ON THE GROWTH OF UNIVALENT FUNCTIONS

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1. RESULTS

1.1. We shall assume that the function

$$f(z) = z + a_2 z^2 + \cdots \tag{1.1}$$

is analytic and univalent in $|z| < 1$. Throughout the paper, cap denotes logarithmic capacity.

THEOREM 1. *Let* $A(R) = \{z: |f(z)| \geq R\}$. *Then*

$$\operatorname{cap} A(R) \leq 1/\sqrt{R} \qquad (R > 0). \tag{1.2}$$

Furthermore, either

$$\sqrt{R} \operatorname{cap} A(R) \to 0 \qquad (R \to \infty), \tag{1.3}$$

or there is a starlike univalent function $g(z) = z + \cdots$ *such that*

$$\left|\log \frac{f(z)}{g(z)}\right| \leq K \,|\log(1 - |z|)|^{1/2} \qquad (|z| < 1) \tag{1.4}$$

for some constant K.

Since $|\arg g(z)/z| < \pi$, it follows from (1.4) that

$$|\arg f(z)/z| \leq K_1 \,|\log(1 - |z|)|^{1/2} \qquad (|z| < 1).$$

A slight modification of the proof of Theorem 1 also gives the bound

$$|f(z)| \leq K_2^{1/\lambda} \,|g(z)|^{1-\lambda} (1 - |z|)^{-2\lambda} \qquad (|z| < 1,\ 0 < \lambda \leq 1).$$

The function $f_m(z) = z(1 - z^m)^{-2/m}$ is univalent in $|z| < 1$, and it has the property

$$\liminf_{R \to \infty} \sqrt{R} \operatorname{cap} A_m(R) > 2^{-2/m} \qquad (m = 1, 2, \cdots).$$

Hence (1.2) is essentially best possible. A result of W. K. Hayman and P. B. Kennedy [5] shows that the right-hand side of (1.4) cannot be replaced by anything smaller than $o(|\log(1 - |z|)|^{1/2})$.

1.2. Hayman was the first to prove a regularity theorem of the following kind: Rapid growth of a univalent function implies rather regular behavior. Let

$$M(r) = \max_{|z|=r} |f(z)|.$$

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Then the limit $\alpha = \lim_{r \to 1} (1 - r)^2 M(r)$ exists, and $0 \leq \alpha \leq 1$. The theorem of Hayman [3, Theorem 2.9], [5], states the following.

Let $\alpha > 0$. *Then there exists* ζ_1 $(|\zeta_1| = 1)$ *such that*

$$|f(r\zeta_1)| \sim \alpha(1 - r)^{-2} \qquad (r \to 1),$$

$$|f(z)| \leq K_1(\lambda)\, |z - \zeta_1|^{\lambda - 2}(1 - |z|)^{-\lambda} \qquad (|z| < 1,\ 0 < \lambda \leq 2),$$

and, as $|z| \to 1$ *with* $|z - \zeta_1| \geq \delta > 0$,

$$\log |f(z)| = o(|\log(1 - |z|)|^{1/2}). \tag{1.5}$$

These estimates are best possible.

This theorem forms the basis of Hayman's result [2], [3, Theorem 5.7], that $|a_n|/n \to \alpha$ as $n \to \infty$. It should be pointed out that Hayman's theorem is valid for more general class of areally mean univalent functions [4].

The relation between Hayman's result and Theorem 1 will become clearer from the following consequence of Theorem 1. The first part is due to Hayman [3, Theorem 2.7].

THEOREM 2. (i) *Let there be a finite or infinite number of distinct points* ζ_k *with* $|\zeta_k| = 1$, *and positive numbers* α_k *and* β_k *such that*

$$|f(z)| > \beta_k (1 - |z|)^{-\alpha_k} \tag{1.6}$$

on some curve C_k *that lies in the unit disk except for its endpoint* ζ_k. *Then*

$$\sum_k \alpha_k \leq 2\,. \tag{1.7}$$

(ii) *Let there be points* ζ_k $(k = 1, \cdots, m)$ *with* $|\zeta_k| = 1$, *and positive numbers* α_k *and* β_k *with*

$$\alpha_1 + \cdots + \alpha_m = 2$$

such that (1.6) *holds. Then there exist positive numbers* γ_k *and* γ_k' *such that*

$$\gamma_k (1 - r)^{-\alpha_k} < |f(r\zeta_k)| < \gamma_k' (1 - r)^{-\alpha_k} \tag{1.8}$$

for $1/2 < r < 1$. *Also,*

$$\left|\log \frac{f(z)}{g(z)}\right| < K\, |\log(1 - |z|)|^{1/2} \qquad (|z| < 1), \tag{1.9}$$

where

$$g(z) = z \prod_{k=1}^{m} (1 - \bar{\zeta}_k z)^{-\alpha_k}. \tag{1.10}$$

We call ζ $(|\zeta| = 1)$ a point of maximal growth if there exists a number β $(0 < \beta \leq 1)$ such that $|f(z)| \geq \beta M(r)$ for all points z on some curve ending at ζ.

COROLLARY. *Let there be* m *distinct points* $\zeta_1, \cdots, \zeta_m$ *of maximal growth. Then*

$$M(r) \leq K_0(1 - r)^{-2/m} \qquad (0 < r < 1) \tag{1.11}$$

for some constant K_0. *Furthermore, if*

$$M(r) > \beta_0(1 - r)^{-2/m} \qquad (1/2 < r < 1) \tag{1.12}$$

for some $\beta_0 > 0$, *then* (1.8), (1.9), *and* (1.10) *hold with*

$$\alpha_1 = \cdots = \alpha_m = 2/m.$$

If we take $m = 1$, we obtain Hayman's theorem, though in a slightly different form. For instance, (1.9) is only an O-estimate, whereas (1.5) is an o-estimate. On the other hand, (1.9) gives an upper bound for $|\log f(z)|$, but (1.5) only for $\log|f(z)|$.

2. PROOF OF THEOREM 1

2.1. For a compact plane set E, we define

$$\Delta_n(E) = \max_{z_1,\cdots,z_n \in E} \prod_{\mu=1}^{n} \prod_{\substack{\nu=1 \\ \mu\neq\nu}}^{n} |z_\mu - z_\nu| \qquad (n = 2, 3, \cdots) \tag{2.1}$$

(the points $z_1, \cdots, z_n$ for which the maximum is attained are called nth *Fekete points* of E). Then

$$[\Delta_n(E)]^{1/n(n-1)} \searrow \operatorname{cap} E \qquad (n \to \infty). \tag{2.2}$$

Let $0 < \rho < 1$. We define

$$A(R, \rho) = \{z: |z| \leq \rho, \ |f(z)| \geq R\}.$$

Because $A(R, \rho)$ increases with ρ,

$$\lim_{\rho \to 1} \operatorname{cap} A(R, \rho) = \operatorname{cap} A(R). \tag{2.3}$$

We may assume that $\operatorname{cap} A(R) > 0$. Then $\operatorname{cap} A(R, \rho) \geq \sigma$ for some $\sigma > 0$ and $\rho_0 < \rho < 1$, and therefore, by (2.2),

$$\Delta_n(A(R, \rho)) \geq \sigma^{n(n-1)} \qquad (\rho_0 < \rho < 1). \tag{2.4}$$

2.2. Golusin has proved the following inequality [1, p. 121]: *If*

$$h(\zeta) = \zeta + b_0 + b_1 \zeta^{-1} + \cdots$$

is univalent in $|\zeta| > 1$, *then*

$$(2.5)\quad \left|\sum_{\mu=0}^{n}\sum_{\nu=0}^{n} c_\mu c_\nu \log\frac{h(\zeta_\mu)-h(\zeta_\nu)}{\zeta_\mu-\zeta_\nu}\right| \leq \sum_{\mu=0}^{n}\sum_{\nu=0}^{n} c_\mu \bar{c}_\nu \log\frac{1}{1-1/\zeta_\mu\bar{\zeta}_\nu}$$

for all c_ν *and* $|\zeta_\nu|>1$ $(\nu = 0, 1, \cdots, n)$.

We choose $z_\nu = z_{n\nu}(R, \rho)$ $(\nu = 1, \cdots, n)$ as nth Fekete points of $A(R, \rho)$. We apply Golusin's inequality with $h(\zeta) = 1/f(\zeta^{-1})$ and $\zeta_0 = \zeta = 1/z$, $c_0 = c$, and $\zeta_\nu = 1/z_\nu$, $c_\nu = 1$ $(\nu = 1, \cdots, n)$. Taking the negative real part on the left-hand side of (2.5), we find (with the indices μ and ν ranging from 1 to n)

$$(2.6)\quad \begin{aligned} &-\Re[c^2\log h'(\zeta)] - 2\Re\left[c\sum_\nu \log\frac{h(\zeta)-h(\zeta_\nu)}{\zeta-\zeta_\nu}\right] \\ &\qquad - \sum_{\mu\neq\nu}\sum \log\left|\frac{h(\zeta_\mu)-h(\zeta_\nu)}{\zeta_\mu-\zeta_\nu}\right| - \sum_\nu \log|h'(\zeta_\nu)| \\ &\qquad \leq |c|^2\log\frac{1}{1-|z|^2} + 2\Re\left[c\sum_\nu\log\frac{1}{1-\bar{z}_\nu z}\right] \\ &\qquad + \sum_{\mu\neq\nu}\sum\log\frac{1}{1-z_\mu\bar{z}_\nu} + \sum_\nu\log\frac{1}{1-|z_\nu|^2}. \end{aligned}$$

The case $n = 1$ of (2.5) gives

$$|\log h'(\zeta)| \leq |\log(1-|\zeta|^{-2})| \leq |\log(1-|z|)|.$$

Since $|h(\zeta_\nu)| = 1/|f(z_\nu)| \leq 1/R$,

$$\prod_{\mu\neq\nu}\prod |h(\zeta_\mu)-h(\zeta_\nu)| \leq n^n R^{-n(n-1)}.$$

Also, $|z_\mu - z_\nu| \leq |\zeta_\mu - \zeta_\nu|$, $|z_\mu - z_\nu| \leq |1 - z_\mu\bar{z}_\nu|$, and $|z_\nu| \leq \rho$. We write

$$(2.7)\quad \phi_n(z) = \frac{1}{n}\sum_{\nu=1}^{n}\log\frac{h(\zeta)-h(\zeta_\nu)}{(\zeta-\zeta_\nu)(1-\bar{z}_\nu z)} = \frac{1}{n}\sum_{\nu=1}^{n}\log\frac{(f(z)-f(z_\nu))zz_\nu}{(z-z_\nu)(1-\bar{z}_\nu z)f(z)f(z_\nu)},$$

and from (2.6) we obtain the inequality

$$\begin{aligned} &-2n\Re[c\,\phi_n(z)] \\ &\quad\leq 2|c|^2\log\frac{1}{1-|z|} + n\log n - n(n-1)\log R + 2n\log\frac{1}{1-\rho} - 2\log\Delta_n(A(R,\rho)). \end{aligned}$$

The last term arises because of (2.1) and the choice of the points z_ν. The inequality is valid for all c. We choose $|c| = n\gamma$ and $\arg c = \pi - \arg\phi_n(z)$. After division by n^2, we find that, for $|z| < 1$,

$$2\gamma\,|\phi_n(z)| \le 2\gamma^2\,|\log(1-|z|)| + \frac{\log n}{n} - \left(1-\frac{1}{n}\right)\log R - \frac{2}{n}\,|\log(1-\rho)| - \frac{2}{n^2}\log \Delta_n(A(R,\rho)). \tag{2.8}$$

2.3. It follows from (2.4) and (2.8) (with $\gamma = 1$) that

$$|\phi_n(z)| \le |\log(1-|z|)| + K(R) + \frac{1}{n-1}\,|\log(1-\rho)| \qquad (|z|<1,\ \rho>\rho_0),$$

where $K(R)$ depends only on R and f. By Montel's theorem, some subsequence $\{\phi_{n_k}(z)\}$ converges locally uniformly in $|z|<1$, say to $\phi(z, R, \rho)$, and clearly

$$|\phi(z,R,\rho)| \le |\log(1-|z|)| + K(R) \qquad (|z|<1,\ \rho>\rho_0). \tag{2.9}$$

Let $n = n_k \to \infty$ in (2.8). Because of (2.2), we see that

$$2\gamma\,|\phi(z,R,\rho)| \le 2\gamma^2\,|\log(1-|z|)| - \log R - 2\log \operatorname{cap} A(R,\rho). \tag{2.10}$$

This inequality holds for $\rho<1$ and $\gamma>0$. By (2.9), there exists a sequence $\{\rho_\ell\}$ such that $\rho_\ell \to 1$ and such that $\phi(z, R, \rho_\ell) \to \phi(z, R)$ locally uniformly in $|z|<1$. Let $\rho = \rho_\ell \to 1$ in (2.10). We find that

$$2\gamma\,|\phi(z,R)| \le 2\gamma^2\,|\log(1-|z|)| - 2\log[\sqrt{R}\operatorname{cap} A(R)] \tag{2.11}$$

for $|z|<1$, $\gamma>0$. By making $\gamma\to 0$, we see that $\sqrt{R}\operatorname{cap} A(R) \le 1$, and this proves (1.2). This part of the proof could have been simplified substantially if the aim had only been to prove (1.2) (compare [8]).

2.4. We shall now assume that (1.3) does not hold. Then there exists a sequence $\{R_j\}$ such that $R_j\to\infty$ and

$$\sqrt{R_j}\operatorname{cap} A(R_j) \ge \alpha > 0 \qquad (j = 1, 2, \cdots).$$

Hence (2.11) implies that

$$2\gamma\,|\phi(z,R_j)| \le 2\gamma^2\,|\log(1-|z|)| + 2\log 1/\alpha \qquad (|z|<1). \tag{2.12}$$

Therefore we may assume that $\phi(z, R_j) \to \phi(z)$ locally uniformly in $|z|<1$. We let $j\to\infty$ in (2.12). After dividing by 2γ, we find that

$$|\phi(z)| \le \gamma\,|\log(1-|z|)| + \gamma^{-1}\log 1/\alpha.$$

Finally, the choice $\gamma = \{|\log\alpha|/|\log(1-|z|)|\}^{1/2}$ gives the inequality

$$|\phi(z)| \le 2\,\{|\log\alpha|\cdot|\log(1-|z|)|\}^{1/2} = K\,|\log(1-|z|)|^{1/2} \qquad (|z|<1). \tag{2.13}$$

2.5. We now have to show that

$$\phi(z) = \log[g(z)/f(z)], \tag{2.14}$$

where $g(z) = z + \cdots$ is analytic and starlike in $|z|<1$. From (1.1) and (2.7) we deduce that $\phi_n(0) = 0$. It follows that $\phi(0) = 0$.

Let $|z| < r < 1$. We can choose an R_0 such that $|f(z)| < R_0$ for $|z| < r$. Then $|z_\nu| \geq r$ for $z_\nu \in A(R, \rho)$ if $R > R_0$ and $0 < \rho < 1$. By (2.7),

$$z\,\phi_n'(z) = \frac{1}{n}\sum_{\nu=1}^{n}\left(\frac{z f'(z)}{f(z) - f(z_\nu)} - \frac{z f'(z)}{f(z)} + \frac{z_\nu + z}{2(z_\nu - z)} + \frac{1 + \bar{z}_\nu z}{2(1 - \bar{z}_\nu z)}\right).$$

Because $|z| < r \leq |z_\nu|$ for $R > R_0$, it follows that

$$\Re\left[z\,\phi_n'(z) + \frac{z f'(z)}{f(z)}\right] \geq \frac{1}{n}\sum_{\nu=1}^{n}\Re\,\frac{z f'(z)}{f(z) - f(z_\nu)} \geq -\frac{|f'(z)|}{R - |f(z)|}.$$

We keep z fixed and let $n = n_k \to \infty$ and then $\rho = \rho_\ell \to 1$, as in Section 2.3. We find that, for $R > R_0$ and $|z| < r$,

$$\Re\left[z\,\phi'(z, R) + \frac{z f'(z)}{f(z)}\right] \geq -\frac{|f'(z)|}{R - |f(z)|}.$$

Now let $R = R_j \to \infty$. Then it follows that

$$\Re\left[z\,\phi'(z) + \frac{z f'(z)}{f(z)}\right] \geq 0, \tag{2.15}$$

and this inequality holds for $|z| < r$ for each $r < 1$, hence for all $|z| < 1$. We put $g(z) = f(z)\exp\phi(z)$. Then $g(z)$ is analytic in $|z| < 1$, and $g(z) = z + \cdots$ because $\phi(0) = 0$ and $f'(0) = 1$. Also, (2.15) implies that

$$\Re\, z\,\frac{g'(z)}{g(z)} \geq 0 \qquad (|z| < 1).$$

Hence $g(z)$ is starlike and univalent in $|z| < 1$, and (2.14) follows.

3. PROOF OF THEOREM 2

3.1. LEMMA. *Let* E_k $(k = 1, \cdots, m)$ *be compact plane sets with mutual distances at least* $b > 0$. *Let* $E = E_1 \cup \cdots \cup E_m$ *and* $\operatorname{cap} E < b$. *Then*

$$\frac{1}{\log(b/\operatorname{cap} E)} \geq \sum_{k=1}^{m}\frac{1}{\log(b/\operatorname{cap} E_k)}. \tag{3.1}$$

This lower estimate is the converse of the well-known upper estimate for the capacity of the union [6, p. 127].

Proof. Let n_k $(k = 1, \cdots, m)$ be positive integers, and let $z_{k1}, \cdots, z_{kn_k}$ be n_kth Fekete points of E_k. Let $n = n_1 + \cdots + n_m$. Then (2.1) implies that

$$\Delta_n(E) \geq \prod_{\substack{k=1\\k\neq\ell}}^{m}\prod_{\ell=1}^{m}\left(\prod_{\mu=1}^{n_k}\prod_{\nu=1}^{n_\ell}|z_{k\mu} - z_{\ell\nu}|\right)\cdot\prod_{k=1}^{m}\left(\prod_{\mu=1}^{n_k}\prod_{\substack{\nu=1\\\mu\neq\nu}}^{n_k}|z_{k\mu} - z_{k\nu}|\right)$$

$$\geq b^{n(n-1)} \prod_{k=1}^{m} [b^{-n_k(n_k-1)} \Delta_{n_k}(E_k)].$$

It follows that

$$\frac{1}{n(n-1)} \log \Delta_n(E) - \log b \geq \sum_{k=1}^{m} \frac{n_k(n_k-1)}{n(n-1)} \left(\frac{1}{n_k(n_k-1)} \log \Delta_{n_k}(E_k) - \log b \right).$$

If we let $n_k \to \infty$ so that $n_k/n \to \delta_k$ $(k = 1, \cdots, m)$ with $\delta_1 + \cdots + \delta_m = 1$, then we see from (2.2) that

$$\log \operatorname{cap} E - \log b \geq \sum_{k=1}^{m} \delta_k^2 (\log \operatorname{cap} E_k - \log b).$$

The choice

$$\delta_k = \frac{1}{\log (b/\operatorname{cap} E_k)} \left(\sum_{\ell=1}^{m} \frac{1}{\log (b/\operatorname{cap} E_\ell)} \right)^{-1}$$

gives (3.1).

3.2. Let ζ_k $(k = 1, \cdots, m)$ be any points for which (1.6) is satisfied on C_k. If R is large enough, we can choose s_k $(k = 1, \cdots, m)$ so that $\beta_k(1 - s_k)^{-\alpha_k} = R$. Let C_k be orientated towards ζ_k, and let z_k be the last point of C_k with $|z_k| = s_k$. If E_k is the arc of C_k between z_k and ζ_k, then $|z| \geq s_k$ for $z \in E_k$, hence, by (1.6),

$$|f(z)| > \beta_k(1 - s_k)^{-\alpha_k} = R \qquad (z \in E_k).$$

Consequently $E_k \subset A(R)$. Since E_k is connected,

$$\text{(3.2)} \quad \operatorname{cap} E_k \geq \frac{1}{4} \operatorname{diam} E_k \geq \frac{1}{4} |\zeta_k - z_k| \geq \frac{1}{4}(1 - s_k) = \frac{1}{4}(\beta_k/R)^{1/\alpha_k} > B_1^{-1} R^{-1/\alpha_k}$$

with some constant B_1.

We apply the lemma to $E_1, \cdots, E_m$. The mutual distance is at least $b = \frac{1}{2} \min_{k \neq \ell} |\zeta_k - \zeta_\ell|$ if R is sufficiently large. Therefore (3.1) and (3.2) imply that

$$\frac{1}{\log (b/\operatorname{cap}[E_1 \cup \cdots \cup E_m])} \geq \sum_{k=1}^{m} \frac{\alpha_k}{\log R + B_2}.$$

Hence we see from (1.2) that

$$\text{(3.3)} \qquad R^{-1/2} \geq \operatorname{cap} A(R) \geq \operatorname{cap}(E_1 \cup \cdots \cup E_m) \geq (B_3 R)^{-1/\Sigma_{k=1}^m \alpha_k}.$$

Making $R \to \infty$, we thus find that $\alpha_1 + \cdots + \alpha_m \leq 2$. Since this inequality holds for any m (in case there are infinitely many points ζ_k), we obtain (1.7). This proves part (i) of Theorem 2.

3.3. We shall now prove part (ii). Let us first turn to the lower estimate in (1.8). Suppose this is false, say for $k = 1$, $\zeta_1 = 1$. Then $(1 - r_n)^{\alpha_1} |f(r_n)| \to 0$ for some sequence $r_n \to 1 - 0$. Applying the distortion theorems to the derivative of the function

$$f_n(z) = \left[f\left(\frac{r_n + z}{1 + r_n z}\right) - f(r_n) \right] / [(1 - r_n^2) f'(r_n)] = z + \cdots,$$

we infer that, for any $\sigma < 1$,

$$(1 - |z|)^{\alpha_1} |f(z)| \to 0 \qquad (z \in H_n(\sigma),\ n \to \infty),$$

where $H_n(\sigma) = \left\{ z: \left| \frac{z - r_n}{1 - r_n z} \right| < \sigma \right\}$. It follows from (1.6) that C_1 does not meet $H_n(\sigma)$ for any $\sigma < 1$ and $n > n_0(\sigma)$.

We now proceed as in Section 3.2. Let $z_1 = z_{n1}$ be the last point of C_1 with $|z_{n1}| = r_n$, and let $E_1 = E_{n1}$ be the arc of C_1 between z_{n1} and 1. We choose $R = R_n = \beta_1 (1 - r_n)^{-\alpha_1}$. Again, $E_{n1} \subset A(R_n)$. Because $H_n(\sigma) \cap E_{n1} = \emptyset$ for $\sigma < 1$ and $n > n_0(\sigma)$, geometric considerations show that $\operatorname{diam} E_{n1} / (1 - r_n) \to \infty$ as $n \to \infty$. Hence we find instead of (3.2) that

$$\operatorname{cap} E_{1n} > \lambda_n R_n^{-1/\alpha_1}, \qquad \lambda_n \to \infty,$$

while (3.2) holds for $k = 2, \cdots, m$. Therefore the lemma implies that

$$\operatorname{cap} A(R_n) \geq \lambda_n' R_n^{-1/2}, \text{ with } \lambda_n' \to \infty,$$

in contradiction to (1.2).

Next, we prove the upper estimate in (1.8). Let again $k = 1$, $\zeta_1 = 1$. Suppose $(1 - r_n)^{\alpha_1} |f(r_n)| = \lambda_n \to \infty$ for some sequence $\{r_n\}$ $(r_n \to 1 - 0)$. We choose $R_n = |f(r_n)|$ and then z_{kn} $(k = 2, \cdots, m)$ as before, whereas $z_{1n} = r_n$. Then $z_{1n} \in A(R_n)$. By the maximum principle, there exists a continuum $E_{1n} \subset A(R_n)$ that connects z_{1n} with the circle $|z| = 1$. Then, instead of (3.2),

$$\operatorname{cap} E_{1n} \geq \frac{1}{4} \operatorname{diam} E_{1n} \geq \frac{1}{4}(1 - r_n) \geq \frac{1}{4} \lambda_n^{1/\alpha_1} R_n^{-1/\alpha_1}.$$

Reasoning as in Section 3.2, we deduce that $\operatorname{cap} A(R_n) > \lambda_n' R_n^{-1/2}$, with $\lambda_n' \to \infty$, contrary to (1.2).

3.4. Finally, we have to prove (1.9) together with (1.10). It follows from (3.3) that (1.3) does not hold, so that (1.4) is valid. We have to prove that the starlike function $g(z)$ has the form (1.10).

Every starlike function $g(z) = z + \cdots$ can be represented [7, Lemma 1] in the form

$$g(z) = z \exp\left[- \int_0^{2\pi} \log(1 - e^{-it} z)\, dp(t) \right], \tag{3.4}$$

where $p(t)$ increases and has variation 2. It follows that

$$g(z) = z \prod_k (1 - e^{-i\theta_k} z)^{-p_k} \psi(z).$$

Here θ_k and $p_k > 0$ are the locations and the heights of the jumps, and $p_1 + p_2 + \cdots \leq 2$; furthermore

$$|\psi(z)| = O((1 - |z|)^{-\varepsilon}) \quad (|z| \to 1)$$

for every $\varepsilon > 0$. Hence the left-hand inequality (1.8) shows that $p_k \geq \alpha_k$ ($e^{i\theta_k} = \zeta_k$) if the numbers are suitably ordered. Since

$$2 = \alpha_1 + \cdots + \alpha_m \leq p_1 + \cdots + p_m \leq 2,$$

it follows that $\alpha_k = p_k$ ($k = 1, \cdots, m$). But $p_1 + \cdots + p_m = 2$ implies that $g(z)$ has the form (1.10), because the function $p(t)$ in (3.4) becomes a jump function with $\theta_1, \cdots, \theta_m$ as the only discontinuities.

3.5. *Proof of the Corollary.* Suppose (1.11) does not hold. Then

$$M(r_n) > \lambda_n (1 - r_n)^{-2/m}, \text{ with } \lambda_n \to \infty$$

for some sequence $\{r_n\}$ ($r_n \to 1 - 0$). We choose

$$R_n = \beta M(r_n), \qquad \beta = \min(\beta_1, \cdots, \beta_m)$$

and proceed as in Section 3.2 to show that

$$\operatorname{cap} A(R_n) > \lambda_n^{1/2} (B_3 R_n)^{-1/2},$$

in contradiction to (1.2).

The second part follows immediately from Theorem 2, because of (1.12) and the definition of points of maximal growth.

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