SINGLY-GENERATED LIOUVILLE F-ALGEBRAS

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1. INTRODUCTION

The entire functions of a single complex variable provide an interesting example of a metrizable, complete, locally multiplicatively-convex, topological algebra (an F-algebra). Operations are pointwise addition and multiplication; the topology is the compact-open topology or, what is equivalent, the topology of uniform convergence on compact subsets of the plane. One may ask whether the algebra of entire functions can be abstractly characterized by certain significant features of its topological and algebraic structure. In particular, to what extent does Liouville's theorem, that every bounded entire function is constant, characterize the entire functions among the singly-generated F-algebras?

Arens in [1] shows that a semisimple F-algebra A with identity which is rationally singly-generated by z in A and which has a continuous derivation D such that $Dz = 1$ and such that

$$\|D^k f\|_n \leq k!\, r_n^{-k} \|f\|_{n+1},$$

where $\|\cdot\|_n$ is a pseudonorm and $r_0, r_1, r_2, \cdots$ is a sequence of positive reals, must be (topologically and algebraically) the algebra of all holomorphic functions on some open subset of the plane with the compact-open topology. Thus the topological-algebraic structure, together with the existence of a continuous derivation satisfying Cauchy's inequality, characterizes such algebras. The connection between Liouville's theorem and Cauchy's inequality makes the Liouville property a plausible selection for characterization of the entire functions. Rudin in [4] asks to what extent function algebras, which satisfy a weak maximum modulus principle on well-behaved domains in the plane, are algebras of holomorphic functions thereon; and he obtains a partial answer. Here again, the connection between the maximum modulus assumption and Liouville's theorem is suggestive.

In this paper we consider singly-generated F-algebras and impose conditions to guarantee algebraic and topological equivalence with the algebra of entire functions in the compact-open topology.

2. PRELIMINARIES

Some definitions which will facilitate the exposition are:

DEFINITION 2.1. *An* F*-algebra is singly-generated if it is the completion of the polynomials in one of its elements.*

DEFINITION 2.2. *An* F*-algebra is to be called a Liouville algebra if the spectrum of each non-constant element of the algebra is unbounded.*

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DEFINITION 2.3. *A function algebra* A *is an* F*-algebra such that the pseudonorms* $\|\cdot\|_n$ *which define the topology of* A *satisfy the condition*

$$\|x^2\|_n = \|x\|_n^2 \quad (\text{all } x \in A).$$

Let A be a commutative F-algebra with identity whose topology is given by the family $\{\|\cdot\|_n: n = 1, 2, \cdots\}$ of pseudonorms $\|\cdot\|_n$. (We may assume that the pseudonorms have been so chosen that $\|\cdot\|_n \leq \|\cdot\|_{n+1}$ $(n = 1, 2, \cdots)$.) Let B_n denote the normed algebra $A/\{x \in A: \|x\|_n = 0\}$, and for $n \geq m$, let $\pi_m^n: B_n \to B_m$ be the canonical mapping between the factor algebras. π_m^n $(n \geq m)$ can be extended to a continuous mapping with dense range from the completion A_n of B_n into the completion A_m of B_m. $\{A_n; \pi_m^n\}$ is a dense inverse limit system; and by the fundamental result of Michael [3], A is the inverse limit of $\{A_n; \pi_m^n\}$. Each A_n is a Banach algebra. The maximal closed ideal space M (or the space of non-zero, continuous, complex-valued, algebra homomorphisms) of A is the union of the maximal ideal spaces M_n of A_n. The relative strong topology on M is the direct limit topology derived from the direct limit system $\{M_n; \pi_m^{n*}\}$. As we have chosen the pseudonorms for A, π_m^{n*} $(n \geq m)$ is an injection map. A set $U \subset M$ is open in the direct limit topology if and only if $U \cap M_n$ is relatively weak* open in M_n for each n. The spectrum of x in A is

$$\{z \in \mathfrak{C} \mid x - z \text{ has no inverse}\} = M(x).$$

By the mapping $\hat{x}: M \to \mathfrak{C}$ that is given by $\hat{x}(m) = m(x)$ for $m \in M$ denote the Gelfand transform of $x \in A$. The topological algebra terminology we use is in accord with Michael's [3].

3. SINGLY-GENERATED F-ALGEBRAS

THEOREM 3.1. *If the image in the plane of the maximal closed ideal space* M *of a singly-generated Liouville* F*-algebra* A *with identity has non-empty interior, then* M *can be identified with the set* $\mathfrak{C}$ *of complex numbers.*

Proof. Let α generate A. A is the inverse limit of Banach algebras A_n, each of which is singly-generated by the projection π_n of α in A_n and so is commutative. As is well known, $(\pi_n \alpha)^\wedge$ is a homeomorphism of M_n with a compact non-separating subset D_n of the complex plane. Suppose there is a $z_0 \in \mathfrak{C}$ such that $z_0 \notin \hat{\alpha}(M) = D$. Then for each n, choose an arc γ_n such that γ_n joins z_0 to ∞ and $\gamma_n \cap D_n = \emptyset$. Such arcs exist, since $\mathfrak{C} \setminus D_n$ is open and connected. Construct a single-valued analytic function $s_n: \mathfrak{C} \setminus \gamma_n \to \mathfrak{C}$ with $(s_n(z))^2 = z - z_0$. Having chosen s_n, restrict s_n to a neighborhood V_n of D_n which does not meet $\gamma_n \cup \gamma_{n+1}$, and then continue $s_n | V_n$ analytically to s_{n+1}, which is defined on $\mathfrak{C} \setminus \gamma_{n+1}$. (The assumption that $\|\cdot\|_n \leq \|\cdot\|_{n+1}$ implies $D_n \subset D_{n+1}$!) Proceeding inductively, construct $\{s_n\}_{n=1}^\infty$, thereby obtaining a single-valued square root s defined on $\hat{\alpha}(M) = D$. Let

$$\beta_n = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{s_n(z)}{z - \alpha}\, dz \quad (n = 1, 2, \cdots),$$

where Γ_n is a contour in V_n missing D_n. $\beta = \{\beta_n\}_{n=1}^\infty$ is in A, since $\pi_i^j \beta_j = \beta_i$ for $j \geq i$. Furthermore, $\hat{\beta} = s \circ \hat{\alpha}$ omits all values in some sphere in $\mathfrak{C}$, since D has interior s and does not assume a value and its negative. Then it is clear (for example, by the Riemann Mapping Theorem) that there exists a non-constant bounded analytic function F defined on a neighborhood of $\hat{\beta}(M)$. By Theorem 10.1 of [3], $F \circ \hat{\beta}$ is in $\hat{A}$, which is in contradiction to the assumption that A is a Liouville algebra. This contradiction establishes the theorem.

COROLLARY. *The maximal closed ideal space* M *of a singly-generated Liouville function algebra* A *with identity can be identified with the set* $\mathfrak{C}$ *of complex numbers.*

Proof. A is the inverse limit of singly-generated sup-normed Banach algebras A_n, each of which is isomorphic to the uniform closure of the polynomials defined on the spectrum of the generator. Denote the respective spectra by D_n $(n = 1, 2, \cdots)$, and let

$$D = \bigcup_{n=1}^{\infty} D_n .$$

The maximal ideal space of A_n is homeomorphic to D_n $(n = 1, 2, \cdots)$. If D_n has no interior, then $\hat{A}_n = C(D_n)$, the uniformly closed algebra of all continuous functions on the compact space D_n. Consequently, if no D_n has interior, $\hat{A} = C(D)$, the algebra of all continuous functions on D, which is not a Liouville algebra. Therefore, the fact that some D_n has interior renders the interior of D non-empty, and the above theorem applies.

For singly-generated F-algebras we introduce a condition from which it will follow that M is the complex plane with its Euclidean topology. Here, α, M, D, M_n, and D_n have the same meanings as in the preceding proofs.

CONDITION ϕ. *The intersection of the boundaries of* D_n $(n = 1, 2, \cdots)$ *is empty.*

THEOREM 3.2. *If condition* ϕ *is satisfied, the maximal closed ideal space* M *of a singly-generated Liouville* F*-algebra* A *with identity is homeomorphic to the complex plane with its Euclidean topology.*

Proof. Clearly, as a set, M can be identified with $\mathfrak{C}$. Now let U be a Euclidean open subset of the plane. Each compact M_n is homeomorphic to D_n, so

$$\hat{\alpha}^{-1}(U \cap D_n) = \hat{\alpha}^{-1}(U) \cap M_n$$

is relatively open in M_n for each n. Also, letting W be an open subset of M in the direct limit topology, $m \in W$, and $\hat{\alpha}(m) = z$, we conclude from condition ϕ that z is in $\mathrm{int}(M_n)$ for some n. Hence, there exists a Euclidean neighborhood V of z with $z \in V \subset \hat{\alpha}(W)$. This shows that $\hat{\alpha}(W)$ is a Euclidean neighborhood of z. Therefore $\hat{\alpha}$ is a homeomorphism of M onto the complex plane.

THEOREM 3.3. *Every singly-generated Liouville* F*-algebra* A *with identity is topologically and algebraically isomorphic to the algebra* $\mathrm{Hol}(\mathfrak{C})$ *of entire functions on the complex plane with the compact-open topology, provided that Condition* ϕ *obtains.*

Proof. We first establish the existence of an equivalent system

$$\{ \|\cdot\|_k' : k = 1, 2, \cdots\}$$

of pseudonorms with the property that any weak* compact subset K on M is contained in some set of the form

$$\{m \in M: |m(x)| \leq 1 \quad \text{if} \quad \|x\|'_k \leq 1\}.$$

To this end, put $U_n = \{x \in A: \|x\|_n \leq 1\}$. $\{U_n\}_{n=1}^{\infty}$ is a base for A in the sense that dilations of the sets in $\{U_n\}_{n=1}^{\infty}$ form a fundamental system of neighborhoods of 0 in A. If K is a weak* compact subset of M, then

$$K^0 = \{x \in A: |m(x)| \leq 1 \text{ for } m \in K\}$$

is a convex, absorbing, balanced, idempotent and closed subset of A, and hence is a neighborhood of 0, since A is an F-algebra (therefore A is barrelled); that is, $K^0 \supset \lambda^{-1} U_n$ for $\lambda \geq 1$ and for some n. Therefore

$$K \subset K^{00} \subset (\lambda^{-1} U_n)^0 = \lambda U_n^0 \tag{1}$$

implies

$$K = K \cap M \subset \lambda U_n^0 \cap M,$$

and we also see that

$$(\lambda^{-1} U_n) \cdot (\lambda^{-1} U_n) \subset \lambda^{-1} U_n. \tag{2}$$

Let D_r be the closed disc of radius r centered at the origin of the complex plane. Since M is homeomorphic to the complex plane (Theorem 3.2), $\hat{\alpha}^{-1}(D_r)$ is a weak* compact subset of M; hence, by (1) above, there exist n_1 and $\lambda_1 \geq 1$ such that

$$\hat{\alpha}^{-1}(D_1) \subset \lambda_1 U_{n_1}^0 \cap M \subset \lambda_1 U_{n_1+1}^0 \cap M.$$

Now $\hat{\alpha}(\lambda_1 U_{n_1+1}^0 \cap M)$ is compact and non-separating, so there exists a disc D_{r_1} and corresponding to it $\lambda_2 \geq 1$ and U_{n_2} such that

$$\lambda_1 U_{n_1+1}^0 \cap M \subset \hat{\alpha}^{-1}(D_{r_1}) \subset \lambda_2 U_{n_2}^0 \cap M \tag{3}$$

and

$$\operatorname{int}(\hat{\alpha}(\lambda_1 U_{n_1+1}^0 \cap M)) \subset \operatorname{int} D_{r_1}. \tag{4}$$

Continue in this fashion, constructing a sequence $\{\lambda_k^{-1} U_{n_k}\}_{k=1}^{\infty}$ of neighborhoods of 0 in A. $\{n_k\}_{k=1}^{\infty}$ is a subsequence of $\{n\}_{n=1}^{\infty}$, for otherwise $\hat{\alpha}(M)$ would be bounded, which is impossible. Therefore $\{\lambda_k^{-1} U_{n_k}\}_{k=1}^{\infty}$ is a base for A. For $x \in A$, let

$$\|x\|'_k = \inf\{\mu \geq 0: x \in \mu \lambda_k U_{n_k}\} \quad (k = 1, 2, \cdots).$$

$\{\|\cdot\|'_k: k = 1, 2, \cdots\}$ is a system of pseudonorms equivalent to the system $\{\|\cdot\|_n: n = 1, 2, \cdots\}$ of pseudonorms originally used to define the topology of A. Furthermore, letting $N_k = \{x \in A: \|x\|'_k = 0\}$ and letting A'_k be the completion of the normed algebra A/N_k, we see that

$$A = \text{inverse limit } (A'_k). \tag{5}$$

Denote the maximal ideal space of the Banach algebra A'_k by M'_k. Then

(6) $$M_1' \subset \operatorname{int}(M_2') \subset M_2' \subset \operatorname{int}(M_3') \subset M_3' \subset \cdots,$$

because $M \cap U^0_{n_k} = M_k'$. (See Michael [3; p. 28].)

Having constructed A_k' $(k = 1, 2, \cdots)$ with M_k' satisfying (6), we are in a position to adapt an argument of Arens in [1] to conclude the proof of the theorem. If p is a polynomial in the generator α, then $p \circ \hat{\alpha}$ is in $\hat{A}$; and, since convergence in the given F-algebra topology entails convergence of representing functions on *all* compact subsets of M, there exists for each $\hat{x} \in \hat{A}$ an f such that $f \circ \hat{\alpha} = \hat{x}$ and f is a uniform limit of polynomials on compact subsets of $\hat{\alpha}(M)$, the complex plane. Consequently, f is an entire function. Moreover, $x \to f$ is a continuous homomorphism of A into $\mathrm{Hol}(\mathfrak{C})$. Continuing to follow Arens, we consider the homomorphism of $\mathrm{Hol}(\mathfrak{C})$ into A given by

$$f \to \left\{ \frac{1}{2\pi i} \int_{\Gamma_j} \frac{f(z)}{z - \alpha}\, dz = (a_f)_j \right\}_{j=1}^{\infty},$$

where the Γ_j are chosen to be the boundaries of the discs D_{r_j} $(j = 1, 2, 3, \cdots)$. Let $a_f = \{(a_f)_j\}_{j=1}^{\infty}$. Then $\hat{a}_f(m) = f \circ \hat{\alpha}(m)$. And for $x \in A$, the maps

$$x \to f \to a_f \to f$$

are all isomorphisms, since semi-simplicity follows from the Liouville hypothesis.

The topology on A induced by the topology of uniform convergence of representing functions on compact subsets of M makes A into an F-algebra which is commutative, semi-simple, and all of whose homomorphisms are continuous. Therefore, by Theorem 14.2 of [3], the original topology of A coincides with the compact-open topology. (We could also verify this fact directly.) This concludes the proof.

THEOREM 3.4. A *is a singly-generated Liouville* F-*algebra with identity and with no topological divisors of zero if and only if* A *is topologically and algebraically isomorphic to* $\mathrm{Hol}(\mathfrak{C})$.

Proof. (For the definition of topological divisor of zero see [3; p. 43].) As before, let A = inverse limit (A_n) and M_n be respective maximal ideal spaces. Assume that $\|\cdot\|_1 \leq \|\cdot\|_2 \leq \cdots$. By 1.2 in Arens [2], if there are no topological divisors of zero, then M_k is contained in the interior of M_n for some n, $n > k$. Consequently, Condition ϕ is satisfied, and Theorem 3.3 applies. Conversely, it is well known that the algebra $\mathrm{Hol}(\mathfrak{C})$ has no topological zero divisors.

Regard the maximal closed ideal space M of a singly-generated Liouville F-algebra A as identified with the set $\mathfrak{C}$ of complex numbers. Then each representing function $\hat{f} \in \hat{A}$ is a continuous function from $\mathfrak{C}$ with its direct limit topology (or relative weak* topology) into $\mathfrak{C}$ with its Euclidean topology. If these topologies agree on $\mathfrak{C}$, then the proof and conclusions of Theorem 3.3 remain valid. In particular, if every $\hat{f} \in \hat{A}$ is a continuous function on the complex plane (Euclidean topology), then all mentioned topologies coincide.

THEOREM 3.5. *Every singly-generated Liouville* F-*algebra of continuous function on the complex plane is topologically and algebraically isomorphic to the algebra* $\mathrm{Hol}(\mathfrak{C})$ *of entire functions with the compact-open topology.*

Proof. The direct limit topology on $\mathfrak{C}$ is stronger than the Euclidean topology on $\mathfrak{C}$. But basic weak* neighborhoods of $z_0 \in \mathfrak{C}$ are of the following form:

$$V = \{z \in \mathfrak{C} : |f_i(z) - f_i(z_0)| < \varepsilon,\ \varepsilon > 0,\ f_i \in A,\ 1 \leq i \leq n\}.$$

Since each f_i is continuous, V contains a Euclidean neighborhood of z_0. Thus the weak*, direct limit, and Euclidean topologies agree on $\mathfrak{C}$, and the proof proceeds as in Theorem 3.3.

To construct examples of function algebras, it is natural to begin by specifying a sequence of domains and then to consider Banach algebras of functions which are uniform limits of certain types of functions defined on these domains, thus reversing the procedure of starting with an F-algebra and then realizing it as a suitable inverse limit of Banach algebras.

We call a subset K of $\mathfrak{C}$ a *natural domain,* if K is the closure of a Jordan domain. Let $K_1 \subset K_2 \subset \cdots$ be a sequence of natural domains. Let A_n be the uniform closure of the polynomials on K_n $(n = 1, 2, \cdots)$. Let A be the inverse limit of the Banach algebras A_n. A is a function algebra consisting of all functions on

$$K = \bigcup_{n=1}^{\infty} K_n$$

that are uniform limits on each K_n of polynomials.

THEOREM 3.6. *If* A *is a function algebra of the type constructed above, then* A *is topologically and algebraically isomorphic to* $\mathrm{Hol}(\mathfrak{C})$ *if and only if* A *is a Liouville algebra.*

Proof. The maximal ideal space of A_n is homeomorphic to K_n. Suppose all K_n share a common boundary point $z_0 \in \mathfrak{C}$. Since the boundary of K_n is a Jordan curve, z_0 can be approached by an arc ℓ_n whose other points are exterior to K_n. In the arcwise connected complement of each K_n, there exists an arc γ_n, such that $\ell_n \cup \gamma_n$ joins z_0 to ∞ and misses K_n. As in Theorem 3.1, a square root function s can be constructed. It belongs to A, because each A_n consists precisely of all continuous functions on K_n which are analytic on $\mathrm{int}(K_n)$. The conclusion that $K = \mathfrak{C}$ and $A = \mathrm{Hol}(\mathfrak{C})$ then follows by contradiction of the Liouville assumption and Theorem 3.3, as before.

The converse is obvious.

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