A NONLINEAR PROBLEM IN POTENTIAL THEORY

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1. In this paper we study the following nonlinear boundary-value problem in the unit disc:

$$(1)\qquad \begin{cases} \Delta u + g(x, y, u) = 0 & ((x, y) \in A = [x^2 + y^2 < 1]), \\ u = 0 & ((x, y) \in \partial A = [x^2 + y^2 = 1]), \end{cases}$$

where g is a measurable function of x, y, u satisfying, for some given constants $R_1 > 0$, $R_2 \geq 0$, $L \geq 0$, the inequalities

$$|g(x, y, u)| \leq R_2 \quad \text{for almost all } (x, y) \in A \text{ and for } |u| \leq R_1,$$

$$(2)\qquad |g(x, y, u_1) - g(x, y, u_2)| \leq L\,|u_1 - u_2| \quad \text{for almost all } (x, y) \in A$$

$$\text{and for } |u_1|, |u_2| \leq R_1.$$

We prove that if g satisfies certain additional inequalities limiting its values and its growth with respect to u, then problem (1) has at least one solution $u(x, y)$ $((x, y) \in A)$ such that

(i) $u(x, y)$ is continuous in $A \cup \partial A$ and is zero on ∂A,

(ii) $u(x, y)$ has first-order partial derivatives that are continuous in A,

(iii) Δu, in the sense of the theory of distributions, is a measurable essentially bounded function,

(iv) Δu satisfies (1) a.e. in A.

If g is also sufficiently smooth in (x, y), then u has continuous second-order partial derivatives and (1) holds everywhere in A in the strict sense. The conditions concerning the growth of g are not unreasonably strict. For instance, for the problem

$$\Delta u + f(x, y)\,|u| = h(x, y) \qquad ((x, y) \in A),$$

$$u = 0 \qquad ((x, y) \in \partial A),$$

all that we require of the measurable functions f and h is that they are bounded and that $|f(x, y)| < 4.13$ in A. The example shows that the present requirement concerning the growth of g is far removed from the usually very strict requirements that are necessary in the use of perturbation techniques.

For the above problem in nonlinear partial differential equations, we apply here a process that we discussed in some generality in [2] and [4] and that has been studied, applied, and extended in a number of ways (see [1], [3], [5], [6], [8], [10],

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[12] to [21], [23], [25]), in connection with boundary-value problems in nonlinear ordinary or partial differential equations and other functional equations.

2. Let $S = L_2(A)$ be the Hilbert space of all real functions $u(x, y)$ that are measurable and L^2-integrable in A. Let $u \cdot v$ and $\|u\|$ denote the inner product and the norm in S, respectively. Let S' be the subset of S of all functions $u(x, y)$ that are essentially bounded in A. Let

$$\mu(u) = \text{Ess Sup}\,|u(x, y)| \qquad \text{for all } (x, y) \in A$$

(so that $0 \leq \mu(u) < +\infty$ for every $u \in S'$). We shall often write z for (x, y), or use polar coordinates ρ, θ ($\rho \cos\theta = x$, $\rho \sin\theta = y$, $\rho \geq 0$, $0 \leq \theta \leq 2\pi$).

The familiar linear problem

$$\text{(3)} \qquad \begin{cases} \Delta u + \ell u = 0 & ((x, y) \in A), \\ u = 0 & ((x, y) \in \partial A) \end{cases}$$

has a fundamental system of eigenvalues $\ell_i = \lambda_i^2$ and orthonormal eigenfunctions ϕ_i $(i = 1, 2, \cdots)$, with $0 < \lambda_1^2 < \lambda_2^2 \leq \cdots$. We know that $\{\phi_i\}$ is a complete orthonormal system in $L_2(A)$. In polar coordinates, problem (1) becomes

$$\text{(4)} \qquad \begin{cases} u_{\rho\rho} + \rho^{-1} u_\rho + \rho^{-2} u_{\theta\theta} = g & (0 \leq \rho < 1), \\ u = 0 & (\rho = 1). \end{cases}$$

For problem (3), the eigenfunctions ϕ_i are all the functions of the form

$$\text{(5)} \qquad \begin{cases} \phi_{0m} = \nu_{0m} J_0(\lambda_{0m}\rho), \\ \phi_{nm} = \nu_{nm} J_n(\lambda_{nm}\rho) \cos n\theta, \\ \psi_{nm} = \nu_{nm} J_n(\lambda_{nm}\rho) \sin n\theta, \end{cases}$$

$$(0 \leq \rho \leq 1,\ 0 \leq \theta \leq 2\pi,\ n = 1, 2, \cdots;\ m = 1, 2, \cdots),$$

and the numbers $\ell_i = \lambda_i^2$ are the numbers λ_{0m}^2 and λ_{nm}^2 $(n, m = 1, 2, \cdots)$. Here λ_{nm} is the mth positive zero of the Bessel function J_n, and the normalization factor ν_{nm} is given by

$$\text{(6)} \qquad \begin{cases} \nu_{0m} = \pi^{-1/2} |J_1^{-1}(\lambda_{0m})| & (m = 1, 2, \cdots), \\ \nu_{nm} = 2^{1/2}\pi^{-1/2} |J_{n+1}^{-1}(\lambda_{nm})| & (n, m = 1, 2, \cdots). \end{cases}$$

The smallest eigenvalue $\ell_1 = \lambda_1^2$ and the corresponding (single) eigenfunction ϕ_1 are given by

$$\text{(7)} \qquad \lambda_1 = \lambda_{01} = 2.4048, \qquad \nu_{01} = 1.0868, \qquad \phi_1 = \phi_{01} = \nu_{01} J_0(\lambda_{01}\rho).$$

The next eigenvalues $\ell_2 = \lambda_2^2$ and $\ell_3 = \lambda_3^2$ and the corresponding eigenfunctions are

$$\lambda_2 = \lambda_3 = \lambda_{11} = 3.8317, \tag{8}$$
$$\phi_2 = \phi_{11} = \nu_{11} J_1(\lambda_{11}\rho)\cos\theta, \qquad \phi_3 = \nu_{11} J_1(\lambda_{11}\rho)\sin\theta .$$

More generally, we have the well-known asymptotic formulas

$$\lambda_{nm} \sim \left(n - \frac{1}{2} + 2m\right)\frac{\pi}{2} = \frac{n\pi}{2} - \frac{\pi}{4} + m\pi \qquad (n = 0, 1, \cdots;\ m \text{ large}), \tag{9}$$

$$J_{n+1}(x) \sim \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \left(n + \frac{3}{2}\right)\frac{\pi}{2}\right) \qquad (n = 0, 1, \cdots;\ x > 0, \text{ large}) \tag{10}$$

(E. Jahnke and F. Emde [11, pp. 138 and 143]; G. N. Watson [24, pp. 195 and 506]). Formulas (6), (9), and (10) yield

$$J_{n+1}(\lambda_{nm}) \sim (-1)^{m-1}\left(\frac{2}{\pi\lambda_{nm}}\right)^{1/2} \qquad (n = 0, 1, \cdots;\ m \text{ large}), \tag{11}$$
$$\nu_{0m} \sim 2^{-1}\pi^{1/2}\left(2m - \frac{1}{2}\right)^{1/2}, \qquad \nu_{nm} \sim 2^{-1/2}\pi^{1/2}\left(n + 2m - \frac{1}{2}\right)^{1/2}.$$

Instead of the asymptotic relations (9), (10), (11), we shall use the relations

$$\begin{cases} \lambda_{nm} \geq \lambda_{n1} + (m-1)\pi, \qquad \lambda_{n1} > n \text{ for } n > 1/2 & (m = 1, 2, \cdots), \\ \lambda_{0m} \geq \lambda_{02} + (m-2)(\lambda_{02} - \lambda_{01}) & (m = 1, 2, \cdots), \end{cases} \tag{9'}$$

$$\begin{cases} |J_{n+1}(\lambda_{nm})| = |J_n'(\lambda_{nm})| \geq (1.0694)n^{-1/6}\lambda_{nm}^{-1/2}, \\ |J_1(\lambda_{0m})| = |J_0'(\lambda_{0m})| \geq n^{1/2}\lambda_{0m}^{-1/2}|J_0(\lambda_{01}')|, \end{cases} \tag{10'}$$

$$\begin{cases} \nu_{nm} \leq 2^{1/2}\pi^{-1/2}(1.0694)^{-1}n^{1/6}\lambda_{nm}^{1/2}, \\ \nu_{0m} \leq \pi^{-1/2}n^{-1/2}\lambda_{0m}^{1/2}|J_0(\lambda_{01}')|^{-1}, \end{cases} \tag{11'}$$

where λ_{01}' denotes the first zero of $y'(x)$, for $y(x) = x^{1/2}J_0(x)$, and $n = 0, 1, 2, \cdots$, $m = 1, 2, \cdots$ (see [23], [24]).

3. We shall denote by $\mathscr{S}$ the set of all functions $u(x, y)$ ($(x, y) \in A$) such that $u(x, y)$ is continuous in $A \cup \partial A$ with $u = 0$ on ∂A, u has continuous first-order partial derivatives in A, and Δu (computed in the sense of the theory of distributions) is a measurable, essentially bounded function defined almost everywhere in A. Thus $\mathscr{S} \subset S'$, Δ: $\mathscr{S} \to S'$, $S' \subset S$.

Every element $u(x, y) \in S$ has a Fourier series

$$u(x, y) \sim \sum c_i \phi_i, \tag{12}$$

where $\sum$ ranges over all $i = 1, 2, \cdots$, and where $c_i = u \cdot \phi_i$. Let P: $S \to \mathscr{S}$ be the linear operator defined by

$$Pu = c_1\phi_1 = (u \cdot \phi_1)\phi_1. \tag{13}$$

Also, let H: $S \to S$ be the linear operator defined by

$$(14) \qquad v = Hu \sim -\sum c_i \lambda_i^{-2} \phi_i \qquad ((x, y) \in A).$$

Then, in particular,

$$(15) \qquad H\phi_i = -\lambda_i^2 \phi_i \qquad (i = 1, 2, \cdots).$$

Note that $u \in S$ implies $\sum c_i^2 < +\infty$. Hence, $\lambda_i^2 \geq \lambda_1^2 > 0$ implies $\sum (c_i \lambda_i^{-2})^2 < +\infty$, and therefore $v \in S$, that is, $H: S \to S$.

Let us prove that the series (14) is absolutely and uniformly convergent in $A \cup \partial A$. Indeed,

$$\sum_{i=p}^{q} |c_i \lambda_i^{-2} \phi_i| \leq \left(\sum_{i=p}^{q} c_i^2 \right)^{1/2} \left(\sum_{i=1}^{\infty} \lambda_i^{-4} \phi_i^2 \right)^{1/2}$$

for all integers $0 \leq p \leq q < \infty$. Here the numerical series $\sum_i c_i^2$ is convergent, and the last series, by force of (5), (6), (9), (11), is a minorant of

$$C' \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} (n + 2m - 1/2)^{-3} < +\infty,$$

where C' is some positive constant. Thus, series (14) converges absolutely and uniformly in $A \cup \partial A$, and $v = Hu$ is a continuous function in $A \cup \partial A$ with $v = 0$ on ∂A.

By the theory of distributions (L. Schwartz, [22], Vol. 1, page 82), v is a distribution and Δv (in the sense of the theory of distributions) is also a distribution, and

$$\Delta v = \Delta \left(-\sum_{1}^{\infty} c_i \lambda_i^{-2} \phi_i \right) = \sum_{1}^{\infty} c_i \phi_i \qquad \text{in } A,$$

where equality means equality of distributions, and convergence means weak convergence. Since the last series converges in $L_2(A)$ toward the square-integrable function u, we conclude that $\Delta v = u$ is a square-integrable function.

Let us assume that $u \in S'$, that is, u is measurable and essentially bounded in A. If by $V(x, y)$ $((x, y) \in E_2)$ we denote the Newtonian potential (up to the factor $-1/4\pi$) defined by the mass distribution

$$U(x, y) = \begin{cases} u(x, y) & \text{for } (x, y) \in A, \\ 0 & \text{for } (x, y) \in E_2 - A, \end{cases}$$

then we know that $V(x, y)$ is continuous in E_2 and has uniformly continuous first derivatives in E_2 (see R. Courant and D. Hilbert [9, Vol. 2, p. 246]). If by $V_1(x, y)$ $((x, y) \in A \cup \partial A)$ we denote the harmonic function in A that takes on ∂A the same values as $V(x, y)$, then V_1 is continuous in $A \cup \partial A$ and has continuous partial derivatives of all orders in A.

Now $w = V - V_1$ is continuous in $A \cup \partial A$, is zero on ∂A, has continuous first-order partial derivatives in A, and satisfies the equation $\Delta w = u$ in A, in the sense of the theory of distributions (L. Schwartz [22, Vol. 2, p. 70]). Let us prove that w coincides with the function $v = Hu$ defined by series (14). Indeed, $V + v$ is a distribution in A satisfying the condition

$$\Delta(V - v) = \Delta V - \Delta v = -u + u = 0$$

(in the sense of the theory of distributions). Hence $V - v$ is a harmonic function [22, Vol. 1, p. 140]. Thus $V - v$ and V_1 are continuous functions in $A \cup \partial A$, harmonic in A, and taking the same values on ∂A. By the maximum property of harmonic functions, $V - v = V_1$; that is,

$$w = V - V_1 = v = Hu \quad \text{for } u \in S' \text{ and } w \in \mathscr{S}.$$

Thus $\Delta: \mathscr{S} \to S'$, $H: S' \to \mathscr{S}$, and also

$$\Delta Hu = u \quad \text{for all } u \in S', \qquad H\Delta u = u \quad \text{for all } u \in \mathscr{S}, \tag{16}$$

and

$$\begin{gathered} H(I - P)\Delta u = (I - P)u \quad \text{and} \quad \Delta Pu = P\Delta u \quad \text{for all } u \in \mathscr{S}, \\ \Delta H(I - P)u = (I - P)u \quad \text{for all } u \in S'. \end{gathered} \tag{17}$$

Let us assume now that the function u above satisfies a Hölder condition locally in A; that is, for every closed region $\Gamma \subset A$, let there exist constants K and α ($K \geq 0$, $0 < \alpha \leq 1$) such that

$$|u(z) - u(z')| \leq K|z - z'|^{\alpha} \quad \text{for all } z, z' \in \Gamma.$$

Then the second partial derivatives V_{xx}, V_{yy} of V exist and are continuous in A, V satisfies the equation $\Delta V = u$ in A, and hence $\Delta v = u$ in A in the strong sense (see Courant and Hilbert [9, Vol. 2, p. 249]).

4. For functions $u \in S$ given by (12), we have the relation

$$H(I - P)u \sim -\sum{}' c_i \lambda_i^{-2} \phi_i, \tag{18}$$

where $\sum{}'$ ranges over $i = 2, 3, \cdots$. Therefore

$$\|H(I - P)u\| = \left(\sum{}' c_i^2 \lambda_i^{-4}\right)^{1/2} \leq \lambda_2^{-2} \left(\sum{}' c_i^2\right)^{1/2} \leq k\|u\|. \tag{19}$$

Since $\lambda_2^{-2} = 0.06811$, we may take $k = 0.069$. Now, for every $z \in A$,

$$|H(I - P)u(z)| \leq \sum{}' c_i \lambda_i^{-2} |\phi_i| \leq \left(\sum{}' c_i^2\right)^{1/2} \left(\sum{}' \lambda_i^{-4} \phi_i^2\right)^{1/2} \leq M(z)\|u\|. \tag{20}$$

Also, for $z = (x, y) = (\rho \cos \theta, \rho \sin \theta)$ $(0 \leq \rho \leq 1,\ 0 \leq \theta \leq 2\pi)$,

$$M^2(z) = \sum_{m=2}^{\infty} \lambda_{0m}^{-4} \nu_{0m}^2 J_0^2(\lambda_{0m}\rho) + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \lambda_{nm}^{-4} \nu_{nm}^2 J_n^2(\lambda_{nm}\rho);$$

because $|J_0(x)| \leq 1$ and $|J_n(x)| \leq 2^{-1/2}$ for all $x \geq 0$ and for $n = 1, 2, \cdots$ (see [24, p. 31]), it now follows from (6) that

$$M^2(z) \leq \sum_{m=2}^{\infty} \lambda_{0m}^{-4} \nu_{0m}^2 + 2^{-1} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \lambda_{nm}^{-4} \nu_{nm}^2 \tag{21}$$
$$\leq \pi^{-1}\left(\sum_{m=2}^{\infty} \lambda_{0m}^{-4} J_1^{-2}(\lambda_{0m}) + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \lambda_{nm}^{-4} J_{n+1}^{-2}(\lambda_{nm}) \right).$$

By using formula (21), C. D. Stocking proved in [23] that $M(z) \leq 0.23$. Thus, relations (19) and (20) yield for all $u \in S$ the estimates

$$\|H(I - P)u\| \leq k\,\|u\|, \qquad |H(I - P)u| \leq k'\,\|u\|, \qquad \text{with } k = 0.069,\ k' = 0.230. \tag{22}$$

5. We shall denote by S_1 the subset of S consisting of all functions $u(x, y) \in S'$ with $\mu(u) \leq R_1$, that is, with $|u(x, y)| \leq R_1$ a.e. in A. Then, for $u \in S_1$, the expression $g(x, y, u(x, y))$ is defined (a.e.) in A, and it represents a bounded measurable function in A with $|g(x, y, u(x, y))| \leq R_2$, or $\mu[g(x, y, u(x, y))] \leq R_2$. Let N and F be the nonlinear operators defined in S_1 by

$$\begin{aligned} Nu &= -g(x, y, u(x, y)) \qquad ((x, y) \in A,\ u \in S_1), \\ Fu &= H(I - P)Nu \qquad (u \in S_1). \end{aligned} \tag{23}$$

Then $N\colon S_1 \to S'$ and $F\colon S_1 \to \mathscr{S}$.

Note that, if $u(x, y) \in \mathscr{S} \cap S_1$ and its Δu (in the sense of the theory of distributions) is a function in A satisfying equation (1) a.e. in A, that is, if

$$\Delta u = -g(x, y, u) \quad (\text{or } \Delta u = Nu) \quad \text{a.e. in } A, \tag{24}$$

then

$$\mu(u) \leq R_1, \qquad \mu(Nu) \leq R_2 \qquad \text{for } u \in \mathscr{S},\ \Delta u \in S'.$$

Hence, by applying the operator $H(I - P)$ at the left of (24), we obtain the equation $H(I - P)\Delta u = H(I - P)Nu$; that is, $(I - P)u = Fu$, or, finally,

$$u = Pu + Fu. \tag{25}$$

For every $u \in S_1$ we shall now define the operator $T\colon S_1 \to \mathscr{S}$ by taking

$$v = Tu = Pu + Fu = Pu + H(I - P)Nu. \tag{26}$$

Note that, for $u \in S_1$,

$$PFu = PH(I - P)Nu = 0. \tag{27}$$

If $u^*(x, y)$ $((x, y) \in A,\ u^* \in \mathscr{S} \cap S_1)$ is any approximation to a solution u of (24), then u^* satisfies equations (24) and (25) with errors θ and Θ given by

$$\Delta u^* = Nu^* + \theta \quad (\theta \in S'), \qquad u^* = Pu^* + Fu^* + \Theta \quad (\Theta \in \mathscr{S}), \tag{28}$$

where $\Theta = H(I - P)\theta$ for $\mu(u^*) \leq R_1$, and where u^* satisfies the condition $\mu(Nu^*) \leq R_2$. Also, for every $u \in S_1$, it follows from (22) and (2) that

$$\text{(29)} \quad \begin{cases} \|Fu - Fu^*\| = \|H(I - P)(Nu - Nu^*)\| \leq k\|Nu - Nu^*\| \leq kL\|u - u^*\|, \\ \mu(Fu - Fu^*) = \mu[H(I - P)(Nu - Nu^*)] \leq k'\|Nu - Nu^*\| \leq k'L\|u - u^*\|. \end{cases}$$

We shall take $u^* = \gamma\phi_1 = \gamma\nu_{01}J_0(\lambda_{01}\rho)$, where γ is an undetermined constant such that $|u^*| = |\gamma\phi_1| \leq R_1$ in A. Then the errors θ and Θ are functions of $z \in A$ and γ, and they are given by the equations

$$\theta(z, \gamma) = \Delta u^* - Nu^* = -\gamma\lambda_{01}^2\nu_{01}J_0(\lambda_{01}\rho) + g(z, \gamma\nu_{01}J_0(\lambda_{01}\rho)),$$

$$\Theta(z, \gamma) = H(I - P)\,\theta(z, \gamma).$$

We shall denote by $B(\gamma)$ the expression (with $dz = dx\,dy$)

$$B(\gamma) = \theta(z, \gamma)\cdot\phi_1 = \iint_A \theta(z, \gamma)\phi_1\,dz = -\gamma\lambda_{01}^2\iint_A \phi_1^2\,dz + \iint_A g\,\phi_1\,dz$$

$$= -\gamma\lambda_{01}^2 + \int_0^{2\pi}\int_0^1 g(z, \gamma\nu_{01}J_0(\lambda_{01}\rho))\,\nu_{01}J_0(\lambda_{01}\rho)\,\rho\,d\rho\,d\theta,$$

where $\lambda_{01}^2 = 5.7831$.

For $\gamma = \gamma_0 = 0$, u^* reduces to $u^* = u_0 = 0$, and

$$\text{(30)} \qquad \theta(z, 0) = g(z, 0), \qquad \Theta(z, 0) = H(I - P)g(z, 0).$$

Let

$$\text{(31)} \qquad b = \|\Theta(z, 0)\|, \qquad b' = \mu\Theta(z, 0) = \text{Ess Sup}\,|\Theta(z, 0)|,$$

where the Ess Sup is taken for $z \in A$. Also, for any number $c > 0$, let

$$\text{(32)} \qquad B_{01} = B(c), \qquad B_{02} = B(-c), \qquad \Omega = \min[|B_{01}|, |B_{02}|].$$

6. Let c, d, r, R be constants such that

$$\text{(33)} \qquad 0 < c < d, \qquad r = c\nu_{01} < R \leq R_1.$$

Let V be the set of all functions $u^* = \gamma\phi_1 = \gamma\nu_{01}J_0(\lambda_{01}\rho)$ with $\|u^*\| \leq c$, that is, with $|\gamma| \leq c$. Then

$$\mu(u^*) = \mu[\gamma\nu_{01}J_0(\lambda_{01}\rho)] = |\gamma|\,\nu_{01} \leq r < R \leq R_1.$$

For every $u^* \in V$, let

$$\text{(34)} \qquad S^* = \{u \mid u \in S,\ Pu = u^*,\ \|u\| \leq d,\ \mu(u) \leq R\}.$$

This set is not empty, since u^* belongs to S^*. Also, it is complete (in S, with the norm $\|u\|$ of S). Finally, $S^* \subset S_1$, and hence $T\colon S^* \to \mathscr{S}$. Let us assume that the inequalities

(35) $$kL < 1, \quad c + kLd + b \leq d, \quad r + k'Ld + b' \leq R$$

hold. Then $T\colon S^* \to S^* \cap \mathscr{S}$. Indeed, if $v = Tu$ and $u \in S^*$, then, by force of (26), (27), (34), (35), (29), (31),

$$Pv = P(Pu + Fu) = PPu + PH(I - P)Nu = Pu = u^*,$$

$$\|v\| = \|Pu + Fu\| \leq \|u^*\| + \|H(I - P)(Nu - Nu_0)\| + \|H(I - P)Nu_0\|$$

$$\leq c + kL\,\|u - u_0\| + \|\Theta(z, \gamma_0)\| \leq c + kLd + b \leq d,$$

$$\mu(v) = \mu(Pu + Fu) \leq \mu(u^*) + \mu[H(I - P)(Nu - Nu_0)] + \mu[H(I - P)Nu_0]$$

$$\leq r + k'L\,\|u - u_0\| + b' \leq r + k'Ld + b' \leq R.$$

Thus $T\colon S^* \to S^* \cap \mathscr{S}$. Also, $kL < 1$ implies that $T \mid S^*$ is a contraction. Indeed for $u_i \in S^*$ and $v_i = Tu_i$ $(i = 1, 2)$, we have the relations

$$v_1 = Pu_1 + Fu_1, \quad v_2 = Pu_2 + Fu_2, \quad Pu_1 = Pu_2 = u^*,$$

and

$$\|v_1 - v_2\| = \|Fu_1 - Fu_2\| = \|H(I - P)(Nu_1 - Nu_2)\| \leq k\,\|Nu_1 - Nu_2\| \leq kL\,\|u_1 - u_2\|,$$

where $kL < 1$.

We conclude that, under hypotheses (2), (33), and (35), the restriction $T \mid S^*$ admits a unique fixed element, say $v = Tv \in S^*$, which is determined by u^*, hence by the value of the constant γ $(|\gamma| \leq c)$. In other words, $v \in S^*$ is a function of γ. Actually, v is a continuous function of γ. Indeed, for $|\gamma_1|, |\gamma_2| \leq c$, $u_i^* = \gamma_i \phi_1$, corresponding sets S_i^*, and fixed elements $v_i = Tv_i$ of the maps $T \mid S_i^*$ $(i = 1, 2)$, it is true that $S_1^* \cup S_2^* \subset S_1$, and

$$v_i = Tv_i = Pv_i + Fv_i, \quad Pv_i = u_i^* \quad (i = 1, 2),$$

$$\|Fv_1 - Fv_2\| = \|H(I - P)(Nv_1 - Nv_2)\| \leq kL\,\|v_1 - v_2\|,$$

$$\|v_1 - v_2\| = \|(Pv_1 - Pv_2) + (Fv_1 - Fv_2)\| \leq \|u_1^* - u_2^*\| + kL\,\|v_1 - v_2\|,$$

and finally

$$\|v_1 - v_2\| \leq (1 - kL)^{-1}\,\|u_1^* - u_2^*\| = (1 - kL)^{-1}\,|\gamma_1 - \gamma_2|.$$

If $u^* \in V$ and $v = Tv$ is the corresponding fixed element of $T \mid S^*$, then

(36) $$v = Pv + Fv = Pv + H(I - P)Nv.$$

This implies that $v \in S^* \cap \mathscr{S}$; in other words, v is a continuous function in $A \cup \partial A$, $v = 0$ on ∂A, v has continuous first-order partial derivatives in A, Δv (computed in the sense of the theory of distributions) is a bounded function in A, $|Nv| \leq R_2$, and

$$\Delta(v - Pv) = \Delta(I - P)v = (I - P)\Delta v, \quad \Delta H(I - P)Nv = (I - P)Nv.$$

Thus, v satisfies an equation of the form

$$\Delta v = Nv + D, \qquad \text{where } D = P(\Delta v - Nv),$$

or

$$\Delta v = -g(x, y, v) + D, \qquad D = D(\gamma)\phi_1 = [(\Delta v + g)\cdot\phi_1]\phi_1. \tag{37}$$

This shows that v is a solution of problem (1) provided u^* (that is, γ) is chosen in such a way that $D = 0$, or $(\Delta v + g)\cdot\phi_1 = 0$.

Now assume that $Nu = -g(x, y, u(x, y))$ is a continuous function of (x, y) satisfying locally in A some Hölder condition whenever $u(x, y)$ has the same properties. Since v is continuous in A, together with its first-order partial derivatives, v satisfies *locally* in A a Lipschitz condition (that is, a Hölder condition with exponent one). Therefore Nv satisfies locally some Hölder condition; hence $H(I - P)Nv$ has continuous second-order partial derivatives in A, and hence, by force of (36), v itself has the same property. Therefore Δv is the usual Laplacian.

7. Let

$$\begin{aligned}
u^* &= \gamma\phi_1,\\
u_0 &= \gamma_0\phi_1,\\
v &= Tv \sim \gamma\phi_1 + c_2\phi_2 + c_3\phi_3 + \cdots,\\
\Delta u^* &= -\gamma\lambda_1^2\phi_1,\\
\Delta u_0 &= -\gamma_0\lambda_1^2\phi_1,\\
\Delta v &\sim -\gamma\lambda_1^2\phi_1 - c_2\lambda_2^2\phi_2 - \cdots,\\
Nv &\sim \delta_1\phi_1 + \delta_2\phi_2 + \cdots,\\
Nu^* &\sim \delta_1^*\phi_1 + \delta_2^*\phi_2 + \cdots,\\
Nu_0 &\sim \delta_{10}\phi_1 + \delta_{20}\phi_2 + \cdots.
\end{aligned}$$

Then

$$\begin{aligned}
D(\gamma) &= (\Delta v - Nv)\cdot\phi_1 = -\gamma\lambda_1^2 - \delta_1,\\
B(\gamma) &= (\Delta u^* - Nu^*)\cdot\phi_1 = -\gamma\lambda_1^2 - \delta_1^*,
\end{aligned}$$

$$|D(\gamma) - B(\gamma)| = |\delta_1 - \delta_1^*| = |(Nv - Nu^*)\cdot\phi_1| \le \|Nv - Nu^*\| \le L\,\|v - u^*\|,$$

where $v = Pv + H(I - P)Nv$ and $Pv = u^*$, and hence

$$\begin{aligned}
|D(\gamma) - B(\gamma)| &\le L\,\|v - Pv\| = L\,\|H(I - P)Nv\|\\
&\le L\,\|H(I - P)(Nv - Nu_0) + H(I - P)Nu_0\|\\
&\le L(kL\,\|v - u_0\| + b) \le L(kLd + b),
\end{aligned} \tag{38}$$

for all γ $(-c \le \gamma \le c)$.

Let us assume that, together with (2), (33), (35), the inequalities

$$(39) \qquad L(kLd+b) < \Omega = \min\{|B_{01}|, |B_{02}|\}, \qquad B_{01}B_{02} < 0,$$

hold; then $B_{01} = B(c) \neq 0$, $B_{02} = B(-c) \neq 0$, and B_{01} and B_{02} are of opposite signs. Suppose, for instance, that $B_{01} > 0 > B_{02}$; then, by force of (38),

$$D(c) > B(c) - L(kLd+b) > \Omega - L(kLd+b) > 0,$$

$$D(-c) < B(-c) + L(kLd+b) < -\Omega + L(kLd+b) < 0.$$

Hence $D(\gamma)$ changes sign in the interval $[-c, +c]$, and hence $D(\gamma) = 0$ for some $|\gamma| < c$. That is, problem (1) has at least one solution $v \in \mathcal{S}$, in the sense that v is continuous in $A \cup \partial A$ with $u = 0$ on ∂A, v has continuous first-order partial derivatives in A, and Δv (in the sense of the theory of distributions) is a measurable bounded function in A with $\Delta v = g(x, y, v)$ a.e. in A, and with

$$\|v\| \leq d, \qquad \mu(v) \leq R \leq R_1, \qquad |v \cdot \phi_1| < c.$$

8. Relations (33), (35), (39) should now be rewritten in the form

$$0 < c < d, \qquad r = 1.0868\,c < R \leq R_1, \qquad 0.069\,L < 1,$$

$$(40) \qquad b \leq (1 - 0.069\,L)d - c, \qquad b' \leq R - 1.0868\,c - 0.230\,Ld, \qquad B(c)B(-c) < 0,$$

$$L(0.069\,Ld + b) < \Omega = \min\{|B(c)|, |B(-c)|\},$$

where

$$b = \|H(I - P)g(z, 0)\|, \qquad b' = \operatorname{Sup}|H(I - P)g(z, 0)|,$$

$$(41) \qquad B(\gamma) = -5.7831\,\gamma + \int_0^{2\pi}\int_0^1 g[z, \gamma\nu_{01}J_0(\lambda_{01}\rho)]\,\nu_{01}J_0(\lambda_{01}\rho)\rho\,d\rho\,d\theta.$$

Note that for any set of constants c, d, R with $0 < c < d$, $1.0868\,c < R \leq R_1$, and $B(c)B(-c) < 0$, there always exist constants $0 < b < d - c$, $0 < b' < R - 1.0868\,c$, and we can find constants $L > 0$ satisfying the inequalities

$$0.069\,L < 1, \qquad 0.069\,Ld \leq d - c - b, \qquad 0.230\,Ld \leq R - 1.0868\,c - b',$$

$$L(0.069\,Ld + b) < \Omega = \min\{|B(c)|, |B(-c)|\};$$

all relations (40) are thus satisfied.

9. We can now summarize our result:

THEOREM 1. *If* $g(x, y, u)$ *is a measurable function of* x, y, u *satisfying hypotheses* (2) *in* $A \times [-R_1, R_1]$, *if* c, d, R, L *are constants such that* $0 < c < d$, $1.0868\,c < R \leq R_1$, *and*

$$b = \|H(I - P)g(z, 0)\| < d - c, \qquad b' = \operatorname{Sup}_A|H(I - P)g(z, 0)| < R - 1.0868\,c,$$

$$(42) \qquad B(c)B(-c) < 0, \qquad 0.069\,L < 1, \qquad 0.069\,Ld \leq d - c - b,$$

$$0.230\,Ld \leq R - 1.0868\,c - b', \qquad L(0.069\,Ld + b) < \Omega = \min\{|B(c)|, |B(-c)|\},$$

then there exists at least one function $u(x, y)$, *continuous in* $A \cup \partial A$ *and zero on* ∂A, *such that* $u(x, y)$ *has first-order partial derivatives continuous in* A, Δu *(in the sense of the theory of distributions) is a bounded measurable function in* A, *and* $\Delta u + g(x, y, u) = 0$ a.e. *in* A.

In addition, if g *has the property that* $g(x, y, w(x, y))$ *satisfies a Hölder condition locally whenever* $w(x, y)$ *does, and if* $|w(x, y)| \leq R_1$ *in* A, *then the function* $u(x, y)$ *above has continuous second-order partial derivatives in* A, *and the equation* $\Delta u + g(x, y, u) = 0$ *holds everywhere in* A, *in the usual sense.*

Note that $B(\gamma)$ above is defined by (41) for $|\gamma| \leq c$, and that the third relation in (42) certainly holds whenever

$$(43) \qquad \left| \int_0^{2\pi} \int_0^1 g[x, y, \pm c\, \nu_{01} J_0(\lambda_{01}\rho)]\, \nu_{01} J_0(\lambda_{01}\rho)\rho\, d\rho\, d\theta \right| < 5.7831\, c\,.$$

10. The following example is of interest. Let us consider the boundary value problem

$$(44) \quad \Delta u + g(x, y, u) = 0 \text{ in } A, \qquad u = 0 \text{ on } \partial A, \quad \text{with } g(x, y, u) = f(x, y)\phi(u) + h(x, y),$$

where f and g are bounded and measurable in A, and where ϕ is a preassigned function with $\phi(0) = 0$, $|\phi(u) - \phi(v)| \leq |u - v|$ for all real u and v. If

$$\alpha = \text{Ess Sup}_A |f(x, y)|, \qquad \beta = \text{Ess Sup}_A |h(x, y)|,$$

then $g(z, 0) = h(x, y)$, and

$$b = \|H(I - P)h(x, y)\| \leq \pi^{1/2} k\beta,$$

$$b' = \text{Sup}_A |H(I - P)h(x, y)| \leq \pi^{1/2} k'\beta, \qquad L = \alpha,$$

with $k = 0.069$ and $k' = 0.230$, and R_1 can be taken as large as we wish. If α' and β' denote the constants

$$\alpha' = \int_0^{2\pi} \int_0^1 |f(x, y)|\, \nu_{01}^2 J_0^2(\lambda_{01}\rho)\rho\, d\rho\, d\theta\,,$$

$$\beta' = \int_0^{2\pi} \int_0^1 |h(x, y)|\, \nu_{01} J_0(\lambda_{01}\rho)\rho\, d\rho\, d\theta\,,$$

then

$$0 \leq \alpha' \leq \alpha\, \nu_{01}^2 \int_0^{2\pi} \int_0^1 \rho J_0^2(\lambda_{01}\rho)\, d\rho\, d\theta = \alpha\, \|\phi_{01}\|^2 = \alpha\,,$$

$$0 \leq \beta' \leq 2\pi\beta\, \nu_{01} \int_0^1 \rho J_0(\lambda_{01}\rho)\, d\rho = 2\pi\beta\, \nu_{01}\lambda_{01}^{-1} J_1(\lambda_{01}),$$

and computations show that $\alpha' \leq \alpha$ and $\beta' \leq 1.4742\,\beta$. The relation (43) (and consequently relation (42)) is then certainly satisfied if

$$\alpha c + 1.4742\beta < 5.7831\,c,$$

with

$$|B(c)|,\ |B(-c)| > 5.7831\,c - (\alpha c + 1.4742\beta).$$

The list of inequalities that we need is now

$$0 < c < d, \quad 1.0868\,c < R, \quad 0.1223\beta < d - c, \quad 0.4077\beta < R - 1.0868\,c,$$

$$\alpha c + 1.4742\beta < 5.7831\,c, \quad 0.069\,\alpha < 1, \quad 0.069\,\alpha d \leq d - c - 0.1223\beta,$$

$$0.230\,\alpha d \leq R - 1.0868\,c - 0.4077, \quad \alpha(0.069\,\alpha d + 0.1223\beta) < (5.7831 - \alpha)c - 1.4742\beta.$$

We can satisfy the three relations involving R by taking R sufficiently large, since $R \leq R_1$ and R_1 is arbitrary. The list now reduces to

$$(45)\quad \begin{aligned} &0 < c < d, \quad 0.1223\beta < d - c, \quad \alpha c + 1.4742\beta < 5.7831\,c, \quad 0.069\,\alpha < 1,\\ &0.069\,\alpha d \leq d - c - 0.1223\beta, \quad \alpha(0.069\,\alpha d + 0.1223\beta) < (5.7831 - \alpha)c - 1.4742\beta. \end{aligned}$$

The first relation concerns c and d only, the next gives an upper bound for β, and the others give an upper bound for α; in particular, the fourth one implies that

$$0 \leq \alpha < \alpha_0 = (0.069)^{-1} = 14.492753.$$

On the other hand, given β, we can always determine c and d satisfying the first two inequalities, and the remaining relations then give an upper bound for α. We shall now prove that there exists some $\overline{\alpha}$ $(0 < \overline{\alpha} < \alpha_0)$, namely $\overline{\alpha} = 4.13$ (by defect) such that for any α, β $(0 \leq \alpha < \overline{\alpha},\ 0 \leq \beta < +\infty)$ we can determine constants c and d such that α, β, c, d satisfy all relations (45).

First we consider the case $\beta = 0$. Then, with $c = \sigma d$ $(0 < \sigma < 1,\ d > 0)$, the first two inequalities (45) are obviously satisfied, and so are the third and the fourth, since $0 \leq \alpha < \overline{\alpha} < 5.7831 < \alpha_0$. The fifth and sixth relations (45) now become

$$\alpha \leq (1 - \sigma)\alpha_0, \quad \alpha^2 + \alpha_0\,\sigma\,\alpha - \alpha_0(5.7831\,\sigma) < 0,$$

and finally

$$\alpha < 2^{-1}\alpha_0\,[-\sigma + (\sigma^2 + 1.5961\,\sigma)^{1/2}].$$

The equation $(1 - \sigma)\alpha_0 = 2^{-1}\alpha_0\,[-\sigma + (\sigma^2 + 1.5961\,\sigma)^{1/2}]$ yields the value $\sigma = 0.715$, and the best value of α (namely $\alpha = \overline{\alpha} = 4.13$) obtainable by this argument.

Now we turn to the case where $\beta > 0$ is fixed. All pairs c, d with

$$(46)\qquad 0 < c < d, \quad d > c + 0.1223\beta$$

satisfy the first two relations (45), and the fourth one is also satisfied, since $0 \leq \alpha < \overline{\alpha} < \alpha_0$. Again, for $0 \leq \alpha < \overline{\alpha} = 4.13 < \alpha_0$, the third equation (45) is certainly satisfied if

$$(47)\qquad c > (5.7831 - 4.13)^{-1}\,1.4742\beta, \quad \text{that is, } c > 0.8918\beta.$$

The fifth and sixth relations in (45), with $c = \sigma d$ $(0 < \sigma < 1)$, now become

$$\alpha \leq (1 - \sigma)\alpha_0 - 0.1223\,\alpha_0\beta/d,$$

$$\alpha^2 + \alpha_0(\sigma + 0.1223\beta/d)\alpha - \alpha_0(5.7831\,\sigma - 1.4742\,\beta/d) < 0,$$

or

$$\alpha < (1 - \sigma)\alpha_0 - 1.7725\,\beta/d,$$

(48)
$$\alpha < 2^{-1}\alpha_0[-\sigma - 0.1223\beta/d + ((\sigma + 0.1223\beta/d)^2 + (1.5961\,\sigma - 0.4069\,\beta/d))^{1/2}].$$

As before, we take $c = \sigma d$, $\sigma = 0.715$; then for sufficiently large c and d the relations (46) and (47) are certainly satisfied. On the other hand, as $d \to +\infty$, the right-hand members in relations (48) both approach $\overline{\alpha} = 4.13$. Thus, given $\beta \geq 0$ and $0 \leq \alpha < \overline{\alpha}$, we can always determine constants c and d satisfying all relations (45).

We can now summarize our result concerning example (44). Let $f(x, y)$ and $h(x, y)$ be bounded measurable functions in A, let $\alpha = \text{Ess Sup}\,|f(x, y)|$ in A, and let $\phi(u)$ (u real) be any function such that $\phi(0) = 0$ and $|\phi(u) - \phi(v)| \leq |u - v|$ for all real u and v. Then the problem

$$\Delta u + f(x, y)\,\phi(u) + h(x, y) = 0 \qquad ((x, y) \in A),$$

$$u = 0 \qquad ((x, y) \in \partial A)$$

has at least one solution $u(x, y)$ as in Theorem 1, provided $0 \leq \alpha < \overline{\alpha} = 4.13$.

Note that the constant 4.13 above—though not necessarily the best possible constant—cannot be replaced by a constant greater than $\lambda_{01}^2 = 5.7831$ (the first eigenvalue of the linear problem), since for $f(x, y) = \lambda_{01}^2 = 5.7831$ and $\phi(u) = u$, there exist bounded continuous functions $h(x, y)$ for which the problem above has no solutions.

11. We return to the general problem $\Delta u + g(x, y, u) = 0$ in A, $u = 0$ on ∂A, with $g(x, y, u)$ measurable in $A \times [-R_1, R_1]$ and satisfying hypotheses (2), and, in addition, with g together with g_u absolutely continuous in u for almost all $(x, y) \in A$, and with the second derivative g_{uu} essentially bounded on A. Let B_0, B_1, B_2 be the constants

$$B_0 = \int_0^{2\pi}\int_0^1 g(z, 0)\,\nu_{01} J_0(\lambda_{01}\rho)\,\rho\,d\rho\,d\theta,$$

$$B_1 = \int_0^{2\pi}\int_0^1 g_u(z, 0)\,\nu_{01}^2 J_0^2(\lambda_{01}\rho)\,\rho\,d\rho\,d\theta,$$

$$B_2 = \text{Ess Sup}\,|g_{uu}(z, \gamma\nu_{01} J_0(\lambda_{01}\rho))|,$$

where the Ess Sup is taken for $z \in A$, and where $|\gamma| \leq c$. Then, for $|\gamma| \leq c$,

$$|B(\gamma) - [B_0 + (B_1 - 5.7831)\gamma]| \leq 2^{-1} B_2\gamma^2 \int_0^{2\pi}\int_0^1 \nu_{01}^3 J_0^3(\lambda_{01}\rho)\,\rho\,d\rho\,d\theta$$

$$= \left(\pi\,\nu_{01}^3 \int_0^1 \rho J_0^3(\lambda_{01}\rho)\,d\rho\right) B_2\gamma^2 \leq 0.3931\,B_2\gamma^2,$$

since the last integral was found to be less than 0.09747. Hence

$$|B(\pm c) - [B_0 \pm (B_1 - 5.7831)c]| \leq 0.3931\, B_2\, c^2 .$$

Relations (40) are certainly satisfied if

$$\begin{gathered} 0 < c < d, \qquad 1.0868c < R \leq R_1, \qquad 0.069\,L < 1, \\ b \leq (1 - 0.069\,L)d - c, \qquad b' \leq R - 1.0868c - 0.23\,Ld, \\ [B_0 + (B_1 - 5.7831)c][B_0 - (B_1 - 5.7831)c] < 0, \\ L(0.069\,Ld + b) < \Omega' = \min[|B_0 \pm (B_1 - 5.7831)c| - 0.3931\,B_2\,c^2]. \end{gathered} \tag{49}$$

For instance, we may take $L \leq 4$ (so that $0.069\,L \leq 0.276 < 1$) and $R = R_1$, and we can choose c and d with

$$1.0868c < R, \qquad c \leq 1, \qquad c = (1 - 0.069\,L)d - 0.25\,d = (0.75 - 0.069\,L)d .$$

For

$$\begin{gathered} b \leq 0.25\,d, \qquad b' \leq R - 1.0868c - 0.23\,Ld, \\ |B_0| \leq 0.25\,c, \qquad |B_1| \leq 0.25\,c, \qquad |B_2| \leq 0.25, \end{gathered}$$

it follows that

$$\begin{gathered} B_0 - (B_1 - 5.7831)c > (5.7831 - 0.5)c = 5.2831c > 0, \\ B_0 + (B_1 - 5.7831)c < -(5.7831 - 0.5)c = -5.2831c < 0, \end{gathered}$$

and hence the first three sets of inequalities (49) are satisfied. Finally

$$\begin{aligned} L(0.069\,Ld + b) &\leq L(0.069\,Ld + 0.25\,d) \leq L(0.069\,L + 0.25)(0.75 - 0.069\,L)^{-1}c \\ &\leq 4(0.526)(0.474)^{-1}c \leq 4.44\,c \\ &< 5.184825\,c = 5.2831c - (0.25)(0.3931)c \\ &\leq 5.2831c - 0.3931\,B_2\,c \leq 5.2831c - 0.3931\,B_2\,c^2 \leq \Omega' ; \end{aligned} \tag{50}$$

that is, all inequalities (49) are satisfied.

We can now summarize the result of the present section.

THEOREM 2. *If* $g(x, y, u)$ *is a measurable function of* x, y, u *satisfying hypotheses* (2) *in* $A \times [-R_1, R_1]$, *if in addition* g *together with* g_u *is absolutely continuous in* u *for almost all* $(x, y) \in A$ *and the second derivative* g_{uu} *is essentially bounded on* A, *and if for some constants* L *and* c *we have the relations*

$$\begin{gathered} 0 < L \leq 4, \qquad 0 < c \leq 1, \qquad 1.0868c < R_1, \\ b = \|H(I - P)g(z, 0)\| \leq 4^{-1}(0.75 - 0.069\,L)^{-1}c, \\ b' = \operatorname{Ess\,Sup}_A |H(I - P)g(z, 0)| \leq R_1 - 1.0868c - 0.23L(0.75 - 0.069\,L)^{-1}c, \end{gathered}$$

$$|B_0| = \left| \int_0^{2\pi} \int_0^1 g(z,\, 0)\, \nu_{01} J_0(\lambda_{01}\rho)\rho\, d\rho\, d\theta \right| < 4^{-1} c,$$

$$|B_1| = \left| \int_0^{2\pi} \int_0^1 g_u(z,\, 0)\, \nu_{01}^2 J_{01}^2(\lambda_{01}\rho)\rho\, d\rho\, d\theta \right| < 4^{-1} c,$$

$$|B_2| = \text{Ess Sup}\, |g_{uu}(z,\, \gamma\, \nu_{01} J_0(\lambda_{01}\rho)| < 4^{-1} \qquad (z \in A,\ |\gamma| \le c),$$

then there exists at least one function u(x, y) *having the properties listed in Theorem* 1 *and satisfying* (1).

12. For particular forms of g(x, y, u), we may impose less severe conditions on g than in Theorem 2.

For instance, let $\phi(u)$ $(-\infty < u < +\infty)$ be a function that is absolutely continuous, together with $\phi'(u)$, and satisfies the conditions

$$\phi(0) = 0, \qquad \phi'(0) = 0, \qquad |\phi'(u)| \le 3u^2, \qquad |\phi''(u)| \le 6\,|u|\,.$$

Let f(x, y) and h(x, y) be essentially bounded, measurable functions in A. Let us consider the boundary value problem

$$\begin{gathered} \Delta u + g(x,\, y,\, u) = 0 \quad \text{in } A, \qquad u = 0 \quad \text{on } \partial A, \\ \text{with } g(x,\, y,\, u) = f(x,\, y)\phi(u) + h(x,\, y). \end{gathered} \tag{51}$$

Then

$$g(x,\, y,\, u) = f(x,\, y)\phi(u) + h(x,\, y),$$

$$g_u = f(x,\, y)\phi'(u), \qquad g_{uu} = f(x,\, y)\phi''(u),$$

$$g(x,\, y,\, 0) = h(x,\, y), \qquad g_u(x,\, y,\, 0) = 0\,;$$

hence $B_1 = 0$. If

$$\alpha = \text{Ess Sup}_A\, |f(x,\, y)|\,, \qquad \beta = \text{Ess Sup}_A\, |h(x,\, y)|\,,$$

then we shall assume that $L = 3\alpha R^2$ for $|u| \le R = R_1$ and that

$$|B_0| \le 2\pi\beta\, \nu_{01} \int_0^1 \rho J_0(\lambda_{01}\rho)\, d\rho = 2\pi\, \nu_{01}\lambda_{01}^{-1} J_1(\lambda_{01})\beta \le 1.4742\,\beta,$$

$$|B_2| \le 6\alpha R, \qquad |b| \le \pi^{1/2} k\beta = 0.1223\,\beta, \qquad b' = \pi^{1/2} k'\beta = 0.4077\,\beta.$$

Thus, inequalities (49) are certainly satisfied provided

$$0 < c < d, \qquad 1.0868\,c < R, \qquad 0.069\,(3\alpha R^2) < 1,$$

$$0.1223\,\beta \le [1 - 0.069\,(3\alpha R^2)]\, d - c,$$

$$0.4077\,\beta \le R - 1.0868\,c - 0.230\,(3\alpha R^2 d), \qquad 1.4742\,\beta < 5.7831\,c,$$

$$(3\alpha R^2)[0.069\,(3\alpha R^2 d) + 0.1223\beta] < 5.7831\,c - 1.4742\beta - 0.3931\,(6\alpha R c^2).$$

Note that if c, d, R are prescribed constants satisfying the relations $0 < c < d$ and $R > 1.0868\,c$, then we need only verify that

$$0.1223\beta < d - c, \quad 0.4077\beta < R - 1.0868\,c, \quad 1.4742\beta < 5.7831\,c,$$

$$0.069\,(3\alpha R^2) < 1, \quad 0.069\,(3\alpha R^2 d) < d - c - 0.1223\beta, \tag{52}$$

$$0.230\,(3\alpha R^2 d) < R - 1.0868\,c - 0.4077\beta,$$

$$(0.069)(3\alpha R^2)^2 d + (0.1223)(3\alpha R^2)\beta + (0.3931)(6\alpha R)c^2 < 5.7831\,c - 1.4742\beta.$$

The first three inequalities then give an upper bound for β; if we fix β within such a bound, then the remaining four inequalities (52) give a bound for α. For instance, for $c = 1$, $d = 2$, $R = 2$, we have the inequalities $\beta < 8.17$, $\beta < 2.23$, $\beta < 3.92$. If $\beta = 0.8$, for example (or even if $0 \leq \beta \leq 0.8$), then the remaining four inequalities (52) yield $\alpha < 1.20$, $\alpha < 0.54$, $\alpha < 0.106$, $\alpha < 0.355$. Thus, boundary value problem (51) *certainly has a solution* u(x, y) *as in Theorem* 2, *for all* α *and* β *with* $0 \leq \alpha \leq 0.1$, $0 \leq \beta \leq 0.8$.

On the other hand, given any $\beta \geq 0$, we can always choose constants c, d, R satisfying the conditions $0 < c < d$ and $R > 1.0868\,c$ and the first three relations (52). Then the remaining four relations (52) yield a bound for α (relative to the chosen values c, d, R). We may now summarize our result concerning example (51).

Let f(x, y) and h(x, y) be bounded measurable functions in A, let $\alpha = \text{Ess Sup}\,|f(x, y)|$, $\beta = \text{Ess Sup}\,|h(x, y)|$ in A, and let $\phi(u)$ be any real-valued function that is absolutely continuous together with $\phi'(u)$ and satisfies the conditions

$$\phi(0) = 0, \quad \phi'(0) = 0, \quad |\phi'(u)| \leq 3u^2, \quad |\phi''(u)| \leq 6\,|u|.$$

Then the problem

$$\Delta u + f(x, y)\,\phi(u) + h(x, y) = 0 \qquad ((x, y) \in A),$$

$$u = 0 \qquad ((x, y) \in \partial A)$$

has at least one solution u(x, y) with the properties listed in Theorem 1, provided $0 \leq \alpha < \overline{\alpha}(\beta)$ for some $\overline{\alpha}(\beta) > 0$. For instance, for $\beta = 0.8$ we can certainly take $0 \leq \alpha \leq 0.1$.

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