THE REPRESENTATION OF INTEGERS BY THREE POSITIVE SQUARES

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It is well known that a positive integer n not of the form $4^a(8m + 7)$ can be expressed as a sum of three integer squares. The problem arises of specifying those integers which have a representation in which all three squares are positive. Some authors [1] have considered this in a recent paper with the same title. It is there shown that it suffices to consider the case when n is square-free, and so only when $n \equiv 1, 2 \pmod 4$. All such integers except a finite number are shown to have positive square representations. The result is trivial if n has a prime factor $\equiv 3 \pmod 4$, and so we may exclude this case. It is very difficult to determine the exceptional set explicitly, since the proof depends upon Siegel's estimate for the class number of quadratic fields, and hence even a slight result may not be out of place. We have then the following

THEOREM. *If* n *is square-free, and has no prime factors* $\equiv 3 \pmod 4$, *and if* $n \equiv 1, 2 \pmod 4$, *the exceptional values of* n *are those for which the only non-negative integer solutions of*

$$yz + zx + xy = n \tag{1}$$

are, when $n \equiv 2 \pmod 4$, *given by* $xyz = 0$, *and when* $n \equiv 1 \pmod 4$, *given by* $xyz = 0$ *together with those typified by* $x = d$, $y = d$, $z = (n - d^2)/2d$, *where* d *is any divisor of* n *with* $d^2 < n$.

Non-exceptional values of n are easily found by making the equation (1) have an additional solution. Thus if also $n \equiv 2 \pmod 3$, then $x = 1$, $y = 2$, $z = (n - 2)/3$ is a solution when $(n - 2)/3 > 2$, that is, when $n > 8$. Hence the integers $n \equiv 5 \pmod{12}$, except $n = 5$, are not exceptional.

The number N_3 of solutions of $x^2 + y^2 + z^2 = n$ for general $n > 0$ is $12(2F(n) - G(n))$, where $G(n)$ is the total number of classes of binary quadratic forms of the type $aX^2 + 2bXY + cY^2$ with $n = ac - b^2$, and $F(n)$ is the number of classes in which a and c are not both even. Here the classes $(a, 0, a)$ and $(2a, a, 2a)$ are reckoned as $1/2$ and $1/3$ respectively. When $n \equiv 1, 2 \pmod 4$, $F(n) = G(n)$, and so $N_3 = 12\,G(n)$. These results are essentially due to Gauss.

The number N_2 of solutions of $x^2 + y^2 = n$ when n is square-free, has no prime factors $\equiv 3 \pmod 4$ and $n \equiv 1, 2 \pmod 4$, is $N_2 = 4k$ where k is the number of odd factors of n. The total N_3 includes $3N_2$ solutions for which $xyz = 0$, and so we require that $N_3 > 3N_2$, that is $G(N) > k$.

Many years ago, I proved [2] that for all positive n, the number N of solutions in non-negative integers of (1) is given by $N = 3G(n)$, where a solution with $xyz = 0$ is reckoned as $1/2$.

Suppose first that $n \equiv 2 \pmod 4$. Then there are $6k$ solutions of (1), typified by $z = 0$, $x = d$, $y = n/d$ and by $z = 0$, $x = 2d$, $y = n/2d$, and these contribute $3k$ to N. Hence there are $N_0 = 3G(n) - 3k$ solutions in which $xyz \neq 0$. Then $N_0 > 0$ if and only if $G(n) > k$. This proves one part of the theorem.

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Suppose next that $n \equiv 1 \pmod 4$. There are now $3k$ solutions of (1) typified by $z = 0$, $x = d$, $y = n/d$, contributing $3k/2$ to N. Hence there are $N_0 = 3G(n) - 3k/2$ solutions in which $xyz \neq 0$, and so we require $N_0 > 3k/2$. But there are $3k/2$ solutions of (1) typified by $x = d$, $y = d$, $z = (n - d^2)/2d$ with $d < \sqrt{n}$, and so d takes $k/2$ values. This completes the proof of the theorem.

It is not easy to deal with solutions of (1). I recall that when $n \equiv 3 \pmod 8$ and is square-free, the number of classes of ideals of the quadratic field $R(\sqrt{-m})$ is one if and only if all the solutions are typified by $(0, 1, n)$, $(1, 1, (n - 1)/2)$ and $(1, 3, (n - 3)/4)$.

REFERENCES

1. E. Grosswald, A. Calloway and J. Calloway, *The representation of integers by three positive squares*, Proc. Amer. Math. Soc. 10 (1959), 451-455.

2. L. J. Mordell, *On the number of solutions in positive integers of the equation* $yz + zx + xy = n$, Amer. J. Math. 45 (1923), 1-4.

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Note added October 30, 1960. Other aspects of the question are considered by A. Schinzel in his paper *Sur les sommes des trois carrés*, Bull. Acad. Polon. Sci. Ser. Sci. Math., Astr. Phys.7 (1959), 307-310.