CONCERNING UNCOUNTABLE FAMILIES OF n-CELLS IN E^n

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Bing has shown [1] that if $\{S_\alpha\}_{\alpha \in A}$ is an uncountable family of mutually exclusive closed surfaces in euclidean 3-space E^3, then all except countably many of the S_α are tame. Moreover, Stallings has given an example showing that the hypothesis that each S_α is closed cannot be deleted [6]. It is natural to conjecture that an analogous situation exists in higher-dimensional space; that is, that there do not exist uncountably many mutually exclusive wild closed (n - 1)-manifolds in euclidean n-space E^n.

We consider this conjecture for the case where each of the (n - 1)-manifolds in question is a topological (n - 1)-sphere in E^n. As one might rightfully deduce from the title of this paper, we are not prepared to solve the general problem. However, a recent result of Černavskiĭ [4] enables us to obtain a partial solution.

THEOREM 1. *Suppose that* $\{B_\alpha\}_{\alpha \in A}$ *is an uncountable collection of* n*-cells in* E^n $(n \geq 5)$ *such that* $\mathrm{Bd}\, B_\alpha \cap \mathrm{Bd}\, B_\beta = \emptyset$ *if* $\alpha \neq \beta$. *Then all but countably many of the* B_α *are tame.*

We shall let d denote the standard metric on E^n. The space of continuous functions from the (n - 1)-sphere S^{n-1} into E^n (with the supremum metric) is denoted by M. Since S^{n-1} is compact and E^n is separable, M is a separable metric space. Thus, if $\{h_\alpha\}_{\alpha \in A}$ is an uncountable family of embeddings of S^{n-1} into E^n such that $h_\alpha(S^{n-1}) \cap h_\beta(S^{n-1}) = \emptyset$ for $\alpha \neq \beta$, then for almost all $\alpha \in A$ (that is, for all except countably many $\alpha \in A$),

$$h_\alpha = \lim_{m \to \infty} h_{\alpha_m} = \lim_{m \to \infty} h_{\beta_m},$$

where each $h_{\alpha_m}(S^{n-1})$ is contained in one complementary domain of $h_\alpha(S^{n-1})$ and each $h_{\beta_m}(S^{n-1})$ is contained in the other. (Otherwise, one could produce uncountably many mutually exclusive open subsets of M.) If X is a subset of E^n and if $\varepsilon > 0$, then $f\colon X \to E^n$ is an ε*-mapping* provided that $d(x, f(x)) < \varepsilon$ for each $x \in X$. Given $X \subset Y \subset E^n$, we say that X is an ε*-retract* of Y if there exists an ε-retraction of Y onto X.

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PROPOSITION 1. *Suppose that* S *is an* (n - 1)*-sphere in* E^n *and* $\varepsilon > 0$. *Then there exists an open set* V *containing* S *and a positive number* η *such that if* $h\colon S \to V$ *is an* η*-homeomorphism, then* h(S) *is an* ε*-retract of* V.

Proof. By induction, it is easy to produce positive numbers $\delta_0, \delta_1, \cdots, \delta_n \leq \varepsilon$ and η with $\delta_{i-1} < \delta_i/4$ $(i = 1, \cdots, n)$ such that if $h\colon S \to E^n$ is an η-homeomorphism and X is a subset of h(S) of diameter less than $5\delta_{i-1}$, then X is contained in a cell C in h(S) of diameter less than $\delta_i/2$.

Let T be a triangulation of E^n with mesh less than δ_0, and let N be the union of all simplexes σ in T that meet S. Let V be the interior of N, and let η be

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chosen as above. Assume that η is sufficiently small so that for each η-homeomorphism $h: S \to E^n$, N is contained in $N_{\delta_0}(h(S))$, the δ_0-neighborhood of $h(S)$. Given such a homeomorphism $h: S \to V$, we shall produce the desired retraction $r: V \to h(S)$, basically as it is done in [2].

Let N^k denote the polyhedron of the k-skeleton of N. Define a retraction $r_0: N^0 \cup h(S) \to h(S)$ by taking $r_0(v)$ to be any point of $h(S)$ within δ_0 of v, if $v \in N^0 - h(S)$. Inductively, assume that we have a δ_k-retraction

$$r_k: N^k \cup h(S) \to h(S).$$

Let σ be a $(k+1)$-simplex of T in N^{k+1}. Then

$$\operatorname{diam}[(\sigma \cap h(S)) \cup r_k(\operatorname{Bd} \sigma)] < 2\delta_0 + 3\delta_k < 5\delta_k,$$

so that $(\sigma \cap h(S)) \cup r_k(\operatorname{Bd} \sigma)$ lies in a cell C in h(S) of diameter less than $\delta_{k+1}/2$. By Tietze's Extension Theorem, $(r_k \mid \operatorname{Bd} \sigma) \cup (1 \mid \sigma \cap h(S))$ extends to a map $r_\sigma: \sigma \to C$. Combining all the r_σ for $(k+1)$-simplexes σ in N^{k+1}, we obtain a δ_{k+1}-retraction $r_{k+1}: N^{k+1} \cup h(S) \to h(S)$. The retraction $r = r_n$ restricted to V has the required properties.

The next proposition follows immediately from Proposition 1.

PROPOSITION 2. *Suppose that* U *is a complementary domain of an* $(n - 1)$-*sphere* S *in* E^n. *Then for each* $\varepsilon > 0$ *there exist an open set* V *containing* Cl(U) *and a positive number* η *such that for each* η-*homeomorphism* $h: S \to U$, *the closure of the component of* $E^n - h(S)$ *contained in* U *is an* ε-*retract of* V.

A subset X of E^n is said to be 1-ULC if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that every mapping $f: S^1 \to X$ with $\operatorname{diam} f(S^1) < \delta$ extends to a mapping g of the 2-cell B^2 into X with $\operatorname{diam} g(B^2) < \varepsilon$.

PROPOSITION 3. *Suppose that* U *is a complementary domain of an* $(n - 1)$-*sphere* S *in* E^n *such that for each* $\eta > 0$ *there exists an* η-*homeomorphism* $h: S \to U$. *Then* U *is* 1-ULC.

Proof. Given $\varepsilon > 0$, choose an open set V containing Cl(U) and a positive number η corresponding to $\varepsilon/3$ as in Proposition 2. Let $\delta > 0$ $(\delta \leq \varepsilon/3)$ be chosen so that $N_\delta(\mathrm{Cl}(U)) \subset V$. Suppose that $f: S^1 \to U$ is a mapping with $\operatorname{diam} f(S^1) < \delta$. Let $h: S \to U$ be an η-homeomorphism such that $f(S^1) \subset U_h$, and let $r: V \to \mathrm{Cl}(U_h)$ be an $\varepsilon/3$-retraction, where U_h denotes the component of $E^n - h(S)$ contained in U.

Since $\operatorname{diam} f(S^1) < \delta$, there exists an extension $g': B^2 \to E^n$ of f with $\operatorname{diam} g'(B^2) < \delta$. Thus $g'(B^2)$ is contained in V, and so the composition $g = rg': B^2 \to \mathrm{Cl}(U_h) \subset U$ is the desired extension of f.

As a consequence of Proposition 3 and [5, Theorem 4], we see that if $\{S_\alpha\}_{\alpha \in A}$ is an uncountable family of mutually exclusive $(n - 1)$-spheres in S^n, then almost all the S_α have complementary domains that are open n-cells.

Proof of Theorem 1. Given an uncountable family $\{B_\alpha\}_{\alpha \in A}$ of n-cells in E^n $(n \geq 5)$ with mutually exclusive boundaries, let $\{h_\alpha\}_{\alpha \in A}$ be a collection of embeddings of S^{n-1} into E^n such that $h_\alpha(S^{n-1}) = \operatorname{Bd} B_\alpha$ for each $\alpha \in A$. Then for almost all $\alpha \in A$, $h_\alpha = \lim_{m \to \infty} h_{\alpha_m}$, where

$$h_{\alpha_m}(S^{n-1}) \subset E^n - B_\alpha \quad \text{for } m = 1, 2, \cdots.$$

Hence, $E^n - B_\alpha$ is 1-ULC for almost all $\alpha \in A$.

In a recent paper [4], Černavskiĭ has shown that a compact k-dimensional manifold M with boundary in E^n $(n \geq 5)$ is locally flat in E^n (see [3]) provided M is locally flat at each of its interior points and E^n - M is 1-LC at each point of Bd M. Hence, almost all the B_α are locally flat in E^n, and so, by [3], almost all the B_α are tame.

In view of the result of Černavskiĭ stated above, it is clear that a more general statement holds.

THEOREM 2. *Suppose that* M *is a compact* n*-manifold with boundary, that* $\{M_\alpha\}_{\alpha \in A}$ *is an uncountable family of* n*-manifolds in* E^n $(n \geq 5)$, *each homeomorphic to* M, *and that* Bd $M_\alpha \cap$ Bd $M_\beta = \emptyset$ *if* $\alpha \neq \beta$. *Then all but countably many of the* M_α *are locally flat in* E^n.

As for (n - 1)-manifolds with boundary in E^n, the situation in higher dimensions is the same as that described by Stallings in [6] for n = 3. As the title of [6] indicates, Stallings constructs uncountably many mutually exclusive wild 2-cells in E^3. For an (n - 1)-cell D in E^n, let $D' = D \times [-1, 1]$ in $E^n \times E^1 = E^{n+1}$. Observe that if $E^{n+1} - D'$ is 1-ULC, then the same is true for $E^n - D$. This proves that if $\{B_\alpha\}_{\alpha \in A}$ is an uncountable collection of mutually exclusive wild 2-cells in E^3, then the same is true for the collection $\{B_\alpha \times I^{n-3}\}_{\alpha \in A}$ in $E^n = E^3 \times E^{n-3}$, where $I = [-1, 1]$.

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