R'. *If* R *is an inertial coefficient ring and* R'/N *is separable over* R/p, *then* R' *is quasi-local.*

Proof. Since R'/N is separable over the field R/p, it follows from [9, Theorem 1] that R'/N is a finite-dimensional algebra over R/p. Let $\bar{u}_1, \cdots, \bar{u}_n$ be a basis of R'/N over R/p. Choose $u_1, \cdots, u_n$ in R' such that $\pi(u_i) = \bar{u}_i$, and consider the ring $S = R[u_1, \cdots, u_n]$. Since each u_i is integral over R, the ring S is a finitely generated R-module. If we let q denote the Jacobson radical of S, then it follows from [4, Theorem 1] that $N \cap S = q$. Now π maps S onto R'/N with kernel $N \cap S$. Hence

$$0 \to q \to S \to R'/N \to 0$$

is exact. Since R'/N is separable over R/p, S/q is separable over R/p. Since q contains p, S/q is separable over R. Now R is an inertial coefficient ring; hence there exists an R-separable subalgebra T of S such that $T + q = S$. The subalgebra T is a finitely generated, separable extension of R and hence is an inertial coefficient ring [6, Proposition 3.3]. Moreover, T is clearly semilocal and connected (it has no idempotents other than 0 and 1). It now follows from [6, Theorem 3.6] that T is a local ring. Since S is integral over T, it follows again from [4, Theorem 1] that the maximal ideal of T is $q \cap T$. Therefore we have the relations

$$S/q = T + q/q = T/T \cap q,$$

and $T/T \cap q$ is a field. Hence q is a maximal ideal of S, and thus N is a maximal ideal of R'. The fact that N is maximal implies that R' is quasi-local with unique maximal ideal N. ∎

COROLLARY. *Let* R *and* R' *be as in Theorem* 2. *Then if* $\{x_1, \cdots, x_m\}$ *is some finite collection of elements of* R', *there exists an* R-*subalgebra* S *of* R' *having the following four properties.*

1) $S \supset R[x_1, \cdots, x_m]$.
2) S *is a local ring with maximal ideal* $N \cap S = q$.
3) $S/q = R'/N$.
4) S *contains a separable* R-*algebra* T *having the properties that*
   a) T *is an inertial coefficient ring;*
   b) T *is a local ring with maximal ideal* $T \cap q$;
   c) $T + q = S$ *and* $T/T \cap q = S/q = R'/N$.

Proof. Let $S = R[x_1, \cdots, x_m, u_1, \cdots, u_n]$, and proceed as in Theorem 2. ∎

THEOREM 3. *Let* R *be a local domain and* R' *its derived normal ring. Let* p *and* N *denote the radicals of* R *and* R', *respectively. If* R *is an inertial coefficient ring and* R'/N *is separable over* R/p, *then* R *is a Hensel ring.*

Proof. By Theorem 2, we know that R' is a quasi-local ring and that R' is normal. We shall show first that R' is a Hensel ring. Suppose that R' is not a Hensel ring. Then, by [8, 43.2], there exists a monic polynomial

$$f(x) = x^n + c_1 x^{n-1} + \cdots + c_n$$

in $R'[x]$ such that c_1 is not in N, the coefficients $c_2, \cdots, c_n$ are in N, and $f(x)$ has no linear factor of the form $x + b$ with $b - c_1$ in N. We consider the collection of all such polynomials and select one of minimal degree. Call this polynomial $f(x)$ also. Then $f(x)$ is monically irreducible in $R'[x]$. By the corollary to Theorem 2, we can find an S having properties 1 through 4 and such that $c_1, \cdots, c_n$ are in S. Hence $f(x)$ is in $S[x]$ and is monically irreducible. Since $N \cap S$ is the maximal ideal q of S, it also follows that c_1 is not in q, that $c_2, \cdots, c_n$ are in q, and that $f(x)$ has no linear factor of the form $x + b$ with b in S and $b - c_1$ in q. We also note from the proof of the corollary to Theorem 2 that we can take S to be finitely generated over R.

Now consider $A = S[x]/(f(x))$. The algebra A is finitely generated over S. Moreover, A is connected, since $f(x)$ is irreducible. The algebra A is finitely generated over T, since S is finitely generated over T. Let M denote the Jacobson radical of A. Then the definition of $f(x)$ in $S[x]$ implies that

$$A/M = S/q \oplus S/q = T/T \cap q \oplus T/T \cap q.$$

Hence A/M is separable over $T/T \cap q$. Thus A/M is separable over T. Now T is an inertial coefficient ring; thus we can find a T-separable subalgebra V of A such that $V + M = A$. The subalgebra V is also finitely generated over T and thus is a semilocal ring. Moreover, V is connected, since it is contained in A. Using [6, Proposition 3.3] again, we see that V is an inertial coefficient ring. Using [6, Theorem 3.6], we see that V is a local ring. Since A is integral over T, A is integral over V. Hence $M \cap V$ is the maximal ideal of V. We now have arrived at a contradiction, for we have the relation

$$S/q \oplus S/q = A/M = V + M/M = V/V \cap M,$$

and $V/V \cap M$ is a field. Hence R' must be a Hensel ring.

Since R' dominates R, we may assume, by [7, Theorem 4], that the Henselization R^* of R lies inside R'. Hence we have the inclusions

$$R \subset R^* \subset R' \subset K,$$

where K is the quotient field of R. By [7, Corollary 1], we have that $aR^* \cap R = a$, for all ideals a in R. By [8, 43.8], $\otimes_R R^*$ is exact. It now follows from [8, 18.4] that $K \cap R^* = R$. Hence $R = R^*$, and R is a Hensel ring. ■

COROLLARY. *Let* R *be as in Theorem* 3, *and suppose* R *is split by some ring homomorphism* j. *If* (R, j) *is a strong inertial coefficient ring, then* R *is a Hensel ring.*

Proof. If (R, j) is a strong inertial coefficient ring, then R is an inertial coefficient ring. Now the result follows from Theorem 3. ■

We note that Ingraham proved Theorem 3 under the condition that R be integrally closed. Our Theorem 3 is more general, as can be seen from the following example. Let F be a field, and let F[[x]] denote the ring of formal power series in the variable x. Let R be the subring of F[[x]] consisting of all power series of the form

$$a_0 + a_2 x^2 + a_3 x^3 + \cdots \qquad (a_i \in F).$$

Then R is an example of a local domain for which the hypotheses of Theorem 3 are satisfied; but R is not integrally closed.

REFERENCES

1. S. S. Abhyankar, *Resolution of singularities of embedded algebraic surfaces.* Academic Press, New York, 1966.

2. M. Auslander and O. Goldman, *The Brauer group of a commutative ring.* Trans. Amer. Math. Soc. 97 (1960), 367-409.

3. G. Azumaya, *On maximally central algebras.* Nagoya Math. J. 2 (1951), 119-150.

4. I. S. Cohen and A. Seidenberg, *Prime ideals and integral dependence.* Bull. Amer. Math. Soc. 52 (1946), 252-261.

5. L. E. Dickson, *Algebras and their arithmetics.* Dover Publications, Inc., New York, 1960.

6. E. C. Ingraham, *Inertial subalgebras of algebras over commutative rings.* Trans. Amer. Math. Soc. 124 (1966), 77-93.

7. M. Nagata, *On the theory of Henselian rings. III.* Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 32 (1959), 93-101.

8. M. Nagata, *Local rings*. Interscience Tracts in Pure and Applied Mathematics, No. 13. Interscience Publishers, New York, 1962.

9. A. Rosenberg and D. Zelinsky, *Cohomology of infinite algebras*. Trans. Amer. Math. Soc. 82 (1956), 85-98.

Michigan State University
East Lansing, Michigan 48823