ON ε-MAPS OF POLYHEDRA ONTO MANIFOLDS

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The compact metric n-dimensional absolute neighborhood retracts which may be mapped with arbitrarily small counter-images onto manifolds have been studied in [1] and [2]. The present note deals with polyhedra which possess the mentioned property.

By a closed manifold we mean a compact, connected, locally Euclidean Hausdorff space; no triangulability assumption is made. By $H^q(X; Z_2)$ we denote the q-th Čech cohomology group of the compact space X with the group of integers mod 2 as coefficients. If $\varepsilon > 0$, a continuous map f of a compact metric space X into another space Y is called an ε-map provided the set $f^{-1}(y)$ has diameter less than ε for each $y \in f(X)$.

THEOREM 1. *Let* X *be a compact* n*-dimensional polyhedron such that for every* $\varepsilon > 0$ *there exists an* ε*-map of* X *onto a closed* n*-dimensional manifold (depending on* ε*). Then* X *is a closed pseudo-manifold.*

Proof. By Theorem 3.1 of [2], there exists a $\sigma = \sigma(X) > 0$ such that every closed subset A of X for which diam $A < \sigma$ separates X if and only if $H^{n-1}(A; Z_2) \neq 0$.

Consider an arbitrary k-dimensional simplex T^k of X with $k < n$, and let T^j be a simplex of maximum dimension among the simplexes which have T^k as a face. The dimension of T^j being j, assume that $j < n$. If x is the barycenter of T^j and S is a $(j - 1)$-dimensional sphere centered at x, of radius less than $\sigma/2$, and contained in the interior of T^j, S separates X; and at the same time $H^{n-1}(S; Z_2) = 0$, which is absurd. Consequently, each k-dimensional simplex of X with $k < n$ is a face of at least one n-dimensional simplex of X.

Suppose now a $(n - 1)$-dimensional simplex T^{n-1} of X is a face of ν n-dimensional simplexes of X, $T_1^n, \cdots, T_\nu^n$, with $\nu \geq 3$. Let x be the barycenter of T^{n-1}; and let S be an $(n - 1)$-dimensional sphere centered at x, of radius less than $\sigma/2$, and contained in the union of T^{n-1} and of the interiors of T_1^n and T_2^n. As is easily seen, any two points of X - S may be joined by a path contained in X - S, and at the same time $H^{n-1}(S; Z_2) \neq 0$, which is absurd.

Assume T^{n-1} is a face of a single n-dimensional simplex T^n of X. Let E be a $(n - 1)$-dimensional hemisphere centered at x, of radius less than $\sigma/2$, and contained in the union of T^{n-1} and of the interior of T^n. Then X - E is not connected, and $H^{n-1}(E; Z_2) = 0$, which is again impossible.

Hence each $(n - 1)$-dimensional simplex of X is face of exactly two n-dimensional simplexes of X.

According to Corollary 4.2 of [2], X has the homotopy type of a closed n-dimensional manifold. Hence $H^n(X; Z_2) \approx Z_2$; and, by [4, p. 89], any two n-dimensional simplexes of X may be joined by a sequence of alternate n and $(n - 1)$-dimensional simplexes, each of which is incident to the following one. This completes the proof of the theorem.

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THEOREM 2. *Let* X *be a compact* 3-*dimensional polyhedron such that for every* $\varepsilon > 0$ *there exists an* ε-*map of* X *onto a closed* 3-*dimensional manifold* (*depending on* ε). *Then* X *is a closed manifold.*

Proof. Since X has the homotopy type of a closed 3-dimensional manifold [2], its Euler characteristic is zero. On the other hand, according to Theorem 1, X is a closed pseudo-manifold and may therefore be obtained from a 3-cell by identifications on the boundary. Hence we may apply Theorem I of [4, p. 208], to conclude that X is a closed manifold.

Remark 1. Theorem 2 was conjectured independently by I. Bucur and V. Poenaru.

Remark 2. T. Ganea has shown [2] that any 2-dimensional compact absolute neighborhood retract X such that there exists for each $\varepsilon > 0$ an ε-map of X onto a closed surface is necessarily a closed surface. On the other hand, he has produced [3] an example of a compact 3-dimensional absolute neighborhood retract which may be mapped with arbitrarily small counter-images onto the 3-dimensional sphere but which fails to be a manifold. Hence the triangulability assumption made on X in Theorem 2 cannot be dropped.

REFERENCES

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