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Comparison of the Pluricomplex and the Classical Green Functions on Convex Domains of Finite Type

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1. Introduction

Let D be a bounded domain with Lipschitz boundary in $\mathbf{R}^n$, and let y be a fixed point in D. Then there is a solution $h_y(x)$ to the Dirichlet problem

$$\begin{cases} \Delta u(x) = 0 & \text{in } D, \\ u(x) = -\eta(x-y) & \text{on } \partial D, \end{cases}$$

where

$$\eta(x) = \begin{cases} \log|x| & \text{if } N = 2, \\ -|x|^{2-N} & \text{if } N \geq 3. \end{cases}$$

The function $G_D(x, y) = \eta(x-y) + h_y(x)$ is called the *classical* (*negative*) *Green function* for the Laplacian, with pole at y. It is harmonic in $D \setminus \{y\}$ and tends to zero on the boundary; furthermore, it is symmetric.

Now let D be a bounded domain in $\mathbf{C}^n$. By $\mathrm{PSH}(D)$ we denote the class of plurisubharmonic (psh) functions on D. The *pluricomplex Green function* for D with pole at w is defined by

$$g_D(z, w) = \sup\{\varphi(z) : \varphi \in \mathrm{PSH}(D),\ \varphi \leq 0,\ \varphi(z) \leq \log|z-w| + O(1)\}.$$

This definition was first given by Klimek [5]. It coincides with the classical Green function in the complex plane. The function $g_D(\cdot, w)$ is a negative plurisubharmonic function in D and has a logarithmic pole at w. It is decreasing with respect to holomorphic maps, which implies that it is biholomorphically invariant. If D is hyperconvex, then $g_D(z, w) \to 0$ as $z \to \partial D$ and g_D is continuous on $\bar{D} \times D$ (cf. [3]). The pluricomplex Green function is symmetric for convex domains [7], although it is not symmetric in general [1]. The pluricomplex Green function plays a similar role in the pluripotential theory as the classical Green function in the classical potential theory, so it is interesting to compare the two. In the case when D is strongly pseudoconvex, Carlehed [2] proved that the following holds for all $z, w \in D$:

$$\frac{g_D(z, w)}{G_D(z, w)} \leq C(D)|z-w|^{2n-4}.$$

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In particular, the quotient is bounded. The purpose of this article is to extend this result to certain weakly pseudoconvex domains. A bounded domain D is called *locally convexifiable* if every $p \in \partial D$ has a neighborhood V_p with the properties that $D \cap V_p$ is biholomorphic to a convex domain. A bounded domain is called *locally convexifiable of finite type m* if it is locally convexifiable and of finite type m. Our main result is the following theorem.

THEOREM 1. *Let D be a bounded, locally convexifiable domain of finite type m in $\mathbf{C}^n$. Then*

$$\frac{g_D(z,w)}{G_D(z,w)} \le C(D)|z-w|^{2(n-m)}. \tag{1}$$

In particular, the quotient is bounded if $n \ge m$.

Since any strongly pseudoconvex domain is a locally convexifiable domain of finite type 2, Theorem 1 generalizes the result of Carlehed.

However, this theorem does not hold in general when $n < m$. We shall show that the quotient g_D/G_D is unbounded on the domain

$$D = \{z \in \mathbf{C}^n : |z_1|^2 + |z_2|^m + \cdots + |z_n|^m < 1\},$$

where $m > n$ is even.

2. An Estimate for the Pluricomplex Green Function

In this section we shall prove the following result, which plays an essential role in proving the main theorem.

PROPOSITION 2. *Let D be a bounded, locally convexifiable domain in $\mathbf{C}^n$. Suppose that there exist positive numbers $\alpha > \beta$ and $\alpha \ge 2$ as well as an $r > 0$ such that, for every $p \in \partial D$, there is a holomorphic function h_p on $D \cap B(p,r)$ satisfying*

$$c_1|z-p|^\alpha \le 1 - |h_p(z)| \le c_2|z-p|^\beta \tag{2}$$

for suitable constants $c_2 > c_1 > 0$ (independent of p), where $B(p,r)$ denotes the ball in $\mathbf{C}^n$ that is centered at p with radius r. Then there exists a constant $C > 0$ depending only on $\alpha, \beta, r, c_1, c_2$ such that

$$-g_D(z,w) \le C\frac{\delta_D^\beta(z)\delta_D^\beta(w)}{|z-w|^{2\alpha}}, \tag{3}$$

where $\delta_D(z)$ denotes the Euclidean boundary distance of z.

For the sake of simplicity, we make the following assumption on the diameter of D: $\operatorname{diam}(D) < 1$. In this section, we shall denote by C all the constants depending only on $\alpha, \beta, r, c_1, c_2$. We first prove several lemmas.

LEMMA 3. *For all $z, w \in D$ with $\delta_D^\beta(w) \le a|z-w|^\alpha$, where $a = c_1/(2^{\alpha+1}c_2)$, one has*

$$-g_D(z,w) \le C\frac{\delta_D^\beta(w)}{|z-w|^\alpha}. \tag{4}$$

Proof. Let us fix w for a moment. We take a boundary point $\tilde{w}$ so that $\delta_D(w) = |w - \tilde{w}|$. If $\delta_D(w) \geq r/2$, then $|z - w| \geq \delta_D^{\beta/\alpha}(w)/a^{1/\alpha} \geq C$. By the trivial estimate

$$-g_D(z, w) \leq \log \frac{\operatorname{diam}(D)}{|z - w|},$$

we immediately get (4). Hence we may assume $\delta_D(w) < r/2$. We will first show that

$$-g_{D \cap B(\tilde{w}, r)}(z, w) \leq C \frac{\delta_D^\beta(w)}{|z - w|^\alpha}. \tag{5}$$

Since $|h_{\tilde{w}}| < 1$ on $D \cap B(\tilde{w}, r)$, it follows that

$$\begin{aligned}
-g_{D \cap B(\tilde{w}, r)}(z, w) &\leq -g_\Delta(h_{\tilde{w}}(z), h_{\tilde{w}}(w)) \\
&= -\frac{1}{2} \log \frac{|h_{\tilde{w}}(z) - h_{\tilde{w}}(w)|^2}{|1 - \overline{h_{\tilde{w}}(w)} h_{\tilde{w}}(z)|^2} \\
&= \frac{1}{2} \log\left(1 + \frac{(1 - |h_{\tilde{w}}(z)|^2)(1 - |h_{\tilde{w}}(w)|^2)}{|h_{\tilde{w}}(z) - h_{\tilde{w}}(w)|^2}\right) \\
&\leq \frac{1}{2} \frac{(1 - |h_{\tilde{w}}(z)|^2)(1 - |h_{\tilde{w}}(w)|^2)}{|h_{\tilde{w}}(z) - h_{\tilde{w}}(w)|^2} \\
&\leq 2 \frac{(1 - |h_{\tilde{w}}(z)|)(1 - |h_{\tilde{w}}(w)|)}{|h_{\tilde{w}}(z) - h_{\tilde{w}}(w)|^2},
\end{aligned}$$

where Δ is the unit disc in $\mathbf{C}$. Notice that

$$1 - |h_{\tilde{w}}(w)| \leq c_2 \delta_D^\beta(w)$$

and

$$\begin{aligned}
|h_{\tilde{w}}(z) - h_{\tilde{w}}(w)| &\geq 1 - |h_{\tilde{w}}(z)| - (1 - |h_{\tilde{w}}(w)|) \\
&\geq c_1 |z - \tilde{w}|^\alpha - c_2 |w - \tilde{w}|^\beta \\
&\geq c_1 (|z - w| - \delta_D(w))^\alpha - c_2 \delta_D^\beta(w) \\
&\geq (c_1 (1 - a^{1/\beta})^\alpha - c_2 a) |z - w|^\alpha \\
&\geq (c_1 2^{-\alpha} - c_2 a) |z - w|^\alpha \\
&\geq c_1 2^{-\alpha - 1} |z - w|^\alpha.
\end{aligned}$$

If $|1 - |h_{\tilde{w}}(z)| \leq 2(1 - |h_{\tilde{w}}(w)|)$, then

$$\begin{aligned}
-g_{D \cap B(\tilde{w}, r)}(z, w) &\leq \frac{4(1 - |h_{\tilde{w}}(w)|)^2}{|h_{\tilde{w}}(z) - h_{\tilde{w}}(w)|^2} \\
&\leq C \frac{\delta_D^{2\beta}(w)}{|z - w|^{2\alpha}} \\
&\leq C \frac{\delta_D^\beta(w)}{|z - w|^\alpha},
\end{aligned}$$

because $\delta_D^\beta(w) \leq a |z - w|^\alpha$. Otherwise, one has

$$
\begin{aligned}
|h_{\tilde w}(z) - h_{\tilde w}(w)| &\geq 1 - |h_{\tilde w}(z)| - (1 - |h_{\tilde w}(w)|) \\
&\geq \tfrac{1}{2}(1 - |h_{\tilde w}(z)|).
\end{aligned}
$$

It follows that

$$
-g_{D\cap B(\tilde w, r)}(z, w) \leq \frac{4(1 - |h_{\tilde w}(w)|)}{|h_{\tilde w}(z) - h_{\tilde w}(w)|} \leq C\frac{\delta_D^\beta(w)}{|z-w|^\alpha}.
$$

The rest of the proof is standard. We fix z, w and set

$$
\lambda = \begin{cases} |z-w| & \text{if } |z-w| < r/4, \\ r/4 & \text{otherwise.} \end{cases}
$$

Clearly, one has $B(w, \lambda) \subset B(\tilde w, r)$. Set

$$
b = \inf_{\zeta \in D\cap \partial B(w,\lambda)} g_{D\cap B(\tilde w, r)}(\zeta, w),
$$

$$
v(\zeta) = b\frac{\log(2|\zeta - w|/r)}{\log(2\lambda/r)}.
$$

Then v is psh on D and satisfies

$$
v(\zeta) = \begin{cases} b \leq g_{D\cap B(\tilde w, r)}(\zeta, w) & \text{if } |\zeta - w| = \lambda, \\ v(\zeta) = 0 > g_{D\cap B(\tilde w, r)}(\zeta, w) & \text{if } |\zeta - w| = r/2. \end{cases}
$$

Hence the function

$$
u(\zeta) = \begin{cases} g_{D\cap B(\tilde w, r)}(\zeta, w), & \zeta \in D \cap B(w, \lambda), \\ \max\{v(\zeta), g_{D\cap B(\tilde w, r)}(\zeta, w)\}, & \zeta \in D\cap B(w, r/2)\backslash B(w, \lambda), \\ v(\zeta), & \zeta \in D\backslash B(w, r/2) \end{cases}
$$

is also psh in D and has a logarithmic pole w. Observe that

$$
u(z) \geq -C\frac{\delta_D^\beta(w)}{|z-w|^\alpha}
$$

because of (5). One also has

$$
\sup_{\zeta\in D} u(\zeta) \leq b\frac{\log(2\,\mathrm{diam}(D)/r)}{\log(2\lambda/r)} \leq C\frac{\delta_D^\beta(w)}{|z-w|^\alpha}.
$$

It follows that

$$
\begin{aligned}
g_D(z, w) &\geq u(z) - \sup_{\zeta\in D} u(\zeta) \\
&\geq -C\frac{\delta_D^\beta(w)}{|z-w|^\alpha}.
\end{aligned}
$$

The proof is complete. □

LEMMA 4. *For all* $z, w \in D$,

$$
-g_D(z, w) \leq C\frac{\delta_D^{2\beta/\alpha}(z)}{|z-w|^2}.
$$

Proof. We fix z, w and set $\gamma = |z - w|$, $w' = w + (w - z)/\gamma$, and $R = 1 + 2\gamma$. Then $|w - w'| = 1$ and $z \in B(w', R)$, since $|z - w'| = 1 + \gamma < R$. Without loss of generality, we may assume that $w' = 0$. We make the following claim.

CLAIM. *There is a constant $C' > 0$, depending only on n, such that*

$$-g_{B(0,R)}(\zeta, w) \leq C', \tag{6}$$

$$|d_\zeta g_{B(0,R)}(\zeta, w)| \leq C'/\gamma \tag{7}$$

for all $1 + \gamma/2 \leq |\zeta| \leq 1 + \gamma$. Here d_ζ denotes the derivative w.r.t. ζ.

Remark. The explicit form of $g_{B(0,R)}(\zeta, w)$ shows that it is smooth off the diagonal.

Let us first observe that Lemma 4 follows from the claim. Let $\chi : \mathbf{R} \to [0, 1]$ be a C^∞ function satisfying $\chi \equiv 1$ on $(-\infty, 1/2]$ and $\chi \equiv 0$ on $[1, \infty)$. We set

$$\varrho(\zeta) = \begin{cases} \chi((|\zeta| - 1)/\gamma) g_{B(0,R)}(\zeta, w) & \text{if } |\zeta| \leq 1 + \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

By a straightforward calculation, we obtain

$$\begin{aligned}\partial\bar{\partial}\varrho(\zeta) = {} & g_{B(0,R)}(\zeta, w)\partial\bar{\partial}\chi((|\zeta| - 1)/\gamma) \\ & + \partial g_{B(0,R)}(\zeta, w)\bar{\partial}\chi((|\zeta| - 1)/\gamma) + \partial\chi((|\zeta| - 1)/\gamma)\bar{\partial} g_{B(0,R)}(\zeta, w) \\ & + \chi((|\zeta| - 1)/\gamma)\partial\bar{\partial} g_{B(0,R)}(\zeta, w).\end{aligned}$$

Neglecting the semipositive term $\chi((|\zeta| - 1)/\gamma)\partial\bar{\partial} g_{B(0,R)}(\zeta, w)$, we thus obtain the inequality

$$\partial\bar{\partial}\varrho(\zeta) \geq -\frac{C''}{\gamma^2}\partial\bar{\partial}|\zeta|^2 \tag{8}$$

from (6) and (7) for a suitable constant $C'' > 0$ depending only on n.

Now let $\tilde{z}$ be a boundary point, so that $\delta_D(z) = |z - \tilde{z}|$. We set

$$\varphi_{\tilde{z}} = \max\{|h_{\tilde{z}}| - 1, -\eta\}$$

for sufficiently small positive constant η. Then $\varphi_{\tilde{z}}$ is a well-defined psh function on D with the estimate

$$c_1|\zeta - \tilde{z}|^\alpha \leq -\varphi_{\tilde{z}}(\zeta) \leq c_2|\zeta - \tilde{z}|^\beta,$$

where the constants are still denoted by c_1, c_2 for the sake of simplicity. Let us denote

$$\psi_{\tilde{z}}(\zeta) = -2c_1^{-2/\alpha}(-\varphi_{\tilde{z}}(\zeta))^{2/\alpha} + |\zeta - \tilde{z}|^2.$$

One has $\psi_{\tilde{z}} < 0$ on D, $\psi_{\tilde{z}}(\tilde{z}) = 0$, and $\partial\bar{\partial}\psi_{\tilde{z}} \geq \partial\bar{\partial}|\zeta|^2$ in the sense of distributions because $\alpha \geq 2$. Therefore, by (8), the function $(C''/\gamma^2)\psi_{\tilde{z}} + \varrho$ is negative and psh in D with a logarithmic pole w. Hence

$$\begin{aligned}-g_D(z, w) &\leq -\frac{C''}{\gamma^2}\psi_{\tilde{z}}(z) - \varrho(z) \\ &\leq C\frac{\delta_D^{2\beta/\alpha}(z)}{|z - w|^2}.\end{aligned}$$

□

LEMMA 5. *Let a be as in Lemma 3. Then* (4) *also holds for all $z, w \in D$ with $\delta_D^{\beta}(w) \geq a|z-w|^{\alpha}$.*

Proof. Using the fact that D is locally convexifiable as well as a standard compactness argument, we argue as follows. There exists $r' > 0$ (independent on $p \in \partial D$) such that every $p \in \partial D$ has a neighborhood V_p with the properties that $D \cap V_p$ is biholomorphic to a convex domain and $D \cap B(p, r') \subset D \cap V_p$. Without loss of generality, we may assume that $r = r'$. It follows that $g_{D\cap V_p}$ is symmetric. By Lemma 4, for all $z, w \in D \cap B(p, r)$ we have that

$$\begin{aligned} -g_{D\cap B(p,r)}(z, w) &\leq -g_{D\cap V_p}(z, w) = -g_{D\cap V_p}(w, z) \\ &\leq -g_D(w, z) \leq C\frac{\delta_D^{2\beta/\alpha}(w)}{|z-w|^2}. \end{aligned}$$

Repeating the arguments as in the proof of Lemma 3, one has

$$-g_D(z, w) \leq C\frac{\delta_D^{2\beta/\alpha}(w)}{|z-w|^2},$$

from which (4) immediately follows because $\delta_D^{\beta}(w) \geq a|z-w|^{\alpha}$ and $\alpha \geq 2$. □

Proof of Proposition 2. Combining Lemma 3 with Lemma 5, we see that

$$-g_D(z, w) \leq C\frac{\delta_D^{\beta}(w)}{|z-w|^{\alpha}}$$

holds for all $z, w \in D$. We will follow the argument of Carlehed [2]. When $\delta_D(z) \geq \frac{1}{4}|z-w|$, the proof follows immediately because $\delta_D^{\beta}(z)/|z-w|^{\alpha} \geq C$. It suffices to prove the proposition for the case $\delta_D(z) < \frac{1}{4}|z-w|$. Let $\gamma, \tilde{z}$ be as before. Observe that

(1) $z \in D \cap B(\tilde{z}, \gamma/2)$, since $\delta_D(z) < \gamma/4$; and
(2) $w \notin D \cap B(\tilde{z}, \gamma/2)$, since

$$\begin{aligned} |w-\tilde{z}| &\geq |w-z| - |z-\tilde{z}| \\ &= |z-w| - \delta_D(z) \\ &\geq \tfrac{3}{4}|z-w| > \gamma/2. \end{aligned}$$

If $\zeta \in D \cap \partial B(\tilde{z}, \gamma/2)$, then

$$\begin{aligned} |\zeta - w| &\geq |z-w| - |z-\zeta| \\ &\geq |z-w| - (|z-\tilde{z}| + |\zeta-\tilde{z}|) \\ &\geq \gamma/4. \end{aligned}$$

Let $\varphi_{\tilde{z}}$ be taken as before. Clearly, one has

$$-\frac{\varphi_{\tilde{z}}(\zeta)}{\gamma^{\alpha}} \geq \frac{c_1|\zeta-\tilde{z}|^{\alpha}}{\gamma^{\alpha}} \geq \frac{c_1}{2^{\alpha}}$$

for all $\zeta \in D \cap \partial B(\tilde{z}, \gamma/2)$. Therefore, the inequality

$$g_D(\zeta, w) \geq C \frac{\delta_D^\beta(w)}{|z-w|^\alpha} \frac{\varphi_{\tilde{z}}(\zeta)}{\gamma^\alpha}$$

holds there. The same inequality holds trivially for $\zeta \in \partial D \cap B(\tilde{z}, \gamma/2)$, since $g_D(\zeta, w) = 0$ there; hence it holds for all $\zeta \in \partial(D \cap B(\tilde{z}, \gamma/2))$. Since $g_D(\zeta, w)$ is a maximal plurisubharmonic function of ζ in $D \cap B(\tilde{z}, \gamma/2)$ and since $\tilde{\varphi}_{\tilde{z}}$ is also psh there, the inequality holds true in $D \cap B(\tilde{z}, \gamma/2)$. In particular,

$$g_D(z, w) \geq -C \frac{\delta_D^\beta(z)\delta_D^\beta(w)}{|z-w|^{2\alpha}}.$$

The proof is complete. □

Proof of the Claim. Because the pluricomplex Green function is biholomorphically invariant, we may assume that $w = (t, 0, \ldots, 0)$ with $t > 0$. Furthermore, we can take $R = 1$ under the dilation $\zeta \to \zeta/R$. Then $t = 1/R \geq 1/3$ and $\frac{2}{3}\gamma \leq 1 - t \leq 2\gamma$ since $R \leq 3$. By [2] one has

$$\begin{aligned} -g_{B(0,1)}(\zeta, w) &= \frac{1}{2} \log \frac{|1 - t\zeta^1|^2}{|t - \zeta^1|^2 + q(1-t^2)} \\ &= \frac{1}{2} \log\left(1 + \frac{(1-|\zeta|^2)(1-t^2)}{|t-\zeta^1|^2 + q(1-t^2)}\right) \\ &\leq \frac{1}{2} \frac{(1-|\zeta|^2)(1-t^2)}{|t-\zeta^1|^2 + q(1-t^2)}, \end{aligned}$$

where $\zeta = (\zeta^1, \zeta^2, \ldots, \zeta^{2n}) \in \mathbf{R}^{2n}$ and $q = q(\zeta) = |\zeta^2|^2 + \cdots + |\zeta^{2n}|^2$. If $|t - \zeta^1| > \gamma/4$, then

$$|t - \zeta^1|^2 + q(1-t^2) > \gamma^2/16.$$

Otherwise,

$$\begin{aligned} q &= |\zeta|^2 - |\zeta^1|^2 \\ &\geq (t + \gamma/2)^2 - (t + \gamma/4)^2 \\ &\geq (\gamma/2)t \\ &\geq \gamma/6 \end{aligned}$$

for all $t + \gamma/2 \leq |\zeta| \leq t + \gamma$. It follows that

$$|t - \zeta^1|^2 + q(1-t^2) \geq q(1-t^2) \geq \tfrac{2}{3}q\gamma \geq \gamma^2/9.$$

Hence (6) is valid because $1 - |\zeta| \leq 1 - t - \gamma/2 \leq 2\gamma$. By the Cauchy–Schwarz inequality, one has

$$\begin{aligned} |d_\zeta g_{B(0,1)}(\zeta, w)| &\leq \frac{t|d\zeta^1|}{1 - t\zeta^1} + \frac{|t-\zeta^1||d\zeta^1| + \sum_{k=2}^{2n} |\zeta^k||d\zeta^k|(1-t^2)}{|t-\zeta^1|^2 + q(1-t^2)} \\ &\leq \frac{1}{1-t} + \frac{\sqrt{1 + (2n-1)(1-t^2)}}{\sqrt{|t-\zeta^1|^2 + q(1-t^2)}} \\ &\leq \frac{C_0}{\gamma}, \end{aligned}$$

where $C_0 > 0$ is a constant depending only on n. The proof is complete. □

3. Proof of Theorem 1

We recall at first some basic facts for convex domains of finite type. Assume $D = \{\rho(z) < 0\}$ to be a bounded convex domain of finite type m with a defining function ρ. Let us make precise the finite-type hypothesis: For each $p \in \partial D$ and each complex line L in the complex tangent space at p, there is a unit direction v in L such that

$$\sum_{i=2}^{m} |D_v^i \rho(p)| \neq 0.$$

Here $D_v^i \rho(p)$ denotes the ith directional derivative of ρ at p. On the other hand, if L is transverse then of course $D_v(p) \neq 0$ for some v. By continuity and compactness we can write the finite-type assumption as follows: If

$$a_{ij}(p, v) = \frac{\partial^{i+j}}{\partial \lambda^i \partial \bar{\lambda}^j} \rho(p + \lambda v)|_{\lambda=0}, \quad p \in \partial D, \ |v| = 1,$$

then

$$\sum_{1 \leq i+j \leq m} |a_{ij}(p, v)| \geq c(D) > 0.$$

The following deep result was proved by Diederich and Fornæss.

THEOREM [4]. *Let n_p be the normal unit vector to ∂D at the boundary point p, and let v be a complex tangential unit vector. Then there exists a holomorphic supporting function $S_p(z)$ at p with the estimate*

$$\operatorname{Re} S_p(z) \leq \frac{\operatorname{Re} \mu}{2} - \frac{K}{2}(\operatorname{Im} \mu)^2 - \hat{c} \sum_{k=2}^{m} \sum_{i+j=k} |a_{ij}(p, v)||\lambda|^k$$

if we write $z = p + \mu n_p + \lambda v$ with $\lambda, \mu \in \mathbf{C}$. Here $K, \hat{c} > 0$ are constants independent of p, v.

For each $p \in \partial D$, we define $h_p = e^{S_p}$. Then

$$c_1 |z - p|^m \leq 1 - |h_p(z)| \leq c_2 |z - p|$$

for suitable constants $c_1, c_2 > 0$.

Now we begin to prove our theorem. By hypothesis, the function h_p just defined exists locally. By Proposition 2, one has

$$-g_D(z, w) \leq C(D) \frac{\delta_D(z) \delta_D(w)}{|z - w|^{2m}}. \tag{9}$$

Let us recall some estimates of the classical Green function for bounded domains of $C^{1,1}$ boundary in $\mathbf{C}^n$ with $n \geq 2$ (cf. [2; 8]):

$$-G_D(z, w) \geq \frac{C(D)}{|z - w|^{2n-2}} \quad \text{if } |z - w| < \max\left\{\frac{\delta_D(z)}{2}, \frac{\delta_D(w)}{2}\right\}, \tag{10}$$

$$-G_D(z, w) \geq C(D) \frac{\delta_D(z) \delta_D(w)}{|z - w|^{2n}} \quad \text{if } |z - w| \geq \max\left\{\frac{\delta_D(z)}{2}, \frac{\delta_D(w)}{2}\right\}. \tag{11}$$

We proceed with the proof by examining two cases as follows.

(1) When $|z-w| < \max\{\delta_D(z)/2, \delta_D(w)/2\}$, we use inequality (10) together with the trivial estimate

$$-g_D(z,w) \leq \log \frac{\operatorname{diam}(D)}{|z-w|}.$$

(2) When $|z-w| \geq \max\{\delta_D(z)/2, \delta_D(w)/2\}$, we use (9) and (11).

Thus, the proof of the main theorem is complete. □

4. An Example

Let us consider the domain

$$D = \{z \in \mathbf{C}^n : |z_1|^2 + |z_2|^m + \cdots + |z_n|^m < 1\},$$

where $m > n$ is even. Clearly, D is a convex domain of finite type m. Let $0 < t < 1$ be any positive number and set $H_t = \{z \in \mathbf{C}^n : z_1 = t,\, z_3 = z_4 = \cdots = z_n = 0\}$. Then $D \cap H_t$ is a disc with radius $(1-t^2)^{1/m}$. Let $w = w(t) = (t, 0, \dots, 0)$ and

$$z = z(t) = \bigl(t, \tfrac{1}{2}(1-t^2)^{1/m}, 0, \dots, 0\bigr).$$

Then $\delta_D(z) \approx \delta_D(w) \approx 1-t$. By definition of the pluricomplex Green function, one has

$$\begin{aligned} g_D(z,w) &\leq g_{D\cap H_t}(z,w) \\ &= g_\Delta(1/2, 0) \\ &= -\log 2. \end{aligned}$$

We use a similar estimate for the classical Green function (cf. [2; 6]):

$$-G_D(z,w) \leq C(D)\frac{\delta_D(z)\delta_D(w)}{|z-w|^{2n}}.$$

Hence

$$\begin{aligned} \frac{g_D(z,w)}{G_D(z,w)} &\geq C(D)\frac{|z-w|^{2n}}{\delta_D(z)\delta_D(w)} \\ &\geq C(D)(1-t)^{2(n/m-1)} \\ &\to \infty \end{aligned}$$

as $t \to 1$, because $n < m$.

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