SOME CHARACTERIZATIONS OF THE LAGUERRE AND HERMITE POLYNOMIALS

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1. Recently Carlitz [1] proved the following representation for the Laguerre polynomials (for its definition see [2, p. 200]):

$$L_n^{(\alpha)}(x) = (n!)^{-1} \prod_{j=1}^{n} (xD - x + \alpha + j) \cdot 1 \tag{1.1}$$

where $D = d/dx$.

Let b_n $(n = 1, 2, 3, \cdots)$ be a sequence of numbers, and let

$$p_n(x) = \prod_{j=1}^{n} (xD + x + b_j) \cdot 1 \qquad (n = 1, 2, 3, \cdots), \tag{1.2}$$

$$p_0(x) = 1.$$

Obviously $p_n(x)$ is of degree exactly n. We propose here to prove the following theorem.

THEOREM 1. *If the set* $\{p_n(x)\}$, *defined by means of* (1.2), *is a set of orthogonal polynomials* [2, p. 147], *then* $p_n(x)$ *is the nth Laguerre polynomial.*

Proof. Since the set $\{p_n(x)\}$ is simple and orthogonal and since the coefficient of x^n in $p_n(x)$ is one, there is a three-term recurrence relation [2, p. 151]

$$\begin{gathered} p_{n+1}(x) = (x + B_n)p_n(x) + C_n p_{n-1}(x) \qquad (n \geq 0), \\ p_{-1}(x) = 0, \;\; p_0(x) = 1, \;\; C_n \neq 0. \end{gathered} \tag{1.3}$$

We see from (1.2) and (1.3) that

$$\begin{aligned} p_{n+1}(x) &= (xD + x + b_{n+1})p_n(x) = (x + n + b_{n+1})p_n(x) + (xD - n)p_n(x) \\ &= (x + B_n)p_n(x) + C_n p_{n-1}(x). \end{aligned} \tag{1.4}$$

Since $(xD - n)p_n(x)$ is a polynomial of degree $n - 1$, it follows that

$$B_n = n + b_{n+1} \tag{1.5}$$

and

$$(xD - n)p_n(x) = C_n p_{n-1}(x). \tag{1.6}$$

We also get from (1.2) and (1.3), respectively, the relations

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$$p_n(0) = b_1 b_2 \cdots b_n$$

$$b_1 b_2 \cdots b_{n+1} = (n + b_{n+1}) b_1 b_2 \cdots b_n + C_n b_1 b_2 \cdots b_{n-1}.$$

Hence,

$$C_n = -n b_n. \tag{1.7}$$

We now put $p_n(x) = \Sigma_{k=0}^{n} a_k^{(n)} x^k$, $a_n^{(n)} = 1$, and substitute in (1.6) and (1.7). We find that

$$a_k^{(n)} = \frac{n}{n-k} b_n a_k^{(n-1)},$$

whose solution is

$$a_k^{(n)} = \binom{n}{k} \frac{b_1 b_2 \cdots b_n}{b_1 b_2 \cdots b_k}.$$

With this value for $a_k^{(n)}$, it follows from (1.4), on comparing coefficients of x^k, that

$$b_{n+1} = b_k + n - k + 1 \qquad (n \geq k \geq 1).$$

Thus $b_{n+1} = b_1 + n = \alpha + n + 1$.
Consequently, we conclude that

$$p_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(1+\alpha)_n}{(1+\alpha)_k} x^k = n!\, L_n^{(\alpha)}(-x).$$

2. As before let b_n $(n = 1, 2, 3, \cdots)$ be a sequence of numbers. Let us recall that a simple set of polynomials $\{p_n(x)\}$ is said to be an Appell set if $p_n'(x) = p_{n-1}(x)$.

We prove in this section the following result.

THEOREM 2. *Let* $\{F_n(x)\}$ *be the simple set of polynomials defined by means of the relation*

$$x^n F_n(x) = \frac{(-1)^n}{n!} \prod_{j=1}^{n} (xD - x^2 + \alpha x + b_j) \cdot 1 \quad (n \geq 1), \tag{2.1}$$

$$F_0(x) = 1.$$

If $\{F_n(x)\}$ *is a set of Appell polynomials, then* $F_n(x)$ *can be reduced to the Hermite polynomial* [2, p. 187]. *In fact*

$$F_n(x) = (n!)^{-1} 2^{-n/2} H_n\left(\frac{x-\alpha}{\sqrt{2}}\right).$$

Proof. From the Appell property it follows that

$$x(n+1)F_{n+1}(x) = x^2 F_n(x) - (\alpha x + n + b_{n+1})F_n(x) - xF_{n-1}(x). \tag{2.2}$$

Now Sheffer [3] proved that a necessary and sufficient condition for a set $\{p_n(x)\}$ to be an Appell set is that it satisfy the recurrence relation

$$n p_n(x) = (c_0 + x)p_{n-1}(x) + c_1 p_{n-2}(x) + \cdots + c_{n-1} p_0(x).$$

Thus it follows from (2.2) and Sheffer's theorem that, in order for $\{F_n(x)\}$ to be an Appell set, it must be that $b_{n+1} + n = 0$. In this case we find the relations

$$(n + 1)F_{n+1}(x) = (x - \alpha)F_n(x) - F_{n-1}(x).$$

If we let $G_n(x) = n!\, F_n(x + \alpha)$, we conclude that

$$G_{n+1}(x) = xG_n(x) - n G_{n-1}(x), \quad G_0(x) = 1.$$

This completes the proof of Theorem 2.

Evidently we have proved the operational representation

$$\prod_{j=1}^{n} (xD - 2x^2 - j + 1) \cdot 1 = (-1)^n x^n H_n(x). \tag{2.3}$$

This formula is a consequence of Rodriques' formula for the Hermite polynomials. Indeed, by the shift rule, the left hand side of (2.3) is nothing but

$$e^{x^2} \prod_{j=1}^{n} (xD - j + 1) \cdot e^{-x^2} = e^{x^2} x^n D^n \cdot e^{-x^2}.$$

REFERENCES

1. L. Carlitz, *A note on the Laguerre polynomials*, Michigan Math. J. 7 (1960), 219-223.

2. E. D. Rainville, *Special Functions*, The Macmillan Co., New York, 1960.

3. I. M. Sheffer, *A differential equation for Appell polynomials*, Bull. Amer. Math. Soc. 41 (1935), 914-923.

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