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A QUATERNARY RELATION AS THE PRIMITIVE NOTION IN SEVERAL GEOMETRIES*

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INTRODUCTION

Since the beginning of the axiomatization of geometry there has been a diversity of views not only about the axioms to be used and the relations which should be regarded as primitive notions but also about which objects should be considered as the variables.

In the Euclidean case Pasch [14] uses points, linear-segments (*gerade Strecke*), and planar-segments (*ebene Fläche*) as undefined objects, then defines lines and planes. The linear-segments and planar-segments are certain sets of points. Peano [15] and Schur [19] follow Pasch but use only points and linear-segments as undefined. Hilbert [7] uses a different approach and accepts points, lines, segments of lines, and planes all as primitive, but for simplicity connects certain sets of points with the other objects, for example, a line is identified with the set of all points lying on it. Veblen [24] accepts both points and lines as primitive. Pieri [16] uses only points and a primitive ternary relation on points.

From the viewpoint of foundations the less complicated point approach of Pieri is much more satisfactory since the cumbersome and somewhat artificial identification of primitives is avoided. Points and relations on points alone are used by Huntington [8] in his study of betweenness and Moore [12] for his development of the incidence and order axioms of Hilbert. Recent works on axiom systems for Euclidean geometry, as Scott [21], Tarski [23], and Beth and Tarski [4], use the point approach.

The object of this work is to establish axiom systems for affine, projective, and hyperbolic geometries using only points as variables. In addition in order to make it possible to study relations between the three systems in each case the primitive notion is to be a single quaternary relation on the points, $\mathbf{t}(ABCD)$. The quaternary will vary from system to system, but in all cases will at least have as one realization that A, B, C, and D are the vertices of a non-degenerate quadrilateral.

In order to show that the proposed systems are those desired their equivalence[1] will be proved with known systems, Artin's [2] for affine and projective geometries, and DeBaggis [5] for hyperbolic geometry. These latter systems use points and lines as variables making the usual identification of a line with the set of all points lying on it. Because of this the proofs of equivalence must be formalized within a first-order predicate calculus that has been augmented by a fragment of set theory.

The notation used will be that of Peano-Russell with a slight simplification in the rules for dots. Anyone familiar with the Peano-Russell symbolism will have no difficulty reading the notation.

The logical constants used are the sentential connectives—negation $\sim$,

1. A proof of the equivalence of two axiom systems which contain defined terms having different definitions in each system means that assuming the definitions and axioms of one system all the definitions and axioms of the other must be proved as theorems and conversely.

implication $\supset$, alternation $\vee$, and conjunction . ; the quantifiers—universal [ ], and existential [$\exists$]; and two binary predicates—identity =, and diversity $\neq$. As non-logical constants there will be several binary, ternary, and quaternary relations which will have first or second-order variables as terms. Except for $\mathbf{t}(ABCD)$ in the proposed systems and $\mathbf{i}(Af)$ in the systems used for comparison all of the non-logical constants will be defined terms. The first-order variables A, B, C, ... will range over the fix set α, the points. The second-order variables f, g, h, ... will range over certain sets of points. The collection of second-order variables, lines, will be called β. The additional constant ε of set membership will be assumed as well as the following usual definition of equality of sets:

$$DO\ \ [fg] .\because. f = g . \equiv : [A] : A\, \varepsilon f\ . \equiv .\ A \varepsilon g$$

CHAPTER I: AFFINE GEOMETRY

In Chapter I two systems for affine geometry, (**A***) a point geometry and (**A**) a point-line geometry, are formalized and shown to be equivalent.

System (**A***) assumes one set of objects. The set is denoted by α and its members by A, B, C, ... One quaternary relation $\mathbf{t}(ABCD)$ is used as the primitive notion. In (**A***) $\mathbf{t}(ABCD)$ means intuitively that the points A, B, C, and D are the vertices of a trapezoid having the segments AB and CD as bases. Two other relations are defined, a binary relation on points $\mathbf{r}(AB)$ which means intuitively that there exist two points C and D such that the four points A, B, C, and D in some order satisfy the relation $\mathbf{t}$, and a ternary $\mathbf{c}(CAB)$ which means that A, B, and C are collinear. A system of axioms for affine geometry is then given using these relations. (**A***) also includes several auxiliary definitions which will be used in order to prove the equivalence between systems (**A***) and (**A**).

System (**A**) is a formalization of affine geometry as given by Artin [2]. Artin assumes two sets, points and lines, and a primitive binary relation of incidence between a point P and a line ℓ: P lies on ℓ. In (**A**) these sets will be denoted by α and β respectively and the relation by $\mathbf{i}(P\ell)$. Artin defines parallelism between lines: If ℓ and m are two lines such that either $\ell = m$ or that no point P lies on both ℓ and m, then we call ℓ and m parallel and write $\ell \parallel m$. Collinear points are understood to mean points lying on the same line. The following axioms are then given for affine geometry.

Axiom 1. *Given two distinct points P and Q, there exists a unique line ℓ such that P lies on ℓ and Q lies on ℓ. We write $\ell = P + Q$.*

Axiom 2. *Given a point P and a line ℓ, there exists one and only one line m such that P lies on m and such that $m \parallel \ell$.*

Axiom 3. *There exist three non-collinear points.*

Axiom 4.[2] *Let* ℓ_1, ℓ_2, ℓ_3, *be distinct lines which meet at a point* P. *Let* Q, Q' *be points on* ℓ_1, R, R' *points on* ℓ_2, *and* S, S' *points on* ℓ_3 *which are distinct from* P. *Assume* $Q+R \| Q'+R'$ *and* $Q+S \| Q'+S'$, *then* $R+S \| R'+S'$.

It should be noted that Artin's affine geometry does not contain the betweenness properties included by some authors, but consists of only the incidence and parallel properties of Hilbert's geometry.

In the formalization of Artin's system the second class of objects β will be identified with certain subsets of the first class of objects α, i.e. a line will be assumed to be identical with the set of all points lying on it. This identification occurs in axiom *A1*. Only by this identification will it be possible to compare the point geometry of (**A***) with the point-line approach of (**A**).

1.1 SYSTEM (A*)

Definitions

D1 $[A]: A \varepsilon \alpha . \equiv . [\exists BCD]. \mathbf{t}(ABCD)$

D2 $[AB] .\therefore \mathbf{r}(AB) . \equiv : [\exists CD]: \mathbf{t}(ABCD) . \vee . \mathbf{t}(ACBD) . \vee . \mathbf{t}(CBAD)$

D3 $[ABC] .\therefore \mathbf{c}(ABC) . \equiv : \mathbf{r}(BC): A = B . \vee . A = C . \vee . [\exists XY]. \mathbf{t}(BCXY). \mathbf{t}(BAXY) . \mathbf{t}(CAXY)$

Axioms

A1 $[\exists ABCD] . \mathbf{t}(ABCD)$

A2 $[ABCD]: \mathbf{t}(ABCD) . \supset . A \neq B$

A3(a) $[ABCD]: \mathbf{t}(ABCD) . \supset . \mathbf{t}(DCAB)$

A3(b) $[ABCD]: \mathbf{t}(ABCD) . \supset . \mathbf{t}(ABDC)$

A4 $[ABCRSTUVWXYZ] :: \mathbf{t}(ARST) . \mathbf{t}(BUVW) . \mathbf{t}(CXYZ) . \supset .\therefore \sim(\mathbf{c}(CAB)) . \equiv : [\exists D] . \mathbf{t}(ABCD) . \vee . A = B$

A5 $[ABCMN]: A \neq B . \mathbf{c}(AMN) . \mathbf{c}(BMN) . \mathbf{c}(CMN) . \supset . \mathbf{c}(CAB)$

A6 $[ABCDEFG]: \mathbf{t}(ABCD) . \mathbf{t}(ABEF) . \mathbf{t}(CDEG) . \supset . \mathbf{t}(CDEF)$

A7 $[ABCD] .\therefore \mathbf{t}(ABCD) . \equiv : \mathbf{r}(AB) . \mathbf{r}(CD) : [M] : \mathbf{c}(MAB) . \supset . \sim(\mathbf{c}(MCD))$

A8 $[ABCDEFGHX]: \mathbf{t}(ABCD) . \mathbf{t}(BEDF) . \mathbf{t}(AECG) . \mathbf{t}(AEFH) . \mathbf{c}(XCA) . \mathbf{c}(XBD) . \mathbf{c}(XEF) . \supset . \mathbf{t}(AECF)$

Auxiliary Definitions

D4 $[ABC]: A \varepsilon \mathbf{R}\{BC\} . \equiv . \mathbf{c}(ABC)$

D5 $[ABCD]: \mathbf{s}(ABCD) . \equiv . \mathbf{c}(ACD) . \mathbf{c}(BCD) . A \neq B$

D6 $[f] :: f \varepsilon \beta . \equiv .\therefore [\exists AB] .\therefore \mathbf{r}(AB) : [C] : \mathbf{c}(CAB) . \equiv . C \varepsilon f$

D7 $[Af]: \mathbf{i}(Af) . \equiv . A \varepsilon \alpha . A \varepsilon f . f \varepsilon \beta$

D8 $[fg] .\therefore \mathbf{p}(fg) . \equiv : f \varepsilon \beta . g \varepsilon \beta : g = f . \vee . \sim([\exists A] . \mathbf{i}(Ag) . \mathbf{i}(Af))$

D9 $[ABC]: \mathbf{k}(ABC) . \equiv . [\exists g] . \mathbf{i}(Ag) . \mathbf{i}(Bg) . \mathbf{i}(Cg)$

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2. Axiom 4 is actually equivalent to Artin's 4b. (See Chapter II of [2]). However, by an unpublished result of H. Zassenhaus, it is known that 4b implies Artin's 4a, so 4a is not included in (**A**).

1.2 SYSTEM (A)

Definitions

D10 $[A]:A\varepsilon\alpha\,.\equiv.\,[\exists f]\,.\,\mathbf{i}(Af)$

D11 $[f]:f\varepsilon\beta\,.\equiv.\,[\exists A]\,.\,\mathbf{i}(Af)$

D9 $[ABC]:\mathbf{k}(ABC)\,.\equiv.\,[\exists g]\,.\,\mathbf{i}(Ag)\,.\,\mathbf{i}(Bg)\,.\,\mathbf{i}(Cg)$

D12 $[ABC]:A\varepsilon\mathbf{R}\{BC\}\,.\equiv.\,B\neq C\,.\,\mathbf{k}(ABC)$

D8 $[fg]\therefore\,\mathbf{p}(fg)\,.\equiv:\,f\varepsilon\beta\,.\,g\varepsilon\beta:g=f\,.\vee.\ \sim([\exists A]\,.\,\mathbf{i}(Ag)\,.\,\mathbf{i}(Af))$

Axioms

AI $[AB]\therefore\,A\varepsilon\alpha\,.\,B\varepsilon\alpha\,.\,A\neq B.\supset:[\exists f]\,.\,\mathbf{i}(Af)\,.\,\mathbf{i}(Bf)\,.\,f=\mathbf{R}\{AB\}:[gh]:\ \mathbf{i}(Ag)\,.\,\mathbf{i}(Bg)\,.\,\mathbf{i}(Ah)\,.\,\mathbf{i}(Bh)\,.\supset.\,g=h$

AII $[Af]\therefore\,A\varepsilon\alpha\,.\,f\varepsilon\beta.\supset:[\exists g]\,.\,\mathbf{i}(Ag)\,.\mathbf{p}(fg):[hk]:\mathbf{i}(Ah]\,.\mathbf{p}(fh)\,.\,\mathbf{i}(Ak)\,.\,\mathbf{p}(fk)\,.\supset.\ h=k$

AIII $[\exists ABC]:A\varepsilon\alpha\,.\,B\varepsilon\alpha\,.\,C\varepsilon\alpha\,.\,A\neq B\,.\,A\neq C.\,B\neq C\,.\ \sim(\mathbf{k}(ABC))$

AIV $[ABCDEFGfgh\ell tmnrs]:\mathbf{k}(ABC)\,.\,\mathbf{k}(ADE)\,.\,\mathbf{k}(AFG)\,.\,A\neq B\,.\,A\neq C\,.\,B\neq C.\,A\neq D\,.\,A\neq E\,.\,D\neq E\,.\,A\neq F\,.\,A\neq G\,.\,F\neq G\,.\,\mathbf{i}(Af)\,.\,\mathbf{i}(Bf)\,.\,\mathbf{i}(Ag)\,.\,\mathbf{i}(Dg)\,.\mathbf{i}(Ah)\,.\mathbf{i}(Fh)\,.\,f\neq g\,.\,f\neq h\,.\,g\neq h\,.\,\mathbf{i}(B\ell)\,.\,\mathbf{i}(D\ell)\,.\,\mathbf{i}(Ct)\,.\,\mathbf{i}(Et)\,.\,\mathbf{p}(\ell t)\,.\,\mathbf{i}(Dm)\,.\,\mathbf{i}(Fm)\,.\ \mathbf{i}(En)\,.\,\mathbf{i}(Gn)\,.\,\mathbf{p}(mn)\,.\,\mathbf{i}(Br)\,.\,\mathbf{i}(Fr)\,.\,\mathbf{i}(Cs)\,.\,\mathbf{i}(Gs)\,.\supset.\,\mathbf{p}(rs)$

Auxiliary Definitions

D13 $[ABCD]:\mathbf{t}(ABCD)\,.\equiv.\,A\neq B\,.\,C\neq D\,.\,[\exists fg]\,.\,\mathbf{p}(fg)\,.\,f\neq g\,.\,\mathbf{i}(Af)\,.\,\mathbf{i}(Bf)\,.\mathbf{i}(Cg)\,.\,\mathbf{i}(Dg)$

D2 $[AB]\therefore\,\mathbf{r}(AB)\,.\equiv:\,[\exists CD]:\mathbf{t}(ABCD)\,.\vee.\,\mathbf{t}(ACBD)\,.\vee.\,\mathbf{t}(CBAD)$

D3 $[ABC]\therefore\mathbf{c}(ABC)\,.\equiv:\,\mathbf{r}(BC):A=B\,.\vee.\,A=C.\vee.\,[\exists XY]\,.\,\mathbf{t}(BCXY).\mathbf{t}(BAXY)\,.\,\mathbf{t}(CAXY)$

D5 $[ABCD]:\mathbf{s}(ABCD)\,.\equiv.\,\mathbf{c}(ACD)\,.\,\mathbf{c}(BCD)\,.\,A\neq B$

1.3 SYSTEM (A*) IMPLIES SYSTEM (A)

L1	$[ABCD]:\mathbf{t}(ABCD)\,.\supset.\,\mathbf{t}(BADC)$	
PF	$[ABCD]:$ Hp(1) [3]$.\supset.$	
	2) $\mathbf{t}(DCAB)\,.$	$[A3(a),\ \bar{1}]$
	$\mathbf{t}(BADC)$	$[A3(a),\ \bar{2}]$
L2	$[ABCD]:\mathbf{t}(ABCD)\,.\supset.\,\mathbf{t}(CDBA)$	$[L1,\ A3(a)]$
L3	$[ABCD]:\mathbf{t}(ABCD)\,.\supset.\,\mathbf{t}(DCBA)$	$[A3(a),\ A3(b)]$
L4	$[ABCD]:\mathbf{t}(ABCD)\,.\supset.\,\mathbf{t}(BACD)$	$[L1,\ A3(b)]$
L5	$[ABCD):\mathbf{t}(ABCD)\,.\supset.\,\mathbf{t}(CDAB)$	$[L4,\ L2]$
L6	$[ABCD]:\mathbf{t}(ABCD)\,.\equiv.\,\mathbf{t}(DCAB)$	$[A3(a);\ L2]$
L7	$[ABCD]:\mathbf{t}(ABCD)\,.\equiv.\ \mathbf{t}(BADC)$	$[L1;\ L1]$
L8	$[ABCD]:\mathbf{t}(ABCD)\,.\equiv.\,\mathbf{t}(CDBA)$	$[L2;\ A3(a)]$
L9	$[ABCD]:\mathbf{t}(ABCD)\,.\equiv.\,\mathbf{t}(ABDC)$	$[A3(b);\ A3(b)]$

3. Symbols Hp(1), Hp(1-2) a.s.o. indicate how many components are in the antecedent of the thesis to be proved.

L10	$[ABCD]:\mathbf{t}(ABCD).\equiv.\mathbf{t}(DCBA)$	$[L3, L3]$
L11	$[ABCD]:\mathbf{t}(ABCD).\equiv.\mathbf{t}(BACD)$	$[L4, L4]$
L12	$[ABCD]:\mathbf{t}(ABCD).\equiv.\mathbf{t}(CDAB)$	$[L5, L5]$
L13	$[ABCD]:\mathbf{t}(ABCD).\supset.C\neq D$	$[L5, A2]$
L14	$[ABCD]:\mathbf{t}(ABCD).\supset.A\neq C$	
PF	$[ABCD]:\mathrm{Hp}(1).\supset.$	
	2) $\quad\mathbf{t}(BACD).$	$[L4, 1]$
	3) $\quad\mathbf{t}(CDAB).$	$[L5, 1]$
	4) $\quad\sim(\mathbf{c}(CAB)).$	$[A4, 1, 1, 2, 3]$
	5) $\quad\mathbf{r}(AB).$	$[D2, 1]$
	$\quad A\neq C$	$[D3, 4, 5]$
L15	$[ABCD]:\mathbf{t}(ABCD).\supset.B\neq C$	$[L4, L14]$
L16	$[ABCD]:\mathbf{t}(ABCD).\supset.A\neq D$	$[A3(b), L14]$
L17	$[ABCD]:\mathbf{t}(ABCD).\supset.B\neq D$	$[L1, L14]$
L18	$[AB]:A\varepsilon\alpha.B\varepsilon\alpha.A\neq B.\sim(\mathbf{r}(AB)).\supset.\mathbf{r}(AB)$	
PF	$[AB]\therefore\mathrm{Hp}(1\text{-}4).\supset:$	
	5) $\quad\sim(\mathbf{c}(AAB)).$	$[D3, 4]$
	$\quad[\exists CDE].$	
	6) $\qquad\mathbf{t}(ACDE).$	$[D1, 1]$
	$\qquad[\exists FGH].$	
	7) $\qquad\qquad\mathbf{t}(BFGH):$	$[D1, 2]$
	8) $\quad[\exists J]:\mathbf{t}(ABAJ).\vee.A=B:$	$[A4, 6, 7, 6, 5]$
	$\quad\mathbf{r}(AB)$	$[8, L14, 3]$
L19	$[AB]:A\varepsilon\alpha.B\varepsilon\alpha.A\neq B.\equiv.\mathbf{r}(AB)$	$[L18; D2, D1, L4, A2, L5, L14, L2, L4, L15]$
L20	$[ABC]:\mathbf{c}(ABC).\supset.[\exists MNOUVWPRS].\mathbf{t}(AMNO).\mathbf{t}(BUVW).\mathbf{t}(CPRS)$	
PF	$[ABC]\therefore\mathrm{Hp}(1).\supset:$	
	2) $\quad\mathbf{r}(BC).$	$[D3, 1]$ (2, 3)
	3) $\quad A=B.\vee.A=C.\vee.[\exists KJ].\mathbf{t}(BAKJ):$	
	4) $\quad A=B.\vee.A=C.\vee.[\exists KJ].\mathbf{t}(ABKJ):$	$[3, L4]$
	$\quad[\exists XY]:$	
	5) $\qquad\mathbf{t}(BCXY).\vee.\mathbf{t}(BXCY).\vee.\mathbf{t}(XCBY):$	$[D2, 2]$
	6) $\qquad\mathbf{t}(BCXY).\vee.\mathbf{t}(BXCY).\vee.\mathbf{t}(BYXC):$	$[5, L5]$
	7) $\qquad\mathbf{t}(CBXY).\vee.\mathbf{t}(CYBX).\vee.\mathbf{t}(CXBY):$	$[6, L4, L5, A3(a)]$
	$\quad[\exists MNOUVWPRS].\mathbf{t}(AMNO).\mathbf{t}(BUVW).\mathbf{t}(CPRS)$	$[4, 6, 7]$
L21	$[ABCD]:A\neq B.A\neq C.\mathbf{c}(ACB).\mathbf{t}(ABCD).\supset.\sim(\mathbf{t}(ABCD))$	
PF	$[ABCD]:\mathrm{Hp}(1\text{-}4).\supset.$	
	$\quad[\exists XY].$	
	5) $\qquad\mathbf{t}(CBXY).$	$[D3, 1, 2, 3]$ (5, 6)
	6) $\qquad\mathbf{t}(BAXY).$	
	7) $\qquad\mathbf{t}(XYCB).$	$[L5, 5]$
	8) $\qquad\mathbf{t}(XYAB).$	$[L2, 6]$
	9) $\quad\mathbf{t}(ABCB).$	$[A6, 8, 7, 4]$
	$\quad\sim(\mathbf{t}(ABCD))$	$[L17, 9]$
L22	$[ABC]:A\neq B.A\neq C.\mathbf{c}(ACB).\supset.\mathbf{c}(CAB)$	
PF	$[ABC]:\mathrm{Hp}(1\text{-}3).\supset.$	
	4) $\quad[D].\sim(\mathbf{t}(ABCD)).$	$[L21, 1, 2, 3]$
	$\quad\mathbf{c}(CAB)$	$[A4, L20, 3, 4, 1]$

L23 $[ABC]:\mathsf{c}(CAB).\supset.\mathsf{c}(CBA)$
PF $[ABC]\therefore \mathrm{Hp}(1).\supset:$
2) $\mathsf{r}(AB).$ [*D3*, 1]
3) $\mathsf{r}(BA):$ [*L19*, 2]
4) $C=A.\vee.C=B.\vee.[\exists XY].\mathsf{t}(ABXY).\mathsf{t}(ACXY).\mathsf{t}(BCXY):$ [*D3*, *1*]
5) $C=B.\vee.C=A.\vee.[\exists XY].\mathsf{t}(BAXY).\mathsf{t}(BCXY).\mathsf{t}(ACXY):$ [4, *L4*]
$\mathsf{c}(CBA)$ [*D3*, 3, 5]
L24 $[ABC]:A\neq C.B\neq C.\mathsf{c}(CAB).\supset.\mathsf{c}(BAC)$
PF $[ABC]:\mathrm{Hp}(1\text{-}3).\supset.$
4) $\mathsf{c}(CBA)$ [*L23*, 3]
5) $\mathsf{c}(BCA).$ [*L22*, 1, 2, 4]
$\mathsf{c}(BAC)$ [*L23*, 5]
L25 $[ABC]:A\neq B.C\neq B.\mathsf{c}(BAC).\supset.\mathsf{c}(ACB)$
PF $[ABC]:\mathrm{Hp}(1\text{-}3).\supset.$
4) $\mathsf{c}(ABC).$ [*L22*, 2, 1, 3]
$\mathsf{c}(ACB)$ [*L23*, 4]
L26 $[ABC]\therefore A\neq B.A\neq C.B\neq C.\supset:\mathsf{c}(ACB).\equiv.\mathsf{c}(CAB):\mathsf{c}(ACB).\equiv.$
$\mathsf{c}(BAC)$ [*L22*, *L24*, *L25*]
L27 $[AB]:\mathsf{r}(AB).\supset.\mathsf{s}(ABBA)$
PF $[AB]:\mathrm{Hp}(1).\supset.$
2) $A\neq B.$ [*L19*, 1]
3) $\mathsf{r}(BA).$ [*L19*, 1]
4) $\mathsf{c}(ABA).$ [*D3*, 3]
5) $\mathsf{c}(BBA).$ [*D3*, 3]
$\mathsf{s}(ABBA)$ [*D5*, 4, 5, 2]
L28 $[AB]:\mathsf{r}(AB).\supset.\mathsf{s}(ABAB)$ [*D3*, *L19*, *D5*]
L29 $[ABCD]:\mathsf{s}(ABCD).\supset.\mathsf{s}(CDAB)$
PF $[ABCD]:\mathrm{Hp}(1).\supset.$
2) $A\neq B.$ [*D5*, 1]
3) $\mathsf{c}(ACD).$ [*D5*, 1]
4) $\mathsf{c}(BCD).$ [*D5*, 1]
5) $\mathsf{r}(CD).$ [*D3*, 3]
6) $C\neq D.$ [*L19*, 5]
7) $\mathsf{c}(CCD).$ [*D3*, 5]
8) $\mathsf{c}(CAB).$ [*A5*, 2, 3, 4, 7]
9) $\mathsf{c}(DCD).$ [*D3*, 5]
10) $\mathsf{c}(DAB).$ [*A5*, 2, 3, 4, 9]
$\mathsf{s}(CDAB)$ [*D5*, 8, 10, 6]
L30 $[ABCDEF]:\mathsf{s}(ABCD).\mathsf{s}(CDEF).\supset.\mathsf{s}(ABEF)$
PF $[ABCDEF]:\mathrm{Hp}(1\text{-}2).\supset.$
3) $A\neq B.$ [*D5*, 1]
4) $\mathsf{c}(ACD).$ [*D5*, 1]
5) $\mathsf{c}(BCD).$ [*D5*, 1]
6) $\mathsf{s}(EFCD).$ [*L29*, 2]
7) $E\neq F.$ [*D5*, 6]
8) $\mathsf{c}(ECD).$ [*D5*, 6]
9) $\mathsf{c}(FCD).$ [*D5*, 6]
10) $\mathsf{c}(EAB).$ [*A5*, 3, 4, 5, 8]

11) $\mathbf{c}(FAB)$. [$A5$, 3, 4, 5, 9]
12) $\mathbf{s}(EFAB)$. [$D5$, 10, 11, 7]
$\mathbf{s}(ABEF)$ [$L29$, 12]

$L31$ $[ABCDM]:\mathbf{s}(ABCD).M\varepsilon\mathbf{R}\{AB\}.\supset.M\varepsilon\mathbf{R}\{CD\}$
PF $[ABCDM]:$ Hp(1-2)$.\supset.$
3) $\mathbf{c}(MAB)$. [$D4$, 2]
4) $\mathbf{s}(CDAB)$. [$L29$, 1]
5) $C\neq D$. [$D5$, 4]
6) $\mathbf{c}(CAB)$. [$D5$, 4]
7) $\mathbf{c}(DAB)$. [$D5$, 4]
8) $\mathbf{c}(MCD)$. [$A5$, 5, 6, 7, 3]
$M\varepsilon\mathbf{R}\{CD\}$ [$D4$, 8]

$L32$ $[ABCD]:\mathbf{s}(ABCD).\supset.\mathbf{R}\{AB\}=\mathbf{R}\{CD\}$
PF $[ABCD]\therefore$ Hp(1)$.\supset:$
2) $\mathbf{s}(CDAB):$ [$L29$, 1]
3) $[M]:M\varepsilon\mathbf{R}\{AB\}.\equiv.M\varepsilon\mathbf{R}\{CD\}:$ [$L31$, *1*; $L31$, 2]
$\mathbf{R}\{AB\}=\mathbf{R}\{CD\}$ [$D0$, 3]

$L33$ $[ABCD]:A\varepsilon\alpha.B\varepsilon\alpha.A\neq B.\mathbf{R}\{AB\}=\mathbf{R}\{CD\}.\supset.\mathbf{s}(ABCD)$
PF $[ABCD]:$ Hp(1-4)$.\supset.$
5) $A\varepsilon\mathbf{R}\{AB\}$. [$D4$, $L19$, 1, 2, 3, $D3$]
6) $B\varepsilon\mathbf{R}\{AB\}$. [$D4$, $L19$, 1, 2, 3, $D3$]
7) $\mathbf{c}(ACD)$. [$D4$, 4, 5, $D0$]
8) $\mathbf{c}(BCD)$. [$D4$, 4, 6, $D0$]
$\mathbf{s}(ABCD)$ [$D5$, 7, 8, 3]

$L34$ $[AB]:A\varepsilon\alpha.B\varepsilon\alpha.A\neq B.\supset.[\exists f].\mathbf{i}(Af).\mathbf{i}(Bf).f=\mathbf{R}\{AB\}$
PF $[AB]:$ Hp(1-3)$.\supset.$
4) $\mathbf{r}(AB)$. [$L19$, 1, 2, 3]
5) $A\varepsilon\mathbf{R}\{AB\}$. [$D4$, 4, $D3$]
6) $B\varepsilon\mathbf{R}\{AB\}$. [$D4$, 4, $D3$]
7) $\mathbf{R}\{AB\}\varepsilon\beta$. [$D6$, $D4$, 4]
$[\exists f].\mathbf{i}(Af).\mathbf{i}(Bf).f=\mathbf{R}\{AB\}$ [$D7$, 1, 5, 7, $D7$, 2, 6, 7]

$L35$ $[ABCfg]:A\neq B.\mathbf{i}(Af).\mathbf{i}(Bf).\mathbf{i}(Ag).\mathbf{i}(Bg).C\varepsilon f.\supset.C\varepsilon g$
PF $[ABCfg]::$ Hp(1-6)$.\supset\therefore$
7) $f\varepsilon\beta$. [$D7$, 2]
8) $g\varepsilon\beta$. [$D7$, 4]
9) $A\varepsilon f$. [$D7$, 2]
10) $B\varepsilon f$. [$D7$, 3]
11) $A\varepsilon g$. [$D7$, 4]
12) $B\varepsilon g\therefore$ [$D7$, 5]
$[\exists MN]\therefore$
13) $[K]:\mathbf{c}(KMN).\equiv.K\varepsilon f:$ [$D6$, 7]
14) $\mathbf{c}(AMN)$. [*13*, 9]
15) $\mathbf{c}(BMN)$. [13, 10]
16) $\mathbf{c}(CMN)\therefore$ [13, 6]
17) $\mathbf{c}(CAB)$. [$A5$, 1, 14, 15, 16]
18) $C\varepsilon\mathbf{R}\{AB\}\therefore$ [$D4$, 17]
$[\exists RS]\therefore$
19) $[T]:\mathbf{c}(TRS).\equiv.T\varepsilon g:$ [$D6$, 8]

20) $\mathsf{c}(ARS)$. [19, 11]
21) $\mathsf{c}(BRS)$. [19, 12]
22) $\mathsf{s}(ABRS)$. [$D5$, 20, 21, 1]
23) $C\varepsilon\mathbf{R}\{RS\}$. [$L31$, 22, 18]
24) $\mathsf{c}(CRS)\therefore$ [$D4$, 23]
$C\varepsilon g$ [19, 24]

AI $[AB]\therefore A\varepsilon\alpha\,.\,B\varepsilon\alpha\,.\,A\neq B\,.\supset:[\exists f]\,.\,\mathsf{i}(Af)\,.\,\mathsf{i}(Bf)\,.\,f=\mathbf{R}\{AB\}:[gh]:$
$\mathsf{i}(Ag)\,.\,\mathsf{i}(Bg)\,.\,\mathsf{i}(Ah)\,.\,\mathsf{i}(Bh)\,.\supset.\,g=h$ [$L34$, $L35$, $D0$]

L36 $[ABCf]\therefore[M]:\mathsf{c}(MBC)\,.\equiv.\,M\varepsilon f:\mathsf{c}(ABC)\,.\supset.\,\mathsf{i}(Af)\,.\,\mathsf{p}(ff)$
PF $[ABCf]\therefore$ Hp(1-2). $\supset$:
3) $A\varepsilon f$. [1, 2]
4) $\mathsf{r}(BC)$. [$D3$, 2]
5) $f\varepsilon\beta$. [$D6$, 4, 1]
$\mathsf{i}(Af)\,.\,\mathsf{p}(ff)$ [$D7$, $L20$, $D1$, 2, 3, 5; $D8$, 5]

L37 $[ABCDNf]\therefore\mathsf{t}(ABCD):[M]:\mathsf{c}(MAB)\,.\equiv.\,M\varepsilon f:\mathsf{i}(Nf)\,.\supset.\,\sim(\mathsf{i}(N\mathbf{R}\{CD\}))$
PF $[ABCDNf]\therefore$ Hp(1-3). $\supset$:
4) $N\varepsilon f$. [$D7$, 3]
5) $\mathsf{c}(NAB)$. [2, 4]
6) $\sim(\mathsf{c}(NCD))$. [$A7$, 1, 5]
7) $\sim(N\varepsilon\mathbf{R}\{CD\})$. [$D4$, 6]
$\sim(\mathsf{i}(N\mathbf{R}\{CD\}))$ [$D7$, 7]

L38 $[ABCDf]\therefore[M]:\mathsf{c}(MAB)\,.\equiv.\,M\varepsilon f:\mathsf{t}(ABCD)\,.\supset.\,\mathsf{i}(C\mathbf{R}\{CD\})\,.\,\mathsf{p}(f\mathbf{R}\{CD\})$
PF $[ABCDf]\therefore$ Hp(1-2). $\supset$:
3) $\sim(\mathsf{c}(CAB))$. [$A4$, 2, $L4$, $L2$]
4) $\mathsf{r}(CD)$. [$D2$, $L2$, 2]
5) $\mathsf{r}(AB)$. [$D2$, 2]
6) $f\varepsilon\beta$. [$D6$, 5, 1]
7) $\mathbf{R}\{CD\}\,\varepsilon\beta$. [$D6$, $D4$, 4]
8) $C\varepsilon\mathbf{R}\{CD\}$: [$D4$, $D3$, 4]
9) $[N]:\mathsf{i}(Nf)\,.\supset.\,\sim(\mathsf{i}(N\mathbf{R}\{CD\}))$: [$L37$, 2, 1]
$\mathsf{i}(C\mathbf{R}\{CD\})\,.\,\mathsf{p}(f\mathbf{R}\{CD\})$ [$D7$, $L19$, 4, 8, 7; $D8$, 6, 7, 9]

L39 $[Af]:A\varepsilon\alpha\,.\,f\varepsilon\beta\,.\supset.\,[\exists g]\,.\,\mathsf{i}(Ag)\,.\,\mathsf{p}(fg)$
PF $[Af]::$ Hp(1-2) $.\supset\therefore$
$[\exists MN]\therefore$
3) $\mathsf{r}(MN)$:
4) $[K]:\mathsf{c}(KMN)\,.\equiv.\,K\varepsilon f$:
(3–4) [$D6$, 2]
5) $M\neq N$: [$D2$, $A2$, $L14$, $L15$, 3]
6) $\mathsf{c}(AMN)\,.\vee.\,[\exists X]\,.\,\mathsf{t}(MNAX)\because$ [$A4$, $L19$, 3, $D1$, 1, 5]
$[\exists g]\,.\,\mathsf{i}(Ag)\,.\,\mathsf{p}(fg)$ [6, $L36$, 4, $L38$, 4]

L40 $[ABCDE]:\mathsf{t}(ABCD)\,.\,\mathsf{c}(EAB)\,.\,\mathsf{c}(ECD)\,.\supset.\,\sim(\mathsf{t}(ABCD))$
PF $[ABCDE]$: Hp(1-3) $.\supset.$
4) $\sim(\mathsf{c}(ECD))$. [$A7$, 1, 2]
$\sim(\mathsf{t}(ABCD))$ [3, 4]

L41 $[Afhk]:\mathsf{i}(Ah)\,.\,\mathsf{p}(fh)\,.\,\mathsf{i}(Ak)\,.\,\mathsf{p}(fk)\,.\,f\neq h\,.\,f\neq k\,.\supset.\,h=k$
PF $[Afhk]::$ Hp(1-6) $.\supset::$
7) $f\varepsilon\beta$.
8) $h\varepsilon\beta$.
(7–8) [$D8$, 2]
9) $k\varepsilon\beta$. [$D8$, 4]

10)	$[R]:\mathsf{i}(Rf).\supset.\sim(\mathsf{i}(Rh)):$	[D8, 2, 5]
11)	$[R]:\mathsf{i}(Rf).\supset.\sim(\mathsf{i}(Rk))::.$	[D8, 4, 6]
	$[\exists MN]::.$	
12)	$\mathsf{r}(MN):$	[D6, 7]
13)	$[K]:\mathsf{c}(KMN).\equiv.K\varepsilon f::$	
	$[\exists BC]::$	
14)	$\mathsf{r}(BC):$	[D6, 8]
15)	$[S]:\mathsf{c}(SBC).\equiv.S\varepsilon h:$	
16)	$\mathsf{R}\{BC\}=h.$	[D0, D4, 15]
17)	$\mathsf{c}(ABC).$	[15, D7, 1]
18)	$[T]:\mathsf{c}(TMN).\supset.\sim(\mathsf{c}(TBC)):$	[15, D7, 10, L20, D1, 8, D7, L20, D1, 13, 7]
19)	$\mathsf{t}(MNBC)\therefore$	[A7, 12, 14, 18]
	$[\exists DE]\therefore$	
20)	$\mathsf{r}(DE):$	[D6, 9]
21)	$[U]:\mathsf{c}(UDE).\equiv.U\varepsilon k:$	
22)	$\mathsf{R}\{DE\}=k.$	[D0, D4, 21]
23)	$\mathsf{c}(ADE):$	[21, D7, 3]
24)	$[V]:\mathsf{c}(VMN).\supset.\sim(\mathsf{c}(VDE)):$	[21, D7, 11, L20, D1, 9, D7, L20, D1, 13, 7]
25)	$\mathsf{t}(MNDE).$	[A7, 12, 20, 24]
26)	$\sim(\mathsf{t}(BCDE)).$	[L40, 17, 23]
27)	$\mathsf{c}(DBC).$	[A4, A6, 19, 25, 26, L19, D1, 14]
28)	$\mathsf{t}(MNED).$	[L9, 25]
29)	$\sim(\mathsf{t}(BCED)).$	[L9, 26]
30)	$\mathsf{c}(EBC).$	[A4, A6, 19, 28, 29, L19, D1, 14]
31)	$D\neq E.$	[L19, 20]
32)	$\mathsf{s}(DEBC).$	[D5, 27, 30, 31]
33)	$\mathsf{R}\{DE\}=\mathsf{R}\{BC\}::.$	[L32, 32]
	$h=k$	[16, 22, 33]

L42 $[Afhk]:\mathsf{i}(Ah).\mathsf{p}(fh).\mathsf{i}(Ak).\mathsf{p}(fk).f=h.f\neq k.\supset.h=k$

PF $[Afhk]::.\mathrm{Hp}(1\text{-}6).\supset::$

7)	$h\varepsilon\beta.$	[D8, 2]
8)	$k\varepsilon\beta:$	[D8, 4]
9)	$[R]:\mathsf{i}(Rf).\supset.\sim(\mathsf{i}(Rk)):$	[D8, 4, 6]
10)	$[R]:\mathsf{i}(Rh).\supset.\sim(\mathsf{i}(Rk))::$	[9, 5]
	$[\exists BC]::$	
11)	$\mathsf{r}(BC):$	[D6, 7]
12)	$[K]:\mathsf{c}(KBC).\equiv.K\varepsilon h:$	
13)	$\mathsf{c}(ABC)\therefore$	[12, D7, 1]
	$[\exists DE]\therefore$	
14)	$\mathsf{r}(DE):$	[D6, 8]
15)	$[K]:\mathsf{c}(KDE).\equiv.K\varepsilon k:$	
16)	$\mathsf{c}(ADE):$	[15, D7, 3]
17)	$[R]:\mathsf{c}(RBC).\supset.\sim(\mathsf{c}(RDE)):$	[15, D7, 10, L20, D1, 8, D7, L20, D1, 12, 7]
18)	$\mathsf{t}(BCDE).$	[A7, 11, 14, 17]

19) $\sim(\mathsf{t}(BCDE))$. [*L40*, 13, 16]
20) $f=k::$ [18, 19]
$h=k$ [20, 5]

L43 $[Afhk]:\mathsf{i}(Ah).\mathsf{p}(fh).\mathsf{i}(Ak).\mathsf{p}(fk).\supset.h=k$ [*L41*, *L42*]

AII $[Af]\therefore A\varepsilon\alpha.f\varepsilon\beta.\supset:[\exists g].\mathsf{i}(Ag).\mathsf{p}(fg):[hk]:\mathsf{i}(Ah).\mathsf{p}(fh).$
$\mathsf{i}(Ak).\mathsf{p}(fk).\supset.h=k$ [*L39*, *L43*]

L44 $[ABCD]:\mathsf{t}(ABCD).\mathsf{k}(ABC).\supset.\sim(\mathsf{k}(ABC))$
PF $[ABCD]\because\mathrm{Hp}(1\text{-}2).\supset::$
$[\exists g]::$
3) $\mathsf{i}(Ag)$.
4) $\mathsf{i}(Bg)$. [*D9*, 2]
5) $\mathsf{i}(Cg)$.
$[\exists XY]\therefore$
6) $\mathsf{r}(XY):$ [*D6*, *D7*, 3]
7) $[R]:\mathsf{c}(RXY).\equiv.R\varepsilon g:$
8) $\mathsf{c}(AXY)$. [7, *D7*, 3]
9) $\mathsf{c}(BXY)$. [7, *D7*, 4]
10) $\mathsf{c}(CXY)$. [7, *D7*, 5]
11) $\mathsf{s}(ABXY)$. [*D5*, 8, 9, *A2*, 1]
12) $\mathsf{s}(XYAB)::$ [*L29*, 11]
13) $\mathsf{c}(CAB)$. [*D4*, *D0*, *L32*, 12, *D4*, 10]
14) $\mathsf{r}(AB)$. [*D2*, 1]
15) $\mathsf{c}(CCD)$. [*D3*, *D2*, *L2*, 1]
16) $\mathsf{r}(CD)$. [*D3*, 15]
17) $\sim(\mathsf{t}(ABCD))$. [*A7*, 14, 16, 13, 15]
$\sim(\mathsf{k}(ABC))$ [1, 17]

AIII $[\exists ABC]:A\varepsilon\alpha.B\varepsilon\alpha.C\varepsilon\alpha.A\neq B.A\neq C.B\neq C.\sim(\mathsf{k}(ABC))$
[*L44*, *A1*, *D1*, *L1*, *L2*, *A2*, *L14*, *L15*]

L45 $[ABC]:\mathsf{k}(ABC).B\neq C.\supset.\mathsf{c}(ABC)$
PF $[ABC]\because\mathrm{Hp}(1\text{-}2).\supset::$
$[\exists g]::$
3) $\mathsf{i}(Ag)$.
4) $\mathsf{i}(Bg)$. [*D9*, 1]
5) $\mathsf{i}(Cg)\therefore$
$[\exists MN]\therefore$
6) $[K]:\mathsf{c}(KMN).\equiv.K\varepsilon g:$ [*D6*, *D7*, 3]
7) $\mathsf{c}(AMN)$. [6, *D7*, 3]
8) $\mathsf{c}(BMN)$. [6, *D7*, 4]
9) $\mathsf{c}(CMN)::$ [6, *D7*, 5]
$\mathsf{c}(ABC)$ [*A5*, 2, 8, 9, 7]

L46 $[ABf]:A\neq B.\mathsf{i}(Af).\mathsf{i}(Bf).\supset.\mathsf{R}\{AB\}=f$
PF $[ABf]::\mathrm{Hp}(1\text{-}3).\supset\therefore$
$[\exists MN]\therefore$
4) $[K]:\mathsf{c}(KMN).\equiv.K\varepsilon f:$ [*D6*, *D7*, 2]
5) $\mathsf{c}(AMN)$. [4, *D7*, 2]
6) $\mathsf{c}(BMN)$. [4, *D7*, 3]
7) $\mathsf{R}\{AB\}=\mathsf{R}\{MN\}\therefore$ [*L32*, *D5*, 5, 6, 1]
$\mathsf{R}\{AB\}=f$ [*D0*, *D4*, 4, 7]

L47 $[ABCf]: A \neq B . B \neq C . \mathbf{i}(Af) . \mathbf{i}(Bf) . \mathbf{k}(ABC) . \supset . \mathbf{R}\{BC\} = f$

PF $[ABCf]: \mathrm{Hp}(1\text{-}5) . \supset .$

6) $\mathbf{R}\{AB\} = f .$ [*L46*, 1, 3, 4]

$[\exists g] .$

7) $\mathbf{i}(Ag) .$

8) $\mathbf{i}(Bg) .$ [*D9*, 5]

9) $\mathbf{i}(Cg) .$

10) $\mathbf{R}\{AB\} = g .$ [*L46*, 1, 7, 8]

11) $\mathbf{R}\{BC\} = g .$ [*L46*, 2, 8, 9]

$\mathbf{R}\{BC\} = f .$ [6, 10, 11]

L48 $[ABCfg]: \mathbf{i}(Af) . \mathbf{i}(Bf) . \mathbf{i}(Ag) . \mathbf{i}(Cg) . f \neq g . A \neq B . A \neq C . \supset .$
$\mathbf{r}(BC)$

PF $[ABCfg]: \mathrm{Hp}(1\text{-}7) . \supset .$

8) $\mathbf{R}\{AB\} = f .$ [*L46*, 6, 1, 2]

9) $\mathbf{R}\{AC\} = g .$ [*L46*, 7, 3, 4]

10) $B \neq C .$ [5, 8, 9]

$\mathbf{r}(BC)$ [*L19*, *D7*, 2, 4, 10]

L49 $[ABCDEfg]: \mathbf{k}(ABC) . \mathbf{k}(ADE) . A \neq B . A \neq C . B \neq C . A \neq D . A \neq E . D \neq E .$
$\mathbf{i}(Af) . \mathbf{i}(Bf) . \mathbf{i}(Ag) . \mathbf{i}(Dg) . f \neq g . \supset . \mathbf{r}(CE)$

PF $[ABCDEfg]: \mathrm{Hp}(1\text{-}13) . \supset .$

14) $\mathbf{c}(ABC) .$ [*L45*, 1, 5]

15) $C \varepsilon \mathbf{R}\{BC\} .$ [*D4*, *D3*, 14]

16) $\mathbf{R}\{BC\} = f .$ [*L47*, 3, 5, 9, 10, 1]

17) $\mathbf{i}(Cf) .$ [*D7*, *D9*, *D7*, 1, *D0*, 15, 16, *D7*, 9]

18) $\mathbf{c}(ADE) .$ [*L45*, 2, 8]

19) $E \varepsilon \mathbf{R}\{DE\} .$ [*D4*, *D3*, 18]

20) $\mathbf{R}\{DE\} = g .$ [*L47*, 6, 8, 11, 12, 2]

21) $\mathbf{i}(Eg) .$ [*D7*, *D9*, *D7*, 2, *D0*, 19, 20, *D7*, 11]

$\mathbf{r}(CE)$ [*L48*, 9, 17, 11, 21, 13, 4, 7]

L50 $[ABCDEfg]: \mathbf{k}(ABC) . \mathbf{k}(ADE) . A \neq B . B \neq C . A \neq D . D \neq E . \mathbf{i}(Af) . \mathbf{i}(Bf) . \mathbf{i}(Ag) .$
$\mathbf{i}(Dg) . f \neq g . \mathbf{R}\{BD\} = \mathbf{R}\{CE\} . \supset . \mathbf{R}\{BD\} \neq \mathbf{R}\{CE\}$

PF $[ABCDEfg]: \mathrm{Hp}(1\text{-}12) . \supset .$

13) $\mathbf{r}(BD) .$ [*L48*, 7, 8, 9, 10, 11, 3, 5]

14) $\mathbf{s}(BDCE) .$ [*L33*, *L19*, 3, 12]

15) $\mathbf{c}(CBD) .$ [*L29*, *D5*, 14]

16) $\mathbf{R}\{BC\} = \mathbf{R}\{BD) .$ [*L32*, *D5*, *D3*, 13, 15, 4]

17) $\mathbf{c}(DCE) .$ [*D5*, 14]

18) $\mathbf{R}\{DE\} = \mathbf{R}\{CE\} .$ [*L32*, *D5*, 17, *D3*, 17, 6]

19) $\mathbf{R}\{BC\} = f .$ [*L47*, 3, 4, 7, 8, 1]

20) $\mathbf{R}\{DE\} = g .$ [*L47*, 5, 6, 9, 10, 2]

21) $f = g .$ [19, 20, 16, 18, 12]

$\mathbf{R}\{BD\} \neq \mathbf{R}\{CE\}$ [11, 21]

L51 $[ABCDEfgmn]: \mathbf{k}(ABC) . \mathbf{k}(ADE) . A \neq B . A \neq C . B \neq C . A \neq D . A \neq E . D \neq E .$
$\mathbf{i}(Af) . \mathbf{i}(Bf) . \mathbf{i}(Ag) . \mathbf{i}(Dg) . f \neq g . \mathbf{i}(Bm) . \mathbf{i}(Dm) . \mathbf{i}(Cn) . \mathbf{i}(En) . \mathbf{p}(mn) . \supset .$
$\mathbf{t}(BDCE)$

PF $[ABCDEfgmn] .\therefore \mathrm{Hp}(1\text{-}18) . \supset :$

19) $\mathbf{r}(BD) .$ [*L48*, 9, 10, 11, 12, 13, 3, 6]

20) $\mathbf{r}(CE) .$ [*L49*, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]

21) $\mathsf{R}\{BD\} \neq \mathsf{R}\{CE\}$. [*L50*, 1, 2, 3, 5, 6, 8, 9, 10, 11, 12, 13]
22) $B \neq D$. [*L19*, 19]
23) $C \neq E$. [*L19*, 20]
24) $\mathsf{R}\{BD\} = m$. [*L46*, 22, 14, 15]
25) $\mathsf{R}\{CE\} = n$. [*L46*, 23, 16, 17]
26) $\mathsf{p}(\mathsf{R}\{BD\}\,\mathsf{R}\{CE\})$: [18, 24, 25]
27) $[N] : \mathsf{i}(N\mathsf{R}\{BD\}) . \supset . \sim(\mathsf{i}(N\mathsf{R}\{CE\}))$: [*D8*, 26, 21]
28) $[N] : \mathsf{c}(NBD) . \supset . \sim(\mathsf{c}(NCE))$:
[27, *D7*, *L20*, *D1*, *D4*, *D7*, 14, 24]
$\mathsf{t}(BDCE)$ [*A7*, 19, 20, 28]

L52 $[ABCDfg] : \mathsf{k}(ABC) . A \neq B . B \neq C . A \neq D . \mathsf{i}(Af) . \mathsf{i}(Bf) . \mathsf{i}(Ag) . \mathsf{i}(Dg) . f \neq g .$
$\mathsf{c}(CBD) . \supset . \sim(\mathsf{c}(CBD))$

PF $[ABCDfg] : \mathrm{Hp}(1\text{-}10) . \supset .$
11) $\mathsf{s}(CBBD)$. [*D5*, 10, *D3*, 10, 3]
12) $\mathsf{c}(DCB)$. [*D5*, *L29*, 11]
13) $\mathsf{c}(DBC)$. [*L23*, 12]
14) $\mathsf{c}(ABC)$. [*L45*, 1, 3]
15) $\mathsf{s}(ADBC)$. [*D5*, 14, 13, 4]
16) $\mathsf{R}\{AD\} = \mathsf{R}\{BC\}$. [*L32*, 15]
17) $\mathsf{R}\{BC\} = f$. [*L47*, 2, 3, 5, 6, 1]
18) $\mathsf{R}\{AD\} = g$. [*L46*, 4, 7, 8]
19) $f = g$. [16, 17, 18]
$\sim(\mathsf{c}(CBD))$ [9, 19]

L53 $[ABCFXYZfh] : \mathsf{i}(Af) . \mathsf{i}(Bf) . \mathsf{i}(Ah) . \mathsf{i}(Fh) . f \neq h . A \neq B . A \neq F . \mathsf{t}(CXYZ) .$
$\sim(\mathsf{c}(CBF)) . \supset . [\exists\, G] . \mathsf{t}(BFCG)$

PF $[ABCFXYZfh] : \mathrm{Hp}(1\text{-}9) . \supset .$
10) $F \varepsilon \alpha$. [*D7*, 4]
11) $B \varepsilon \alpha$. [*D7*, 2]
12) $\mathsf{r}(BF)$. [*L48*, 1, 2, 3, 4, 5, 6, 7]
13) $B \neq F$. [*L19*, 12]
$[\exists\, G] . \mathsf{t}(BFCG)$ [*A4*, *D1*, 11, 10, 8, 9, 13]

AIV $[ABCDEFGfghltmnrs] : \mathsf{k}(ABC) . \mathsf{k}(ADE) . \mathsf{k}(AFG) . A \neq B . A \neq C . B \neq C . A \neq D$
$. A \neq E . D \neq E . A \neq F . A \neq G . F \neq G . \mathsf{i}(Af) . \mathsf{i}(Bf) . \mathsf{i}(Ag) . \mathsf{i}(Dg) . \mathsf{i}(Ah) . \mathsf{i}(Fh) . f \neq g .$
$f \neq h . g \neq h . \mathsf{i}(Bl) . \mathsf{i}(Dl) . \mathsf{i}(Ct) . \mathsf{i}(Et) . \mathsf{p}(lt) . \mathsf{i}(Dm) . \mathsf{i}(Fm) . \mathsf{i}(En) . \mathsf{i}(Gn) . \mathsf{p}(mn)$
$. \mathsf{i}(Br) . \mathsf{i}(Fr) . \mathsf{i}(Cs) . \mathsf{i}(Gs) . \supset . \mathsf{p}(rs)$

PF $[ABCDEFGfghltmnrs] :: \mathrm{Hp}(1\text{-}35) . \supset .\because$
36) $\mathsf{t}(BDCE)$.
[*L51*, 1, 2, 4, 5, 6, 7, 8, 9, 13, 14, 15, 16, 19, 22, 23, 24, 25, 26]
37) $\mathsf{t}(DFEG)$.
[*L51*, 2, 3, 7, 8, 9, 10, 11, 12, 15, 16, 17, 18, 21, 27, 28, 29, 30, 31]
38) $\sim(\mathsf{c}(CBF))$. [*L52*, 1, 4, 6, 10, 13, 14, 17, 18, 20]
39) $\sim(\mathsf{c}(GFB))$. [*L52*, 3, 10, 11, 4, 17, 18, 13, 14, 20]
40) $\sim(\mathsf{c}(GBF))$. [*L23*, 40]
41) $\mathsf{c}(ABC)$. [*L45*, 1, 6]
42) $\mathsf{c}(ADE)$. [*L45*, 2, 9]
43) $\mathsf{c}(AFG) .\because$ [*L45*, 3, 12]
$[\exists\, H] .\because$

44) $\mathbf{t}(BFCH)$:
[$L53$, 13, 14, 17, 18, 20, 4, 10, $D7$, $D1$, 24, 38]
$[\exists J]$:

45) $\mathbf{t}(BFGJ).\because$
[$L53$, 13, 14, 17, 18, 20, 4, 10, $D7$, $D1$, 30, 40]

46) $\mathbf{t}(BFCG)$: [$A8$, 36, 37, 44, 45, 41, 42, 43]

47) $[V]:\mathbf{c}(VBF).\supset.\sim(\mathbf{c}(VCG))$: [$A7$, 46]

48) $\mathbf{r}(BF)$. [$A7$, 46]

49) $\mathbf{r}(CG)$. [$A7$, 46]

50) $\mathbf{R}\{BF\}\,\varepsilon\beta$. [$D6$, 48, $D4$]

51) $\mathbf{R}\{CG\}\,\varepsilon\beta$: [$D6$, 49, $D4$]

52) $[V]:\mathbf{i}(V\mathbf{R}\{BF\}).\supset.\sim(\mathbf{i}(V\mathbf{R}\{CG\})$:
[$D7$, $D4$, $D3$, $D2$, $D1$, 47, 51]

53) $\mathbf{p}(\mathbf{R}\{BF\}\mathbf{R}\{CG\})$. [$D8$, 50, 51, 52]

54) $\mathbf{R}\{BF\}=r$. [$L46$, $L19$, 48, 32, 33]

55) $\mathbf{R}\{CG)=s$. [$L46$, $L19$, 49, 34, 35]

$\mathbf{p}(rs)$ [53, 54, 55]

L54 $[ABCDfg]\therefore A\neq B.C\neq D.\mathbf{i}(Af).\mathbf{i}(Bf).\mathbf{i}(Cg).\mathbf{i}(Dg)$:
$[N]:\mathbf{i}(Nf).\supset.\sim(\mathbf{i}(Ng)):\supset.\mathbf{t}(ABCD)$

PF $[ABCDfg]\therefore \mathrm{Hp}(1\text{-}7):\supset$:

8) $\mathbf{R}\{AB\}=f$. [$L46$, 1, 3, 4]

9) $\mathbf{R}\{CD\}=g$. [$L46$, 2, 5, 6]

10) $\mathbf{r}(AB)$. [$L19$, $D7$, 3, 4, 1]

11) $\mathbf{r}(CD)$: [$L19$, $D7$, 5, 6, 2]

12) $[N]:\mathbf{c}(NAB).\supset.\sim(\mathbf{c}(NCD))$:
[$D4$, $D7$, 6, 9, $D7$, $L20$, $D4$, 8, $D7$, 3]

$\mathbf{t}(ABCD)$ [$A7$, 10, 11, 12]

L55 $[ABCD]\therefore\mathbf{t}(ABCD).\supset\therefore[\exists fg]\therefore\mathbf{i}(Af).\mathbf{i}(Bf).\mathbf{i}(Cg).\mathbf{i}(Dg)$:
$[N]:\mathbf{i}(Nf).\supset.\sim(\mathbf{i}(Ng))$

PF $[ABCD]::\mathrm{Hp}(1).\supset\therefore$

2) $\mathbf{r}(AB)$. [$A7$, 1]

3) $\mathbf{r}(CD)$. [$A7$, 1]

4) $[N]:\mathbf{c}(NAB).\supset.\sim(\mathbf{c}(NCD))$: [$A7$, 1]

5) $\mathbf{R}\{AB\}\,\varepsilon\beta$. [$D6$, $D4$, 2]

6) $\mathbf{R}\{CD\}\,\varepsilon\beta$. [$D6$, $D4$, 3]

7) $A\varepsilon\mathbf{R}\{AB\}$. [$D4$, $D3$, 2]

8) $B\varepsilon\mathbf{R}\{AB\}$. [$D4$, $D3$, 2]

9) $C\varepsilon\mathbf{R}\{CD\}$. [$D4$, $D3$, 3]

10) $D\varepsilon\mathbf{R}\{CD\}$. [$D4$, $D3$, 3]

11) $[N]:\mathbf{i}(N\mathbf{R}\{AB\}).\supset.\sim(\mathbf{i}(N\mathbf{R}\{CD\}).\because$ [$D7$, $D4$, 8, $D7$, $D4$]
$[\exists fg]\therefore\mathbf{i}(Af).\mathbf{i}(Bf).\mathbf{i}(Cg).\mathbf{i}(Dg):[N]:\mathbf{i}(Nf).\supset.\sim(\mathbf{i}(Ng))$
[$D7$, $L19$, 2, 7, 5, $L19$, 2, 8, 5, $L19$, 3, 9, 6, $L19$, 3, 10, 6, 11]

L56 $[ABCD]::\mathbf{t}(ABCD).\equiv\therefore A\neq B.C\neq D\therefore[\exists fg]\therefore\mathbf{i}(Af).\mathbf{i}(Bf).\mathbf{i}(Cg).\mathbf{i}(Dg)$:
$[N]:\mathbf{i}(Nf).\supset.\sim(\mathbf{i}(Ng))$ [$A2$, $L13$, $L55$; $L54$]

D13 $[ABCD]:\mathbf{t}(ABCD).\equiv.A\neq B.C\neq D.[\exists fg].\mathbf{p}(fg).f\neq g.\mathbf{i}(Af)$.
$\mathbf{i}(Bf).\mathbf{i}(Cg).\mathbf{i}(Dg)$ [$L56$, $D8$, $D7$]

L57 $[A]:A\varepsilon\alpha.\supset.[\exists f].\mathbf{i}(Af)$ [$D1$, $D13$]

D10 $[A]:A\varepsilon\alpha.\equiv.[\exists f].\mathsf{i}(Af)$ [*L57*; *D7*]

L58 $[f]:f\varepsilon\beta.\supset.[\exists A].\mathsf{i}(Af)$

PF $[f]::\mathrm{Hp}(1).\supset\therefore$

$[\exists AB]\therefore$

2) $\mathsf{r}(AB):$

3) $[C]:\mathsf{c}(CAB).\equiv.C\varepsilon f:$ [*D6*, 1]

4) $A\varepsilon\alpha.$ [*L19*, 2]

5) $\mathsf{c}(AAB).$ [*D3*, 2]

6) $A\varepsilon f\therefore$ [5, 3]

$[\exists A].\mathsf{i}(Af)$ [*D7*, 4, 6, 1]

D11 $[f]:f\varepsilon\beta.\equiv.[\exists A].\mathsf{i}(Af)$ [*L58*; *D7*]

L59 $[ABC]:A\varepsilon\mathbf{R}\{BC\}.\supset.B\neq C$

PF $[ABC]:\mathrm{Hp}(1).\supset.$

2) $\mathsf{c}(ABC).$ [*D4*, 1]

3) $\mathsf{r}(BC).$ [*D3*, 2]

$B\neq C$ [*L19*, 3]

L60 $[ABC]:A\varepsilon\mathbf{R}\{BC\}.\supset.\mathsf{k}(ABC)$

PF $[ABC]:\mathrm{Hp}(1).\supset.$

2) $\mathsf{r}(BC).$ [*D4*, *D3*, 1]

3) $\mathbf{R}\{BC\}\varepsilon\beta.$ [*D6*, 2, *D4*]

4) $B\varepsilon\mathbf{R}\{BC\}.$ [*D4*, *D3*, 2]

5) $C\varepsilon\mathbf{R}\{BC\}.$ [*D4*, *D3*, 2]

6) $A\varepsilon\alpha.$ [*D1*, *L20*, *D4*, 1]

7) $B\varepsilon\alpha.$ [*L19*, 2]

8) $C\varepsilon\alpha.$ [*L19*, 2]

$\mathsf{k}(ABC)$ [*D9*, *D7*, 6, 1, 3, *D7*, 7, 4, 3, *D7*, 8, 5, 3]

L61 $[ABC]:B\neq C.\mathsf{k}(ABC).\supset.A\varepsilon\mathbf{R}\{BC\}$

PF $[ABC]\therefore\mathrm{Hp}(1\text{-}2).\supset:$

$[\exists f]:$

3) $\mathsf{i}(Af).$

4) $\mathsf{i}(Bf).$ [*D9*, 2]

5) $\mathsf{i}(Cf).$

6) $f=\mathbf{R}\{BC\}:$ [*L46*, 1, 4, 5]

7) $\mathsf{i}(A\mathbf{R}\{BC\}).$ [3, 6]

$A\varepsilon\mathbf{R}\{BC\}$ [*D7*, 7]

D12 $[ABC]:A\varepsilon\mathbf{R}\{BC\}.\equiv.B\neq C.\mathsf{k}(ABC)$ [*L59*, *L60*; *L61*]

1.4 SYSTEM (A) IMPLIES SYSTEM (A*)

M1 $[f]:f\varepsilon\beta.\supset.\mathsf{p}(ff)$ [*D8*]

M2 $[fg]:\mathsf{p}(fg).\equiv.\mathsf{p}(gf)$ [*D8*]

M3 $[fgh]:\mathsf{p}(fg).\mathsf{p}(gh).\sim(\mathsf{p}(fh)).\supset.\mathsf{p}(fh)$

PF $[fgh]:\mathrm{Hp}(1\text{-}3).\supset.$

4) $g\varepsilon\beta.$ [*D8*, 1]

5) $\mathsf{p}(gf).$ [*M2*, 1]

$[\exists A].$

6) $\mathsf{i}(Af).$

7) $\mathsf{i}(Ah).$ [*D8*, 3, 1, 2]

8) $f\neq h.$

9) $A \varepsilon \alpha .$ [*D10*, 6]
10) $f = h.$ [*AII*, 9, 4, 6, 5, 7, 2]
$\mathsf{p}(fh)$ [8, 10]

M4 $[Afghj] : \mathsf{i}(Af) . \mathsf{i}(Ag) . f \neq g . \mathsf{p}(fh) . \mathsf{p}(gj) . \mathsf{p}(hj) . \supset . \sim(\mathsf{p}(hj))$
PF $[Afghj] : \mathrm{Hp}(1\text{-}6) . \supset .$
7) $\mathsf{p}(jh).$ [*M2*, 6]
8) $\mathsf{p}(gh).$ [*M3*, 5, 7]
9) $\mathsf{p}(hg).$ [*M2*, 8]
10) $\mathsf{p}(fg).$ [*M3*, 4, 9]
11) $\sim([\exists A] . \mathsf{i}(Af) . \mathsf{i}(Ag)).$ [*D8*, 10, 3]
$\sim(\mathsf{p}(hj))$ [11, 1, 2]

A2 $[ABCD] : \mathsf{t}(ABCD) . \supset . A \neq B$ [*D13*]
A3(a) $[ABCD] : \mathsf{t}(ABCD) . \supset . \mathsf{t}(DCAB)$ [*D13*, *M2*]
A3(b) $[ABCD] : \mathsf{t}(ABCD) . \supset . \mathsf{t}(ABDC)$ [*D13*]

Since *A3(a)* and *A3(b)* hold if follows that *L1*-*L12* of **1.3** are valid in **(A)** by the same reasoning. *L14* follows from *D13* and *D8*, so with *A2* and the reasoning of **1.3**, *L13*-*L17* are also valid in **(A)**. For brevity in the following *A3* and *A2* will be cited as reasons for any of *L1*-*L12* and *L13*-*L17* respectively.

M5 $[fgh] : \mathsf{p}(fg) . \sim(\mathsf{p}(fh)) . \mathsf{p}(gh) . \supset . \sim(\mathsf{p}(gh))$
PF $[fgh] : \mathrm{Hp}(1\text{-}3) . \supset .$
4) $\mathsf{p}(fh).$ [*M3*, 1, 3]
$\sim(\mathsf{p}(gh))$ [2, 4]

M6 $[ABC] : A \varepsilon \alpha . B \varepsilon \alpha . C \varepsilon \alpha . A \neq B . C \neq A . \sim(\mathsf{k}(ABC)) . \supset . \sim(\mathsf{p}(\mathbf{R}\{AB\}\mathbf{R}\{AC\}))$
PF $[ABC] : \mathrm{Hp}(1\text{-}6) . \supset .$
7) $\mathsf{i}(A\mathbf{R}\{AB\}).$ [*AI*, 1, 2, 4]
8) $\mathsf{i}(B\mathbf{R}\{AB\}).$ [*AI*, 1, 2, 4]
9) $\mathsf{i}(A\mathbf{R}\{AC\}).$ [*AI*, 1, 3, 5]
10) $\mathsf{i}(C\mathbf{R}\{AC\}).$ [*AI*, 1, 3, 5]
11) $\mathbf{R}\{AB\} \neq \mathbf{R}\{AC\}.$ [*D9*, 6, 7, 8, 10]
$\sim(\mathsf{p}(\mathbf{R}\{AB\}\mathbf{R}\{AC\}))$ [*D8*, 11, 7, 9]

M7 $[ABCf] : \mathsf{i}(Af) . B \varepsilon \alpha . C \varepsilon \alpha . \sim(\mathsf{k}(ABC)) . \mathsf{p}(f\mathbf{R}\{BC\}) . B \neq C .$
$A \neq B . \supset . [\exists D] . A \neq D . \mathsf{i}(Df)$
PF $[ABCf] :: \mathrm{Hp}(1\text{-}7) . \supset .\therefore$
8) $\mathsf{i}(B\mathbf{R}\{BC\}).$ [*AI*, 2, 3, 6]
9) $\mathsf{i}(C\mathbf{R}\{BC\}) .\therefore$ [*AI*, 2, 3, 6]
$[\exists g] .\therefore$
10) $\mathsf{i}(Ag).$
11) $\mathsf{i}(Bg).$ } [*AI*, *D10*, 1, 2, 7]
12) $g \neq \mathbf{R}\{BC\} :$ [*D9*, 4, 10, 11, 9]
$[\exists h] :$
13) $\mathsf{p}(gh).$
14) $\mathsf{i}(Ch).$ } [*AII*, 3, *D11*, 10]
15) $\sim(\mathsf{p}(hf)).$ [*M4*, 11, 8, 12, 13, *M2*, 5]
16) $h \neq g.$ [*D9*, 4, 10, 11, 14]
$[\exists D].$

17) $\mathsf{i}(Df)$.
18) $\mathsf{i}(Dh)$. [*D8*, 15, *D11*, 1, 14]
19) $A \neq D$.·. [*D8*, 13, 16, 10, 18]
$[\exists D] . A \neq D . \mathsf{i}(Df)$ [19, 17]

M8 $[ABCf] : \mathsf{i}(Af) . B \varepsilon \alpha . C \varepsilon \alpha . \sim(\mathsf{k}(ABC)) . \sim(\mathsf{p}(f\mathbf{R}\{BC\}) . B \neq C . \supset .$
$[\exists D] . A \neq D . \mathsf{i}(Df)$

PF $[ABCf] : \mathrm{Hp}(1\text{-}6) . \supset .$
7) $\mathsf{i}(B\mathbf{R}\{BC\})$. [*AI*, 2, 3, 6]
8) $\mathsf{i}(C\mathbf{R}\{BC\})$. [*AI*, 2, 3, 6]
$[\exists D]$.
9) $\mathsf{i}(Df)$.
10) $\mathsf{i}(D\mathbf{R}\{BC\})$. [*D8*, 5, *D11*, 9, 10]
11) $A \neq D$. [*D9*, 4, 10, 7, 8]
$[\exists D] . A \neq D . \mathsf{i}(Df)$ [11, 9]

M9 $[AB] : \mathbf{R}\{AB\} = \mathbf{R}\{BA\}$ [*D0*, *D12*, *D9*]

M10 $[ABCDf] : \mathsf{i}(Af) . B \neq C . C \neq D . D \neq B . \sim(\mathsf{k}(BCD)) . \mathsf{p}(f\mathbf{R}\{BC\}) .$
$\sim(\mathsf{i}(Df)) . B \varepsilon \alpha . C \varepsilon \alpha . D \varepsilon \alpha . \supset . [\exists E] . A \neq E . \mathsf{i}(Ef)$

PF $[ABCDf]$ ∴ $\mathrm{Hp}(1\text{-}10) . \supset ::$
11) $\mathsf{i}(B\mathbf{R}\{BD\})$. [*AI*, 8, 10, 4]
12) $\mathsf{i}(D\mathbf{R}\{BD\})$. [*AI*, 8, 10, 4]
13) $\mathsf{i}(C\mathbf{R}\{CD\})$. [*AI*, 9, 10, 3]
14) $\mathsf{i}(D\mathbf{R}\{CD\})$. [*AI*, 9, 10, 3]
15) $\sim(\mathsf{p}(\mathbf{R}\{BD\}\mathbf{R}\{BC\}))$. [*M6*, 8, 10, 9, 4, 2, *D9*, 5]
16) $\sim(\mathsf{p}(\mathbf{R}\{CD\}\mathbf{R}\{CB\}))$. [*M6*, 9, 10, 8, 3, 2, *D9*, 5]
17) $\sim(\mathsf{p}(\mathbf{R}\{CD\}\mathbf{R}\{BC\}))$. [*M9*, 16]
18) $\sim(\mathsf{p}(\mathbf{R}\{BD\}f)$. [*M2*, *M5*, *M2*, 6, *M2*, 15]
19) $\sim(\mathsf{p}(\mathbf{R}\{CD\}f)$. [*M2*, *M5*, *M2*, 6, *M2*, 17]
20) $\mathbf{R}\{BD\} \neq \mathbf{R}\{CD\} ::$ [*D9*, 5, 11, 13, 14]
$[\exists E] ::$
21) $\mathsf{i}(E\mathbf{R}\{BD\})$.
22) $\mathsf{i}(Ef)$. [*D8*, 18, *D11*, 1, 12]
23) $D \neq E$.·. [7, 22]
$[\exists F]$.·.
24) $\mathsf{i}(F\mathbf{R}\{CD\})$.
25) $\mathsf{i}(Ff)$. [*D8*, 19, *D11*, 1, 13]
26) $D \neq F$. [7, 25]
27) $E \neq F$: [*AI*, 12, 21, 23, 14, 24, 26, 20]
28) $A \neq E . \vee . A \neq F ::$ [27]
$[\exists E] . A \neq E . \mathsf{i}(Ef)$ [28, 22, 25]

M11 $[BCDEFf] : \mathsf{i}(E\mathbf{R}\{BC\}) . \mathsf{i}(F\mathbf{R}\{BD\}) . \mathsf{i}(Ff) . \sim(\mathsf{i}(Bf)) . B \neq C .$
$D \neq B . B \varepsilon \alpha . C \varepsilon \alpha . D \varepsilon \alpha . \sim(\mathsf{k}(BCD)) . E = F . \supset . E \neq F$

PF $[BCDEFf] : \mathrm{Hp}(1\text{-}11) . \supset .$
12) $\mathbf{R}\{BC\} \neq \mathbf{R}\{BD\}$. [*D9*, 10, *AI*, 7, 8, 9, 5, 6]
13) $\mathsf{i}(F\mathbf{R}\{BC\})$. [1, 11]
14) $\mathsf{i}(B\mathbf{R}\{BC\})$. [*AI*, 7, 8, 5]
15) $\mathsf{i}(B\mathbf{R}\{BD\})$. [*AI*, 7, 9, 6]
16) $B \neq F$. [3, 4]
17) $\mathbf{R}\{BC\} = \mathbf{R}\{BD\}$. [*AI*, *D10*, 2, 15, 16, 13, 14]
$E \neq F$ [12, 17]

M12 $[ABCDf]:\mathsf{i}(Af).B\neq C.C\neq D.D\neq B.\sim(\mathsf{k}(BCD)).\sim(\mathsf{i}(Bf)).\sim(\mathsf{i}(Cf)).$
$\sim(\mathsf{p}(f\mathsf{R}\{BC\})).B\varepsilon\alpha.C\varepsilon\alpha.D\varepsilon\alpha.\supset.[\exists E].A\neq E.\mathsf{i}(Ef)$

PF $[ABCDf]\therefore\mathrm{Hp}(1\text{-}11).\supset:$

12) $f\varepsilon\beta.$ [*D11*, 1]

13) $\mathsf{R}\{BC\}\varepsilon\beta.$ [*D11*, *AI*, 9, 10, 2]

14) $\mathsf{R}\{BD\}\varepsilon\beta:$ [*D11*, *AI*, 9, 11, 4]

15) $\mathsf{p}(f\mathsf{R}\{BD\}).\mathsf{v}.[\exists F].\mathsf{i}(Ff).\mathsf{i}(F\mathsf{R}\{BD\}):$ [*D8*, 12, 14]

$[\exists E]:$

16) $\mathsf{i}(Ef).$

17) $\mathsf{i}(E\mathsf{R}\{BC\}).$ [*D8*, 8, 12, 13]

$[\exists F].$

18) $E\neq F.$

19) $\mathsf{i}(Ff):$

[15; *M10*, 1, 4, 3, 2, *D9*, 5, 7, 9, 11, 10; *M11*, 17, 6, 2, 4, 9, 10, 11, 5]

$[\exists E].A\neq E.\mathsf{i}(Ef)$ [16, 18, 19]

M13 $[f]:f\varepsilon\beta.\supset.[\exists AB].A\neq B.\mathsf{i}(Af).\mathsf{i}(Bf)$

PF $[f]::\mathrm{Hp}(1).\supset\therefore$

$[\exists A]\therefore$

2) $\mathsf{i}(Af):$ [*D11*, 1]

$[\exists BCD]:$

3) $B\varepsilon\alpha.$

4) $C\varepsilon\alpha.$

5) $D\varepsilon\alpha.$

6) $B\neq C.$ [*AIII*]

7) $C\neq D.$

8) $D\neq B.$

9) $\sim(\mathsf{k}(BCD)):$

10) $A\neq B.A\neq C.A=D.\mathsf{v}.A\neq C.A\neq D.A=B.\mathsf{v}.A\neq B.A\neq D.$
$A=C.\mathsf{v}.A\neq B.A\neq C.A\neq D:$ [6, 7, 8, *D10*, 2]

11) $\mathsf{i}(Bf).\mathsf{v}.\sim(\mathsf{i}(Bf)):\mathsf{i}(Cf).\mathsf{v}.\sim(\mathsf{i}(Cf)):$
$\mathsf{i}(Df).\mathsf{v}.\sim(\mathsf{i}(Df))\therefore$ [*Logic*]

$[\exists AB].A\neq B.\mathsf{i}(Af).\mathsf{i}(Bf)$

[10, 11; *M7*, 2, 3, 4, *D9*, 9, 6, *M8*, 2, 3, 4, *D9*, 9, 6; *M7*, 2, 4. 5, *D9*, 9, 7, *M8*, 2, 4, 5, *D9*, 9, 7; *M7*, 2, 3, 5, *D9*, 9, 8, *M8*, 2, 3, 5, *D9*, 9, 8; *M10*, 2, 6, 7, 8, 9, *M12*, 2, 6, 7, 8, 9, 3, 4, 5]

M14 $[Af]:\mathsf{i}(Af).\supset.A\varepsilon f$

PF $[Af]:\mathrm{Hp}(1).\supset.$

$[\exists BC].$

2) $B\neq C.$

3) $\mathsf{i}(Bf).$ [*M13*, *D11*, 1]

4) $\mathsf{i}(Cf).$

5) $\mathsf{k}(ABC).$ [*D9*, 1, 3, 4]

6) $A\varepsilon\mathsf{R}\{BC\}.$ [*D12*, 2, 5]

$A\varepsilon f$ [*AI*, *D10*, 3, 4, 2, *D0*, 6]

M15 $[Af]:A\varepsilon f.f\varepsilon\beta.\supset.\mathsf{i}(Af)$

PF $[Af]\therefore\mathrm{Hp}(1\text{-}2).\supset:$

$[\exists BC]:$

3) $\mathsf{i}(Bf)$.
4) $\mathsf{i}(Cf)$. [$M13$, 2]
5) $B \dotplus C$.
6) $f = \mathsf{R}\{BC\}$. [AI, $D10$, 3, 4, 5]
7) $A \varepsilon \bar{\mathsf{R}}\{BC\}$. [$D0$, 1, 6]
8) $\mathsf{k}(ABC)$. [$D12$, 7]
$[\exists g]$.
9) $\mathsf{i}(Ag)$.
10) $\mathsf{i}(Bg)$. [$D9$, 8]
11) $\mathsf{i}(Cg)$.
12) $g = f$: [AI, $D10$, 10, 11, 5, 3, 4]
$\mathsf{i}(Af)$ [9, 12]

$D7$ $[Af]:\mathsf{i}(Af).\equiv.A\varepsilon\alpha.A\varepsilon f.f\varepsilon\beta$ [$D10$, $M14$, $D11$; $M15$]

$M16$ $[ABC]:A\varepsilon\alpha.B\varepsilon\alpha.C\varepsilon\alpha.A \dotplus B.A \dotplus C.B \dotplus C.\sim(\mathsf{k}(ABC)).\supset.[\exists D].$
$\mathsf{t}(ABCD)$

PF $[ABC]::\mathrm{Hp}(1\text{-}7).\supset\because$
$[\exists f]\because$
8) $\mathsf{i}(Af)$.
9) $\mathsf{i}(Bf)$. [AI, 1, 2, 4]
10) $f\varepsilon\beta$. [$D11$, 8]
11) $\sim(\mathsf{i}(Cf))::$ [$D9$, 7, 8, 9]
$[\exists g]::$
12) $\mathsf{i}(Ag)$.
13) $\mathsf{i}(Cg)$. [AI, 1, 3, 5]
14) $C\varepsilon g$. [$D7$, 13]
15) $\sim(C\varepsilon f)$. [$D7$, 11, 3, 10]
16) $f \dotplus g\therefore$ [$D0$, 14, 15]
$[\exists h]\therefore$
17) $\mathsf{i}(Ch)$.
18) $\mathsf{p}(fh)$. [AII, 3, 10]
19) $C\varepsilon h$. [$D7$, 17]
20) $f \dotplus h$. [$D0$, 15, 19]
21) $g\varepsilon\beta$. [$D7$, 12]
22) $\sim(\mathsf{i}(Bg))$. [$D9$, 7, 12, 13]
23) $\sim(B\varepsilon g)$: [$D7$, 22, 2, 21]
$[\exists j]$:
24) $\mathsf{i}(Bj)$.
25) $\mathsf{p}(gj)$. [AII, 2, 21]
26) $g \dotplus j$. [$D0$, 23, 24]
27) $\sim(\mathsf{p}(hj))$. [$M4$, 8, 12, 16, 18, 25]
28) $\sim(C\varepsilon j)$.
[$D8$, 25, 26, $D7$, 3, 13, 21, $D7$, 3, $D11$, 24]
$[\exists D]$:
29) $\mathsf{i}(Dh)$.
30) $\mathsf{i}(Dj)$. [$D8$, 27, $D11$, 17, 24]
31) $D\varepsilon j$. [$D7$, 30]
32) $D \dotplus C\because$ [28, 31]
$[\exists D].\mathsf{t}(ABCD)$ [$D13$, 4, 32, 18, 20, 8, 9, 17, 29]

M17 $[ABCD] \therefore A\varepsilon\alpha . \sim(\mathsf{k}(BCD)) . \mathsf{k}(ABC) . \mathsf{k}(ACD) . \mathsf{k}(ABD) . \supset :$
$\sim(\mathsf{k}(ABC)) . \vee . \sim(\mathsf{k}(ACD)) . \vee . \sim(\mathsf{k}(ABD))$
PF $[ABCD] \therefore \mathrm{Hp}(1\text{-}5) . \supset :$
6) $A \neq B :$ [2, 4]
$[\exists f] :$
7) $\mathsf{i}(Af) .$
8) $\mathsf{i}(Bf) .$ [*D9*, 3]
9) $\mathsf{i}(Cf) .$
$[\exists g] .$
10) $\mathsf{i}(Ag) .$
11) $\mathsf{i}(Bg) .$ [*D9*, 5]
12) $\mathsf{i}(Dg) .$
13) $f = g :$ [*AI*, 1, *D7*, 8, 6, 7, 8, 10, 11]
14) $\mathsf{k}(BCD) :$ [*D9*, 13, 8, 9, 12]
$\sim(\mathsf{k}(ABC)) . \vee . \sim(\mathsf{k}(ACD)) . \vee . \sim(\mathsf{k}(ABD))$ [2, 14]
M18 $[AB] : A\varepsilon\alpha . B\varepsilon\alpha . A \neq B . \supset . \mathsf{k}(ABA)$
PF $[AB] : \mathrm{Hp}(1\text{-}3) . \supset .$
4) $\mathsf{i}(A\mathsf{R}\{AB\}) .$ [*AI*, 1, 2, 3]
5) $\mathsf{i}(B\mathsf{R}\{AB\}) .$ [*AI*, 1, 2, 3]
$\mathsf{k}(ABA)$ [*D9*, 4, 5, 4]
M19 $[AB] : A\varepsilon\alpha . B\varepsilon\alpha . A \neq B . \supset . \mathsf{k}(AAB)$ [*D9*, *M18*]
M20 $[A] : A\varepsilon\alpha . \supset . [\exists BCD] . \mathsf{t}(ABCD)$
PF $[A] :: \mathrm{Hp}(1) . \supset \therefore$
$[\exists BCD] \therefore$
2) $B\varepsilon\alpha .$
3) $C\varepsilon\alpha .$
4) $D\varepsilon\alpha .$
5) $B \neq C .$ [*AIII*]
6) $B \neq D .$
7) $C \neq D .$
8) $\sim(\mathsf{k}(BCD)) :$
9) $\sim(\mathsf{k}(ABC)) . \vee . \sim(\mathsf{k}(ACD)) . \vee . \sim(\mathsf{k}(ABD)) :$ [*M17*, 1, 8]
10) $A \neq B . A \neq C . A = D . \vee . A \neq B . A = C . A \neq D . \vee . A = B . A \neq C . A \neq D . \vee .$
$A \neq B . A \neq C . A \neq D . \therefore$ [5, 6, 7]
$\mathsf{t}(ABCD)$ [*M16*, 10, 9; 1, 2, 3, 5, *M18*, 1, 3, *M18*, 1, 2; *M18*, 1, 2, *M19*, 1, 4, 1, 2, 4, 6; *M19*, 1, 3, 1, 3, 4, 7, *M19*, 1, 4, 1, 2, 3, 5, 1, 3, 4, 7, 1, 2, 4, 6]
M21 $[ABCD] : \mathsf{t}(ABCD) . \supset . A\varepsilon\alpha .$ [*D13*, *D10*]
D1 $[ABCD] : A\varepsilon\alpha . \equiv . [\exists BCD] . \mathsf{t}(ABCD)$ [*M20*, *M21*]
A1 $[\exists ABCD] . \mathsf{t}(ABCD)$ [*M16*, *AIII*]
M22 $[ABCDEf] \therefore \mathsf{i}(Af) . \mathsf{i}(Bf) . A \neq B . \sim(\mathsf{k}(CDE)) . \mathsf{k}(ABC) . \mathsf{k}(ABD) .$
$\mathsf{k}(ABE) . \supset : \sim(\mathsf{k}(ABC)) . \vee . \sim(\mathsf{k}(ABD)) . \vee . \sim(\mathsf{k}(ABE))$
PF $[ABCDEf] :: \mathrm{Hp}(1\text{-}7) . \supset \therefore$
$[\exists g] \therefore$
8) $\mathsf{i}(Ag) .$
9) $\mathsf{i}(Bg) .$ [*D9*, 5]
10) $\mathsf{i}(Cg) .$
11) $g = f :$ [*AI*, *D7*, 1, 2, 3, 8, 9]
$[\exists h] :$

12) $\mathsf{i}(Ah)$. [D9, 6]
13) $\mathsf{i}(Bh)$.
14) $\mathsf{i}(Dh)$.
15) $h = f$. [AI, D7, 1, 2, 3, 12, 13]
$[\exists j]$.
16) $\mathsf{i}(Aj)$. [D9, 7]
17) $\mathsf{i}(Bj)$.
18) $\mathsf{i}(Dj)$.
19) $j = f$.∴ [AI, D7, 1, 2, 3, 16, 17]
20) $\mathsf{k}(CDE)$: [D9, 10, 12, 18, 11, 15, 19]
$\sim(\mathsf{k}(ABC)) . \mathsf{v} . \sim(\mathsf{k}(ABD)) . \mathsf{v} . \sim(\mathsf{k}(ABE))$ [4, 20]

M23 $[ABf] : \mathsf{i}(Af) . \mathsf{i}(Bf) . A \neq B . \supset . [\exists C] . \sim(\mathsf{i}(Cf)) . C \varepsilon \alpha$
PF $[ABf]$ :: Hp(1-3) . ⊃ ∴
$[\exists CDE]$ ∴
4) $C \varepsilon \alpha$. [AIII]
5) $D \varepsilon \alpha$.
6) $E \varepsilon \alpha$.
7) $\sim(\mathsf{k}(CDE))$:
8) $\sim(\mathsf{k}(ABC)) . \mathsf{v} . \sim(\mathsf{k}(ABD)) . \mathsf{v} . \sim(\mathsf{k}(ABE))$: [M22, 1, 2, 3, 7]
9) $\sim(\mathsf{i}(Cf)) . \mathsf{v} . \sim(\mathsf{i}(Df)) . \mathsf{v} . \sim(\mathsf{i}(Ef))$ ∴ [8, D9, 1, 2]
$[\exists C] . \sim(\mathsf{i}(Cf)) . C \varepsilon \alpha$ [9, 4, 5, 6]

M24 $[ABf] : A \neq B . \mathsf{i}(Af) . \mathsf{i}(Bf) . \supset . [\exists CD] . \mathsf{t}(ABCD)$
PF $[ABf]$ :: Hp(1-3) . ⊃ ∴
$[\exists E]$ ∴
4) $\sim(\mathsf{i}(Ef))$. [M23, 2, 3, 1]
5) $E \varepsilon \alpha$:
$[\exists g]$:
6) $\mathsf{i}(Eg)$. [AII, 5, D11, 2]
7) $\mathsf{p}(fg)$.
8) $f \neq g$. [4, 6]
$[\exists CD]$.
9) $C \neq D$. [M13, D8, 7]
10) $\mathsf{i}(Cg)$.
11) $\mathsf{i}(Dg)$.∴
$\mathsf{t}(ABCD)$ [D13, 1, 9, 7, 8, 2, 3, 10, 11]

M25 $[ABC] : A \varepsilon \mathsf{R}\{BC\} . \supset . \mathsf{c}(ABC)$
PF $[ABC]$ ⸬ Hp(1) . ⊃ ∵
2) $B \neq C$. [D12, 1]
3) $\mathsf{k}(ABC)$ ∵
$[\exists f]$ ∵
4) $\mathsf{i}(Af)$. [D9, 3]
5) $\mathsf{i}(Bf)$.
6) $\mathsf{i}(Cf)$ ::
$[\exists DE]$ ::
7) $\mathsf{t}(BCDE)$. [M24, 2, 5, 6]
8) $\mathsf{r}(BC)$. [D2, 7]
9) $D \neq E$ ∴ [D13, 7]
$[\exists gh]$ ∴

10) $\mathsf{p}(gh).$

11) $g \neq h.$

12) $\mathsf{i}(Bg).$

13) $\mathsf{i}(Cg).$ [*D13*, 7]

14) $\mathsf{i}(Dh).$

15) $\mathsf{i}(Eh).$

16) $g = f.$ [*AI*, *D10*, 5, 6, 2, 12, 13]

17) $\mathsf{i}(Ag):$ [4, 16]

18) $A = B . \vee . A = C . \vee . \mathsf{t}(BCDE) . \mathsf{t}(BADE) . \mathsf{t}(CADE) \therefore$ [*D13*, 2, 9, 10, 11, 12, 13, 17, 14, 15]

$\mathsf{c}(ABC)$ [*D3*, 8, 18]

M26 $[ABC] : \mathsf{c}(ABC) . \supset . A \varepsilon \mathsf{R}\{BC\}$

PF $[ABC] \therefore \mathrm{Hp}(1) . \supset :$

2) $\mathsf{r}(BC):$ [*D3*, 1]

3) $A = B . \vee . A = C . \vee . [\exists XY] . \mathsf{t}(BCXY) . \mathsf{t}(BAXY) . \mathsf{t}(CAXY):$

4) $B \neq C.$ [*A2*, *D2*, *A3*, 2]

5) $\mathsf{i}(B\mathsf{R}\{BC\}).$ [*AI*, *D1*, *D2*, *A3*, 2, 4]

6) $\mathsf{i}(C\mathsf{R}\{BC\}).$ [*AI*, *D1*, *D2*, *A3*, 2, 4]

7) $\mathsf{i}(A\mathsf{R}\{BC\}).$ [3, 5, 6, *D13*, *AI*, *AII*, *D1*, *A3*, *D11*]

8) $\mathsf{k}(ABC):$ [*D9*, 7, 5, 6]

$A \varepsilon \mathsf{R}\{BC\}$ [*D12*, 4, 8]

D4 $[ABC] : A \varepsilon \mathsf{R}\{BC\} . \equiv . \mathsf{c}(ABC)$ [*M25*; *M26*]

M27 $[ABf] \therefore \mathsf{r}(AB) : [K] : \mathsf{c}(KAB) . \equiv . K \varepsilon f : \supset . f \varepsilon \beta$

PF $[ABf] :: \mathrm{Hp}(1\text{-}2) : \supset :$

$[\exists CD] \therefore$

3) $\mathsf{t}(ABCD) . \vee . \mathsf{t}(ACBD) . \vee . \mathsf{t}(CBAD):$ [*D2*, 1]

4) $\mathsf{t}(ABCD) . \vee . \mathsf{t}(ACBD) . \vee . \mathsf{t}(ADCB):$ [3, *A3*]

5) $A \neq B.$ [4, *A2*]

6) $A \varepsilon \alpha.$ [*D1*, *A3*, 3]

7) $B \varepsilon \alpha.$ [*D1*, *A3*, 4]

$[\exists g].$

8) $\mathsf{i}(Ag).$

9) $\mathsf{i}(Bg).$ [*AI*, 6, 7, 5]

10) $g = \mathsf{R}\{AB\}.$

11) $g = f.$ [*D0*, *D4*, 2, 10]

12) $g \varepsilon \beta \therefore$ [*D7*, 8]

$f \varepsilon \beta$ [11, 12]

M28 $[DEf] :: \mathsf{i}(Df) . \mathsf{i}(Ef) . D \neq E . \supset \therefore [\exists AB] \therefore \mathsf{r}(AB) : [C] : \mathsf{c}(CAB) . \equiv . C \varepsilon f$

PF $[DEf] :: \mathrm{Hp}(1\text{-}3) . \supset \therefore$

4) $D \varepsilon \alpha.$ [*D10*, 1]

5) $E \varepsilon \alpha.$ [*D10*, 2]

6) $D \varepsilon \mathsf{R}\{DE\}.$ [*D12*, 3, *D9*, 1, 2]

7) $\mathsf{r}(DE) \therefore$ [*D3*, *D4*, 6]

$[\exists g] \therefore$

8) $\mathsf{i}(Dg).$

9) $\mathsf{i}(Eg).$ [*AI*, 4, 5, 3]

10) $g = \mathsf{R}\{DE\}:$

11) $g = f.$ [*AI*, 4, 5, 3, 8, 9, 1, 2]

	12)	$f=\mathbf{R}\{DE\}:$	[10, 11]
	13)	$[K]:\mathbf{c}(KDE).\equiv.K\varepsilon f\therefore$	[$D4$, $D12$, $D0$]
		$[\exists AB]\therefore\mathbf{r}(AB):[C]:\mathbf{c}(CAB).\equiv.C\varepsilon f$	[7, 13]
$D6$		$[f]::f\varepsilon\beta.\equiv\therefore[\exists AB]\therefore\mathbf{r}(AB):[C]:\mathbf{c}(CAB).\equiv.C\varepsilon f$	[$M28$, $M13$; $M27$]
$M29$		$[ABC]:\sim(\mathbf{c}(CAB)).A\neq B.\supset.\sim(\mathbf{k}(ABC))$	
PF		$[ABC]:\mathrm{Hp}(1\text{-}2).\supset.$	
	3)	$\sim(C\varepsilon\mathbf{R}\{BC\}).$	[$D4$, 1]
	4)	$\sim(\mathbf{k}(CAB)).$	[$D12$, 3, 2]
		$\sim(\mathbf{k}(ABC))$	[$D9$, 4]
$M30$		$[ABCRSTUVWXYZ]:\mathbf{t}(ARST).\mathbf{t}(BUVW).\mathbf{t}(CXYZ).\sim(\mathbf{c}(CAB)).A\neq B.\supset.[\exists D].\mathbf{t}(ABCD)$	
PF		$[ABCRSTUVWXYZ]\because\mathrm{Hp}(1\text{-}5).\supset::$	
	6)	$A\varepsilon\alpha.$	[$D1$, 1]
	7)	$B\varepsilon\alpha.$	[$D1$, 2]
	8)	$C\varepsilon\alpha::$	[$D1$, 3]
		$[\exists f]::$	
	9)	$\mathbf{i}(Af).$	[$A1$, 6, 7, 5]
	10)	$\mathbf{i}(Bf).$	
	11)	$f\varepsilon\beta\therefore$	[$D11$, 10]
		$[\exists DE]\therefore$	
	12)	$\mathbf{r}(DE):$	[$D6$, 11]
	13)	$[K]:\mathbf{c}(KDE).\equiv.K\varepsilon f:$	
	14)	$\mathbf{R}\{DE\}=f.$	[$D0$, 13, $D4$]
	15)	$\mathbf{i}(A\mathbf{R}\{DE\}).$	[9, 14]
	16)	$\mathbf{i}(B\mathbf{R}\{DE\}).$	[10, 14]
	17)	$\mathbf{R}\{AB\}=\mathbf{R}\{DE\}.$	[$A1$, 6, 7, 5, 15, 16]
	18)	$A\varepsilon f.$	[$D7$, 9]
	19)	$\mathbf{c}(ADE).$	[13, 18]
	20)	$A\varepsilon\mathbf{R}\{DE\}\therefore$	[$D4$, 19]
	21)	$A\varepsilon\mathbf{R}\{AB\}.$	[20, 17]
	22)	$\mathbf{c}(AAB).$	[$D4$, 21]
	23)	$\mathbf{r}(AB).$	[$D3$, 22]
	24)	$C\neq A.$	[$D3$, 4, 23]
	25)	$C\neq B.$	[$D3$, 4, 23]
	26)	$\sim(\mathbf{k}(ABC)).$	[$M29$, 4, 25]
		$[\exists D].\mathbf{t}(ABCD)$	[$M16$, 6, 7, 8, 5, 24, 25, 26]
$M31$		$[ABCD]:\mathbf{t}(ABCD).\supset.\sim(\mathbf{c}(CAB))$	
PF		$[ABCD]:\mathrm{Hp}(1).\supset.$	
		$[\exists fg].$	
	2)	$\mathbf{i}(Af).$	[$D13$, 1]
	3)	$\mathbf{i}(Bf).$	
	4)	$\mathbf{i}(Cg).$	
	5)	$\mathbf{p}(fg).$	
	6)	$f\neq g.$	
	7)	$\mathbf{R}\{AB\}=f.$	[$A1$, $D7$, 2, 3, $A2$, 1]
	8)	$\sim(\mathbf{i}(Cf)).$	[$D8$, 5, 6, 4]
	9)	$C\varepsilon\alpha.$	[$D10$, 4]

10) $\sim(C\varepsilon f)$. [$D7$, 8, 9, $D7$, 2]
$\sim(\mathsf{c}(CAB))$ [$D4$, 10, 7]

A4 $[ABCRSTUVWXYZ]::\mathsf{t}(ARST).\mathsf{t}(BUVW).\mathsf{t}(CXYZ).\supset.\therefore$
$\sim(\mathsf{c}(CAB)).\equiv.[\exists D].\mathsf{t}(ABCD).\vee.A=B$ [$M30$; $M31$, $D3$, $D2$, $A3$, $A2$]

L18-L20 now hold in (**A**), the proofs are the same as in 1.3.

A5 $[ABCMN]:A\neq B.\mathsf{c}(AMN).\mathsf{c}(BMN).\mathsf{c}(CMN).\supset.\mathsf{c}(CAB)$
PF $[ABCMN].\therefore.\mathrm{Hp}(1\text{-}4).\supset:$
5) $\mathsf{r}(MN)$. [$D3$, 2]
6) $\mathsf{R}\{MN\}\,\varepsilon\,\beta$. [$D6$, $D4$, 6]
7) $A\varepsilon\alpha$. [$D1$, $L20$, 2]
8) $B\varepsilon\alpha$. [$D1$, $L20$, 3]
9) $\mathsf{i}(A\mathsf{R}\{MN\})$. [$D7$, 8, $D4$, 2, 6]
10) $\mathsf{i}(B\mathsf{R}\{MN\})$. [$D7$, 9, $D4$, 3, 6]
11) $\mathsf{R}\{AB\}=\mathsf{R}\{MN\}$: [$AI$, 7, 8, 1, 9, 10]
12) $[K]:\mathsf{c}(KMN).\equiv.K\varepsilon\mathsf{R}\{AB\}$: [$D0$, $D4$, 11]
$\mathsf{c}(CAB)$ [$D4$, 4, 12]

M32 $[ABCDfg]:\mathsf{t}(ABCD).\mathsf{i}(Af).\mathsf{i}(Bf).\mathsf{i}(Cg).\supset.f\neq g$
PF $[ABCDfg].\therefore.\mathrm{Hp}(1\text{-}4).\supset:$
5) $\sim(\mathsf{c}(CAB))$: [$A4$, 1]
6) $[K]:\mathsf{c}(KAB).\equiv.K\varepsilon f$: [$D6$, $D7$, 2, AI, $D7$, 2, 3, $A2$, $D4$]
7) $\sim(C\varepsilon f)$. [6, 5]
8) $C\varepsilon g$. [$D7$, 4]
$f\neq g$ [7, 8]

A6 $[ABCDEFG].\mathsf{t}(ABCD).\mathsf{t}(ABEF).\mathsf{t}(CDEG).\supset.\mathsf{t}(CDEF)$
PF $[ABCDEFG].\therefore.\mathrm{Hp}(1\text{-}3).\supset:$
4) $A\neq B$. [$D13$, 1]
5) $C\neq D$. [$D13$, 1]
6) $E\neq F$. [$D13$, 2]
$[\exists fg]$:
7) $\mathsf{p}(fg)$.
8) $f\neq g$.
9) $\mathsf{i}(Af)$. [$D13$, 1]
10) $\mathsf{i}(Bf)$.
11) $\mathsf{i}(Cg)$.
12) $\mathsf{i}(Dg)$.
$[\exists hj]$.
13) $\mathsf{p}(hj)$.
14) $h\neq j$.
15) $\mathsf{i}(Ah)$.
16) $\mathsf{i}(Bh)$. [$D13$, 2]
17) $\mathsf{i}(Ej)$.
18) $\mathsf{i}(Fj)$.
19) $f=h$. [AI, $D7$, 9, 10, 4, 15, 16]
20) $\mathsf{p}(gj)$. [$M3$, $M2$, 19, 13, 7]
21) $g\neq j$: [$M32$, 3, 11, 12, 17]
$\mathsf{t}(CDEF)$ [$D13$, 5, 6, 20, 21, 11, 12, 17, 18]

M33		$[ABCD] \therefore \mathsf{t}(ABCD) . \supset : \mathsf{r}(AB) . \mathsf{r}(CD) : [M] : \mathsf{c}(MAB) . \supset . \sim(\mathsf{c}(MCD))$	
PF		$[ABCD] \therefore \mathrm{Hp}(1) . \supset :$	
	2)	$A \neq B.$	$[D13, 1]$
	3)	$C \neq D.$	$[D13, 1]$
		$[\exists fg]:$	
	4)	$\mathsf{p}(fg).$	
	5)	$f \neq g.$	
	6)	$\mathsf{i}(Af).$	$[D13, 1]$
	7)	$\mathsf{i}(Bf).$	
	8)	$\mathsf{i}(Cg).$	
	9)	$\mathsf{i}(Dg).$	
	10)	$\mathsf{R}\{AB\} = f.$	$[AI, D7, 6, 7, 2]$
	11)	$\mathsf{R}\{CD\} = g.$	$[AI, D7, 8, 9, 3]$
	12)	$\mathsf{r}(AB).$	$[D3, D4, 10, D7, 6]$
	13)	$\mathsf{r}(CD):$	$[D3, D4, 11, D7, 8]$
	14)	$[M] : \mathsf{c}(MAB) . \supset . \sim(\mathsf{c}(MCD)):$	
			$[D4, 10, D8, 4, 5, D7, 6, D0, 11, D4]$
		$\mathsf{r}(AB) . \mathsf{r}(CD) : [M] : \mathsf{c}(MAB) . \supset . \sim (\mathsf{c}(MCD)$	$[12, 13, 14]$
M34		$[ABCD] \therefore \mathsf{r}(AB) . \mathsf{r}(CD) : [M] : \mathsf{c}(MAB) . \supset . \sim(\mathsf{c}(MCD)) : \supset . \mathsf{t}(ABCD)$	
PF		$[ABCD] \therefore \mathrm{Hp}(1\text{-}3) : \supset .$	
	4)	$A \neq B.$	$[A2, D2, A3, 1]$
	5)	$C \neq D.$	$[A2, D2, A3, 2]$
	6)	$A \varepsilon \alpha .$	$[D1, D2, A3, 1]$
	7)	$B \varepsilon \alpha .$	$[D1, D2, A3, 1]$
	8)	$C \varepsilon \alpha .$	$[D1, D2, A3, 2]$
	9)	$D \varepsilon \alpha .$	$[D1, D2, A3, 2]$
	10)	$\mathsf{i}(A\mathsf{R}\{AB\}).$	$[AI, 6, 7, 4]$
	11)	$\mathsf{i}(B\mathsf{R}\{AB\}).$	$[AI, 6, 7, 4]$
	12)	$\mathsf{i}(C\mathsf{R}\{CD\}).$	$[AI, 8, 9, 5]$
	13)	$\mathsf{i}(D\mathsf{R}\{CD\}).$	$[AI, 8, 9, 5]$
	14)	$\mathsf{R}\{AB\} \neq \mathsf{R}\{CD\} .$	$[D0, D4, 3, D7, 10]$
	15))	$\mathsf{p}(\mathsf{R}\{AB\}\mathsf{R}\{CD\}).$	$[D8, D7, 10, 12, 14, D4, 3]$
		$\mathsf{t}(ABCD)$	$[D13, 4, 5, 15, 14, 10, 11, 12, 13]$
A7		$[ABCD] \therefore \mathsf{t}(ABCD) . \equiv : \mathsf{r}(AB) . \mathsf{r}(CD) : [M] : \mathsf{c}(MAB) . \supset .$	
		$\sim(\mathsf{c}(MCD))$	$[M33; M34]$
M35		$[ABCD] : \mathsf{t}(ABCD) . \supset . \mathsf{p}(\mathsf{R}\{AB\}\mathsf{R}\{CD\}) . \mathsf{R}\{AB\} \neq \mathsf{R}\{CD\} .$	
		$\mathsf{i}(A\mathsf{R}\{AB\}) . \mathsf{i}(B\mathsf{R}\{AB\}) . \mathsf{i}(C\mathsf{R}\{CD\}) . \mathsf{i}(D\mathsf{R}\{CD\})$	
PF		$[ABCD] : \mathrm{Hp}(1) . \supset .$	
	2)	$A \neq B.$	$[D13, 1]$
	3)	$C \neq D.$	$[D13, 1]$
		$[\exists fg].$	
	4)	$p(fg).$	
	5)	$f \neq g.$	
	6)	$\mathsf{i}(Af).$	$[D13, 1]$
	7)	$\mathsf{i}(Bf).$	
	8)	$\mathsf{i}(Cg).$	
	9)	$\mathsf{i}(Dg).$	

10) $A \varepsilon \alpha$. [$D7$, 6]
11) $B \varepsilon \alpha$. [$D7$, 7]
12) $C \varepsilon \alpha$. [$D7$, 8]
13) $D \varepsilon \alpha$. [$D7$, 9]
14) $\mathsf{R}\{AB\} = f$. [AI, 10, 11, 2]
15) $\mathsf{R}\{CD\} = g$. [AI, 12, 13, 3]
$\mathsf{p}(\mathsf{R}\{AB\}\mathsf{R}\{CD\}) . \mathsf{R}\{AB\} \neq \mathsf{R}\{CD\} . \mathsf{i}(A\mathsf{R}\{AB\})$.
$\mathsf{i}(B\mathsf{R}\{AB\}) . \mathsf{i}(C\mathsf{R}\{CD\}) . \mathsf{i}(D\mathsf{R}\{CD\})$ [4, 5, 6, 7, 8, 9, 14, 15]

M36 $[ABCD] : \mathsf{t}(ABCD) . \supset . \mathsf{i}(B\mathsf{R}\{BC\})$
PF $[ABCD] : \mathrm{Hp}(1) . \supset .$
2) $\mathsf{r}(CB)$. [$D2$, 1]
3) $B \varepsilon \alpha$. [$D1$, $A3$, 1]
4) $\mathsf{c}(BCB)$. [$D3$, 2]
5) $B \varepsilon \mathsf{R}\{CB\}$. [$D4$, 4]
6) $\mathsf{R}\{CB\} \varepsilon \beta$. [$D6$, 2, $D4$]
7) $\mathsf{i}(B\mathsf{R}\{CB\})$. [$D7$, 3, 5, 6]
$\mathsf{i}(B\mathsf{R}\{BC\})$ [7, $M9$]

M37 $[BDEFX] : \mathsf{t}(BEDF) . \mathsf{i}(X\mathsf{R}\{EF\}) . X = B . \supset . X \neq B$
PF $[BDEFX] : \mathrm{Hp}(1\text{-}3) . \supset .$
4) $\mathsf{t}(XEDF)$. [1, 4]
5) $X \neq E$. [$D13$, 5]
6) $\mathsf{p}(\mathsf{R}\{XE\}\mathsf{R}\{DF\})$. [$M35$, 4]
7) $\mathsf{R}\{XE\} \neq \mathsf{R}\{DF\}$. [$M35$, 4]
8) $\mathsf{i}(E\mathsf{R}\{EF\})$. [$M36$, $A3$, 1]
9) $\mathsf{i}(X\mathsf{R}\{XE\})$. [$M35$, 4]
10) $\mathsf{i}(E\mathsf{R}\{XE\})$. [$M35$, 4]
11) $\mathsf{R}\{XE\} = \mathsf{R}\{EF\}$. [$AI$, $D7$, 2, 8, 5, 9, 10]
12) $\mathsf{i}(F\mathsf{R}\{EF\})$. [$M36$, $A3$, 1]
13) $\mathsf{i}(F\mathsf{R}\{XE\})$. [$D7$, $D1$, $A3$, 1, $D7$, 12, 11, $D7$, 2]
14) $\mathsf{i}(F\mathsf{R}\{DF\})$. [$M35$, 4]
$X \neq B$ [$D8$, 6, 7, 13, 14]

M38 $[ABCD] : \mathsf{t}(ABCD) . \mathsf{R}\{AC\} = \mathsf{R}\{BD\} . \supset . \mathsf{R}\{AC\} \neq \mathsf{R}\{BD\}$
PF $[ABCD] : \mathrm{Hp}(1\text{-}2) . \supset .$
3) $\mathsf{R}\{AB\} \neq \mathsf{R}\{CD\}$. [$M35$, 1]
4) $\mathsf{i}(A\mathsf{R}\{AC\})$. [$M36$, $A3$, 1]
5) $\mathsf{i}(B\mathsf{R}\{BD\})$. [$M36$, $A3$, 1]
6) $\mathsf{i}(B\mathsf{R}\{AC\})$. [5, 2]
7) $A \neq B$. [$A2$, 1]
8) $\mathsf{R}\{AB\} = \mathsf{R}\{AC\}$. [$AI$, $D1$, $A3$, 1, 7, 4, 6]
9) $\mathsf{i}(D\mathsf{R}\{BD\})$. [$M36$, $A3$, 1]
10) $\mathsf{i}(C\mathsf{R}\{AC\})$. [$M36$, $A3$, 1]
11) $\mathsf{i}(C\mathsf{R}\{BD\})$. [10, 2]
12) $D \neq C$. [$A2$, $A3$, 1]
13) $\mathsf{R}\{CD\} = \mathsf{R}\{BD\}$. [$AI$, $D1$, $A3$, 1, 12, 11, 9]
14) $\mathsf{R}\{AB\} = \mathsf{R}\{CD\}$. [8, 2, 13]
$\mathsf{R}\{AC\} \neq \mathsf{R}\{BD\}$ [3, 14]

M39 $[ACEFGH]:\mathsf{t}(AECG).\mathsf{t}(AEFH).\mathsf{R}\{EF\}=\mathsf{R}\{AC\}.\supset.\mathsf{R}\{EF\}\neq\mathsf{R}\{AC\}$

PF $[ACEFGH]:\mathrm{Hp}(1\text{-}3).\supset.$

4) $\mathsf{i}(E\mathsf{R}\{EF\})$. [*M36*, 2]

5) $\mathsf{i}(A\mathsf{R}\{AC\})$. [*M36*, *A3*, 1]

6) $\mathsf{i}(A\mathsf{R}\{EF\})$. [5, 3]

7) $\mathsf{R}\{AE\}=\mathsf{R}\{EF\}$. [*A1*, *D1*, *A3*, 1, *A2*, 1, 6, 4]

8) $\mathsf{R}\{AE\}=\mathsf{R}\{AC\}$. [3, 7]

9) $\mathsf{i}(C\mathsf{R}\{AC\})$. [*M36*, *A3*, 1]

10) $\mathsf{i}(C\mathsf{R}\{AE\})$. [8, 9]

11) $\mathsf{c}(CAE)$. [*D4*, *D7*, 10]

12) $\sim(\mathsf{c}(CAE))$. [*A4*, *D1*, *A3*, 1]

$\mathsf{R}\{EF\}\neq\mathsf{R}\{AC\}$ [11, 12]

A8 $[ABCDEFGHX]:\mathsf{t}(ABCD).\mathsf{t}(BEDF).\mathsf{t}(AECG).\mathsf{t}(AEFH).$
$\mathsf{c}(XCA).\mathsf{c}(XBD).\mathsf{c}(XEF).\supset.\mathsf{t}(AECF)$

PF $[ABCDEFGHX]:\mathrm{Hp}(1\text{-}7).\supset.$

8) $A\neq B$. [*D13*, 1]

9) $C\neq D$. [*D13*, 1]

10) $\mathsf{p}(\mathsf{R}\{AB\}\mathsf{R}\{CD\})$. [*M35*, 1]

11) $\mathsf{R}\{AB\}\neq\mathsf{R}\{CD\})$. [*M35*, 1]

12) $\mathsf{i}(A\mathsf{R}\{AB\})$. [*M35*, 1]

13) $\mathsf{i}(B\mathsf{R}\{AB\})$. [*M35*, 1]

14) $\mathsf{i}(C\mathsf{R}\{CD\})$. [*M35*, 1]

15) $\mathsf{i}(D\mathsf{R}\{CD\})$. [*M35*, 1]

16) $B\neq E$. [*D13*, 2]

17) $D\neq F$. [*D13*, 2]

18) $\mathsf{p}(\mathsf{R}\{BE\}\mathsf{R}\{DF\})$. [*M35*, 2]

19) $\mathsf{R}\{BE\}\neq\mathsf{R}\{DF\}$. [*M35*, 2]

20) $\mathsf{i}(B\mathsf{R}\{BE\})$. [*M35*, 2]

21) $\mathsf{i}(E\mathsf{R}\{BE\})$. [*M35*, 2]

22) $\mathsf{i}(D\mathsf{R}\{DF\})$. [*M35*, 2]

23) $\mathsf{i}(F\mathsf{R}\{DF\})$. [*M35*, 2]

24) $\mathsf{i}(E\mathsf{R}\{AE\})$. [*M35*, 3]

25) $\mathsf{i}(A\mathsf{R}\{AE\})$. [*M35*, 3]

26) $\mathsf{i}(X\mathsf{R}\{CA\})$.
[*D7*, *D1*, *L20*, 5, *D4*, *M9*, 5, *D6*, *D3*, *D2*, 1, *D4*]

27) $\mathsf{i}(C\mathsf{R}\{CA\})$. [*M36*, *A3*, 1]

28) $\mathsf{i}(A\mathsf{R}\{CA\})$. [*M36*, *A3*, 1]

29) $\mathsf{k}(XAC)$. [*D9*, 26, 28, 27]

30) $\mathsf{i}(X\mathsf{R}\{BD\})$. [*D7*, *D1*, *L20*, 6, *D4*, 6, *D6*, *D3*, *D2*, 1, *D4*]

31) $\mathsf{i}(B\mathsf{R}\{BD\})$. [*M36*, *A3*, 2]

32) $\mathsf{i}(D\mathsf{R}\{BD\})$. [*M36*, *A3*, 2]

33) $\mathsf{k}(XBD)$. [*D9*, 30, 31, 32]

34) $\mathsf{i}(X\mathsf{R}\{EF\})$.
[*D7*, *D1*, *L20*, 7, *D4*, 7, *D6*, *D3*, *D2*, *A3*, 2, *D4*]

35) $\mathsf{i}(E\mathsf{R}\{EF\})$. [*M36*, *A3*, 2]

36) $\mathsf{i}(F\mathsf{R}\{EF\})$. [*M36*, *A3*, 2]

37) $\mathsf{k}(XEF)$. [*D9*, 34, 35, 36]

38)	$A \neq C$.	[$A2$, 1]
39)	$E \neq F$.	[$A2$, 2]
40)	$B \neq D$.	[$A2$, 2]
41)	$X \neq B$.	[$M37$, 2, 34]
42)	$X \neq D$.	[$M37$, $A3$, 2, 34]
43)	$X \neq E$.	[$M37$, $A3$, 2, 30]
44)	$X \neq F$.	[$M37$, $A3$, 2, 30]
45)	$X \neq A$.	[$M37$, 1, 30]
46)	$X \neq C$.	[$M37$, $A3$, 1, 26]
47)	$\mathsf{R}\{EF\} \neq \mathsf{R}\{BD\}$.	[$M38$, 2]
48)	$\mathsf{R}\{BD\} \neq \mathsf{R}\{CA\}$.	[$M38$, 1]
49)	$\mathsf{R}\{EF\} \neq \mathsf{R}\{CA\}$.	[$M39$, $M9$, 3, 4]
50)	$F \neq C$.	[AI, $D7$, 26, 27, 32, 34, 36, 49]
51)	$\mathsf{i}(F\mathsf{R}\{CF\})$.	[AI, $D7$, 14, 36, 50]
52)	$\mathsf{i}(C\mathsf{R}\{CF\})$.	
53)	$\mathsf{p}(\mathsf{R}\{AE\}\mathsf{R}\{CF\})$.	

[AIV, 37, 33, 29, 43, 44, 39, 41, 42, 40, 45, 46, 38, 34, 35, 30, 31, 26, 28, 47, 49, 48, 20, 21, 23, 22, 18, 13, 12, 14, 15, 10, 24, 25, 51, 52]

54)	$A \neq E$.	[$A2$, 3]
55)	$\sim(C \varepsilon \mathsf{R}\{AE\})$.	[$D4$, $A4$, $D1$, $A3$, 3]
56)	$C \varepsilon \mathsf{R}\{CF\}$.	[$D7$, 52]
57)	$\mathsf{R}\{AE\} \neq \mathsf{R}\{CF\}$.	[$D0$, 55, 56]
	$\mathsf{t}(AECF)$	[$D13$, 54, 50, 53, 57, 25, 24, 52, 51]

CHAPTER II: PROJECTIVE GEOMETRY

In this chapter two systems for projective geometry, (P*) a point geometry and (P) a point-line geometry, are formalized and shown to be equivalent.

System (P*) assumes one set of objects, points. This set will again be denoted by α, and its members by A, B, C, ... One quaternary relation is used as the primitive notion. Because of the many common properties it shares with the primitive of Chapter I it will be denoted by $\mathsf{t}(ABCD)$, however, it must be remembered throughout that it is not the same relation as that in system (A*). Intuitively, $\mathsf{t}(ABCD)$ in system (P*) means that the points A, B, C, and D are the vertices of a non-degenerate quadrilateral. A binary relation $\mathsf{r}(AB)$ and a ternary relation[4] $\mathsf{c}(CAB)$ are defined having the same intuitive meaning as in (A*), and a system of axioms for projective geometry is given. Several auxiliary definitions are necessary to establish the equivalence of (P*) with (P).

System (P) is a formalization of projective geometry as given by Artin [2]. As in his affine geometry two sets of objects, points and lines, are assumed and the primitive notion of incidence, P lies on ℓ. Again in the

4. In Chapter II (and again in III) the definiens of $D3$ as formulated in 1.1 is satisfied by any three points, hence is not well-defined. The modified definition used in this chapter will be denoted by $D3'$.

formalization the set of points will be represented by α, the lines by β, and incidence by $\mathsf{i}(P\ell)$. Two lines intersect if a common point lies on both of them, and points are collinear if they lie on the same line. Artin gives the following axioms for projective geometry.

Axiom 1. *Two distinct points determine a unique line on which they lie.*

Axiom 2. *Two distinct lines intersect in exactly one point.*

Axiom 3. *Each line contains at least three points and there are three non-collinear points.*

Axiom 4. (The projective Desargues) *Let ℓ_1, ℓ_2, ℓ_3 be distinct lines which meet at a point P. Let Q, Q' be points on ℓ_1, R, R' points on ℓ_2, and S, S' points on ℓ_3 which are distinct from P. Assume $Q+R$ intersects $Q'+R'$ at X, $Q+S$ intersects $Q'+S'$ at Y and $R+S$ intersects $R'+S'$ at Z, then X, Y, and Z are collinear.*

In the formalization a member of the second class of objects, lines, must be identified logically with the set of all points lying on it in order to compare systems (P^*) and (P).

2.1 SYSTEM (P^*)

Definitions

D1 $[A]:A\varepsilon\alpha.\equiv.[\exists BCD].\mathsf{t}(ABCD)$

D2 $[AB]\therefore\mathsf{r}(AB).\equiv:[\exists CD]:\mathsf{t}(ABCD).\vee.\mathsf{t}(ACBD).\vee.\mathsf{t}(CBAD)$

D3' $[ABC]\therefore\mathsf{c}(ABC).\equiv:\mathsf{r}(BC):A=B.\vee.A=C.\vee.[X].\sim(\mathsf{t}(AXBC)).A\varepsilon\alpha$

Axioms

A1 $[\exists ABCD].\mathsf{t}(ABCD)$

A2 $[ABCD]:\mathsf{t}(ABCD).\supset.A\neq B$

A3(a) $[ABCD]:\mathsf{t}(ABCD).\supset.\mathsf{t}(DCAB)$

A3(b) $[ABCD]:\mathsf{t}(ABCD).\supset.\mathsf{t}(ABDC)$

A4' $[ABCD]:\mathsf{t}(ABCD).\supset.[\exists M].\mathsf{c}(MAB).\mathsf{c}(MCD)$

A5 $[ABCMN]:A\neq B.\mathsf{c}(AMN).\mathsf{c}(BMN).\mathsf{c}(CMN).\supset.\mathsf{c}(CAB)$

A6' $[AB]:A\varepsilon\alpha.B\varepsilon\alpha.A\neq B.\supset.[\exists C].\mathsf{c}(CAB).C\neq A.C\neq B$

A7' $[ABCD]:\mathsf{t}(ABCD).\equiv.\mathsf{r}(AB).\mathsf{r}(CD).\sim(\mathsf{c}(ACD)).\sim(\mathsf{c}(BCD)).$
$\sim(\mathsf{c}(CAB)).\sim(\mathsf{c}(DAB))$

A8' $[ABCDEFXIJK]\therefore X\neq A.X\neq B.X\neq C.X\neq D.X\neq E.X\neq F.\mathsf{t}(ABCD).$
$\mathsf{t}(BEDF).\mathsf{t}(AECF).\mathsf{c}(XAC).\mathsf{c}(XBD).\mathsf{c}(XEF).\mathsf{c}(IAB).\mathsf{c}(ICD).$
$\mathsf{c}(JBE).\mathsf{c}(JDF).\mathsf{c}(KAE).\mathsf{c}(KCF).\supset:\mathsf{c}(IJK).\vee.J=K$

Auxiliary Definitions

D4 $[ABC]:A\varepsilon\mathsf{R}\{BC\}.\equiv.\mathsf{c}(ABC)$

D5 $[ABCD]:\mathsf{s}(ABCD).\equiv.\mathsf{c}(ACD).\mathsf{c}(BCD).A\neq B$

D6 $[f]::f\varepsilon\beta.\equiv\therefore[\exists AB]\therefore\mathsf{r}(AB):[C]:\mathsf{c}(CAB).\equiv.C\varepsilon f$

D7 $[Af]:\mathsf{i}(Af).\equiv.A\varepsilon\alpha.A\varepsilon f.f\varepsilon\beta$

D9 $[ABC]:\mathsf{k}(ABC).\equiv.[\exists g].\mathsf{i}(Ag).\mathsf{i}(Bg).\mathsf{i}(Cg)$

2.2 SYSTEM (P)

Definitions

D10 $[A]:A\varepsilon\alpha.\equiv.[\exists f].\mathsf{i}(Af)$
D11 $[f]:f\varepsilon\beta.\equiv.[\exists A].\mathsf{i}(Af)$
D9 $[ABC]:\mathsf{k}(ABC).\equiv.[\exists g].\mathsf{i}(Ag).\mathsf{i}(Bg).\mathsf{i}(Cg)$
D12 $[ABC]:A\varepsilon\mathsf{R}\{BC\}.\equiv.B\neq C.\mathsf{k}(ABC)$

Axioms

AI $[AB]\therefore A\varepsilon\alpha.B\varepsilon\alpha.A\neq B.\supset.[\exists f].\mathsf{i}(Af).\mathsf{i}(Bf).f=\mathsf{R}\{AB\}:[gh]:$ $\mathsf{i}(Ag).\mathsf{i}(Bg).\mathsf{i}(Ah).\mathsf{i}(Bh).\supset.g=h$
AII' $[fg]\therefore f\varepsilon\beta.g\varepsilon\beta.f\neq g.\supset.[\exists A].\mathsf{i}(Af).\mathsf{i}(Ag):[BC]:\mathsf{i}(Bf).\mathsf{i}(Bg).\mathsf{i}(Cf).$ $\mathsf{i}(Cg).\supset.B=C$
AIII $[\exists ABC]:A\varepsilon\alpha.B\varepsilon\alpha.C\varepsilon\alpha.A\neq B.A\neq C.B\neq C.\sim(\mathsf{k}(ABC))$
AIII' $[f]:f\varepsilon\beta.\supset.[\exists ABC].\mathsf{i}(Af).\mathsf{i}(Bf).\mathsf{i}(Cf).A\neq B.B\neq C.C\neq A$
AIV' $[ABCDEFIJKfghlpmnuv]:\mathsf{k}(XAC).\mathsf{k}(XBD).\mathsf{k}(XEF).X\neq A.X\neq C.$ $A\neq C.X\neq B.X\neq D.B\neq D.X\neq E.X\neq F.E\neq F.\mathsf{i}(Xf).\mathsf{i}(Af).\mathsf{i}(Xg).$ $\mathsf{i}(Bg).\mathsf{i}(Xh).\mathsf{i}(Eh).f\neq g.f\neq h.g\neq h.\mathsf{i}(Al).\mathsf{i}(Bl).\mathsf{i}(Cp).$ $\mathsf{i}(Dp).\mathsf{i}(Il).\mathsf{i}(Ip).\mathsf{i}(Bm).\mathsf{i}(Em).\mathsf{i}(Dn).\mathsf{i}(Fn).\mathsf{i}(Jm).$ $\mathsf{i}(Jn).\mathsf{i}(Au).\mathsf{i}(Eu).\mathsf{i}(Cv).\mathsf{i}(Fv).\mathsf{i}(Ku).\mathsf{i}(Kv).\supset.\mathsf{k}(IJK)$

Auxiliary Definitions

D13' $[ABCD]:\mathsf{t}(ABCD).\equiv.A\neq B.C\neq D.[\exists fg].\mathsf{i}(Af).\mathsf{i}(Bf).\mathsf{i}(Cg).$ $\mathsf{i}(Dg).\sim(\mathsf{i}(Ag)).\sim(\mathsf{i}(Bg)).\sim(\mathsf{i}(Cf)).\sim(\mathsf{i}(Df))$
D2 $[AB]\therefore\mathsf{r}(AB).\equiv:[\exists CD]:\mathsf{t}(ABCD).\mathsf{v}.\mathsf{t}(ACBD).\mathsf{v}.\mathsf{t}(CBAD)$
D3' $[ABC]\therefore\mathsf{c}(ABC).\equiv:\mathsf{r}(BC):A=B.\mathsf{v}.A=C.\mathsf{v}.[X].\sim(\mathsf{t}(AXBC)).A\varepsilon\alpha$
D5 $[ABCD]:\mathsf{s}(ABCD).\equiv.\mathsf{c}(ACD).\mathsf{c}(BCD).A\neq B.$

2.3 SYSTEM (P*) IMPLIES SYSTEM (P)

In the following the analogs of *L1-L20, L22, L23, L26, L29-L35, L46,* and *L58-L61* will be used. The proofs are similar to those in 1.3 if *D3'* is substituted for *D3* except for *L14, L18, L22,* and *L23*. *L14* follows immediately from *A7'* and *D3'*, *L18* from *A6'* and *D3'*, *L22* and *L23* are proved below.

L22' $[ABC]:A\neq B.\mathsf{c}(ABC).\supset.\mathsf{c}(CAB)$
PF $[ABC]:\mathrm{Hp}(1\text{-}2).\supset.$
3) $\mathsf{r}(BC).$ [*D3'*, 2]
4) $\mathsf{c}(BBC).$ [*D3'*, 3]
5) $\mathsf{c}(CBC).$ [*D3'*, 3]
$\mathsf{c}(CAB)$ [*A5*, 1, 2, 4, 5]
L22 $[ABC]:A\neq B.A\neq C.\mathsf{c}(ABC).\supset.\mathsf{c}(CAB)$ [*L22'*]
L23 $[ABC]:\mathsf{c}(CAB).\supset.\mathsf{c}(CBA)$
PF $[ABC]:\mathrm{Hp}(1).\supset.$
2) $\mathsf{r}(AB).$ [*D3'*, 1]
3) $\mathsf{r}(BA):$ [*L19*, 2]

4) $C=A.\mathrm{v}.C=B.\mathrm{v}.[X].\sim(\mathbf{t}(CXAB).C\varepsilon\alpha:$ [D3', 1]

5) $C=B.\mathrm{v}.C=A.\mathrm{v}.[X].\sim(\mathbf{t}(CXBA).C\varepsilon\alpha:$ [4, A3(b)]

$\mathbf{c}(CBA)$ [D3', 3, 5]

AI $[AB]\therefore A\varepsilon\alpha.B\varepsilon\alpha.A\neq B.\supset.[\exists f].\mathbf{i}(Af).\mathbf{i}(Bf).f=\mathbf{R}\{AB\}:[gh]:$
$\mathbf{i}(Ag).\mathbf{i}(Bg).\mathbf{i}(Ah).\mathbf{i}(Bh).\supset.g=h$ [L34, L35]

P1 $[ABCDf]:\mathbf{c}(DBC).\mathbf{c}(ADC).D\neq C.D\neq B.f=\mathbf{R}\{BC\}.\supset.A\varepsilon f$

PF $[ABCDf]:\mathrm{Hp}(1\text{-}5).\supset.$

6) $\mathbf{c}(BDC).$ [L22, 4, 3, 1]

7) $\mathbf{r}(DC).$ [D3', 6]

8) $B\neq C.$ [A2, L14, L15, D2, D3', 1]

9) $\mathbf{s}(BCDC).$ [D5, 6, D3', 7, 8]

10) $\mathbf{R}\{BC\}=\mathbf{R}\{DC\}.$ [L32, 9]

11) $f=\mathbf{R}\{DC\}.$ [10, 5]

$A\varepsilon f$ [D0, 11, D4, 2]

P2 $[BCEFH]:\mathbf{r}(BC).\mathbf{R}\{BC\}\neq\mathbf{R}\{EF\}.\mathbf{c}(HBC).H\neq C.\mathbf{s}(HCEF).\supset.$
$\sim(\mathbf{s}(HCEF))$

PF $[BCEFH]:\mathrm{Hp}(1\text{-}5).\supset.$

6) $\mathbf{R}\{HC\}=\mathbf{R}\{EF\}.$ [L32, 5]

7) $\mathbf{s}(HCBC).$ [D5, 3, D3', D2, 1, 4]

8) $\mathbf{R}\{HC\}=\mathbf{R}\{BC\}.$ [L32, 7]

9) $\mathbf{R}\{BC\}=\mathbf{R}\{EF\}.$ [6, 8]

$\sim(\mathbf{s}(HCEF))$ [2, 9]

P3 $[ABf]\therefore\mathbf{r}(AB):[C]:\mathbf{c}(CAB).\equiv.C\varepsilon f:\supset.f=\mathbf{R}\{AB\}$

PF $[ABf]\therefore\mathrm{Hp}(1\text{-}2):\supset.$

3) $A\varepsilon f.$ [2, D3', 1]

4) $B\varepsilon f.$ [2, D3', 1]

5) $A\varepsilon\alpha.$ [D1, L2, D2, 1]

6) $B\varepsilon\alpha.$ [D1, L4, L2, L3, D2, 1]

7) $f\varepsilon\beta.$ [D6, 1, 2]

8) $\mathbf{i}(Af).$ [D7, 5, 3, 7]

9) $\mathbf{i}(Bf).$ [D7, 6, 4, 7]

$f=\mathbf{R}\{AB\}$ [AI, 5, 6, A2, L14, L15, D2, 8, 9]

P4 $[BCEF]:\sim(\mathbf{t}(BCEF)).\mathbf{r}(BC).\mathbf{r}(EF).\supset.[\exists A].\mathbf{c}(ABC).\mathbf{c}(AEF)$

PF $[BCEF]\therefore\mathrm{Hp}(1\text{-}3).\supset:$

4) $\mathbf{c}(BEF).\mathrm{v}.\mathbf{c}(CEF).\mathrm{v}.\mathbf{c}(EBC).\mathrm{v}.\mathbf{c}(FBC):$ [A7', 1, 2, 3]

5) $\mathbf{c}(BBC).$ [D3', 2]

6) $\mathbf{c}(CBC).$ [D3', 2]

7) $\mathbf{c}(EEF).$ [D3', 3]

8) $\mathbf{c}(FEF).$ [D3', 3]

$[\exists A].\mathbf{c}(ABC).\mathbf{c}(AEF)$ [4, 5, 6, 7, 8]

P5 $[fg]:f\varepsilon\beta.g\varepsilon\beta.f\neq g.\supset.[\exists A].\mathbf{i}(Af).\mathbf{i}(Ag)$

PF $[fg]::\mathrm{Hp}(1\text{-}3).\supset::$

$[\exists BC]::$

4) $\mathbf{r}(BC):$

5) $[D]:\mathbf{c}(DBC).\equiv.D\bar{\varepsilon}f:$ [D6, 1] (for 4–5)

6) $f=\mathbf{R}\{BC\}\therefore$ [P3, 4, 5]

$[\exists EF]\therefore$

7) $\mathbf{r}(EF):$

8) $[G]:\mathbf{c}(GEF).\equiv.G\varepsilon g:$ [D6, 2] (for 7–8)

9) $\mathbf{t}(BCEF).\vee.\sim(\mathbf{t}(BCEF))$: [*Logic*]

$[\exists A]$.

$$\left.\begin{array}{ll} 10) & \mathbf{c}(ABC). \\ 11) & \mathbf{c}(AEF). \end{array}\right\} [9;\ A4';\ P4,\ 4,\ 7]$$

12) $A \varepsilon f$. [5, 10]

13) $A \varepsilon g$. [8, 11]

14) $A \varepsilon \alpha$ :: [*D1*, *L2*, *D3'*, *D2*, 10]

$[\exists A].\mathbf{i}(Af).\mathbf{i}(Ag)$ [*D7*, 14, 12, 1, 14, 13, 2]

P6 $[BCfg]:f\varepsilon\beta.g\varepsilon\beta.f\neq g.\mathbf{i}(Bf).\mathbf{i}(Bg).\mathbf{i}(Cf).\mathbf{i}(Cg).B\neq C.\supset.B=C$

PF $[BCfg]$ ∴ Hp(1-8). $\supset$ ::

$[\exists DE]$ ::

9) $[F]:\mathbf{c}(FDE).\equiv.F\varepsilon f$: [*D6*, 1]

10) $\mathbf{c}(BDE)$. [9, *D7*, 4]

11) $\mathbf{c}(CDE)$. [9, *D7*, 6]

12) $\mathbf{s}(BCDE)$ ∴ [*D5*, 10, 11, 8]

$[\exists HI]$ ∴

13) $[J]:\mathbf{c}(JHI).\equiv.J\varepsilon g$: [*D6*, 2]

14) $\mathbf{c}(BHI)$. [13, *D7*, 5]

15) $\mathbf{c}(CHI)$. [13, *D7*, 7]

16) $\mathbf{s}(BCHI)$. [*D5*, 14, 15, 8]

17) $\mathbf{s}(DEHI)$. [*L30*, *L29*, 12, 16]

18) $\mathbf{R}\{DE\}=\mathbf{R}\{HI\}$ :: [*L32*, 17]

19) $f=g$. [*D0*, *D4*, 18, 9, 13]

$B=C$ [3, 19]

AII' $[fg]$ ∴ $f\varepsilon\beta.g\varepsilon\beta.f\neq g.\supset:[\exists A].\mathbf{i}(Af).\mathbf{i}(Ag):[BC]:\mathbf{i}(Bf).\mathbf{i}(Bg).$
$\mathbf{i}(Cf).\mathbf{i}(Cg).\supset.B=C$ [*P5*, *P6*]

P7 $[ABCD]:\mathbf{t}(ABCD).\mathbf{k}(ABD).\mathbf{k}(ABC).\supset.\sim(\mathbf{k}(ABC))$

PF $[ABCD]$ ∴ Hp(1-3). $\supset$:

4) $A\varepsilon\alpha$. [*D1*, 1]

5) $B\varepsilon\alpha$. [*D1*, *L4*, 1]

6) $A\neq B$: [*A2*, 1]

$[\exists f]$:

$$\left.\begin{array}{ll} 7) & \mathbf{i}(Af). \\ 8) & \mathbf{i}(Bf). \\ 9) & \mathbf{i}(Df). \end{array}\right\} [D9,\ 2]$$

$[\exists g]$.

$$\left.\begin{array}{ll} 10) & \mathbf{i}(Ag). \\ 11) & \mathbf{i}(Bg). \\ 12) & \mathbf{i}(Cg). \end{array}\right\} [D9,\ 3]$$

13) $\mathbf{R}\{AB\}=f$. [*AI*, 4, 5, 6, 7, 8]

14) $\mathbf{R}\{AB\}=g$: [*AI*, 4, 5, 6, 10, 11]

15) $C\varepsilon\mathbf{R}\{AB\}$. [*D7*, 12, 14]

16) $D\varepsilon\mathbf{R}\{AB\}$. [*D7*, 9, 13]

17) $\mathbf{c}(CAB)$. [*D4*, 15]

18) $\mathbf{c}(DAB)$. [*D4*, 16]

19) $C\neq D$. [*L13*, 1]

20) $\mathbf{s}(ABCD)$. [*L29*, *D5*, 17, 18, 19]

21) $\sim(\mathbf{s}(ABCD))$. [*D5*, *A7'*, 1]

$\sim(\mathbf{k}(ABC))$ [20, 21]

AIII $[\exists ABC].A\varepsilon\alpha.B\varepsilon\alpha.C\varepsilon\alpha.A\neq B.A\neq C.B\neq C.\sim(\mathbf{k}(ABC))$

PF $[\exists ABCD]\therefore$

1) $\mathbf{t}(ABCD).$ [*A1*]
2) $A\varepsilon\alpha.$ [*D1*, 1]
3) $B\varepsilon\alpha.$ [*D1*, *L4*, 1]
4) $C\varepsilon\alpha.$ [*D1*, *L2*, 1]
5) $D\varepsilon\alpha.$ [*D1*, *L3*, 1]
6) $A\neq B.$ [*A2*, 1]
7) $A\neq C.$ [*L14*, 1]
8) $B\neq C.$ [*L15*, 1]
9) $A\neq D.$ [*L16*, 1]
10) $B\neq D:$ [*L17*, 1]
11) $\mathbf{k}(ABD).\vee.\sim(\mathbf{k}(ABD)):$ [*Logic*]
12) $\sim(\mathbf{k}(ABC)).\vee.\sim(\mathbf{k}(ABD)):$ [11, *D7*, 1]

$[\exists ABC]:A\varepsilon\alpha.B\varepsilon\alpha.C\varepsilon\alpha.A\neq B.A\neq C.B\neq C.\sim(\mathbf{k}(ABC))$ [12, 2, 3, 4, 5, 7, 8, 9, 10]

AIII' $[f]:f\varepsilon\beta.\supset.[\exists ABC].\mathbf{i}(Af).\mathbf{i}(Bf).\mathbf{i}(Cf).A\neq B.B\neq C.C\neq A$

PF $[f]::\mathrm{Hp}(1).\supset\therefore$

$[\exists AB]\therefore$

2) $\mathbf{r}(AB):$
3) $[M]:\mathbf{c}(MAB).\equiv.M\varepsilon f:$ } [*D6*, 1]
4) $A\varepsilon\alpha.$ [*D1*, *L2*, *D2*, 2]
5) $B\varepsilon\alpha.$ [*D1*, *L1*, *L2*, *L3*, *D2*, 2]
6) $A\neq B.$ [*A2*, *L14*, *L15*, *D2*, 2]

$[\exists C].$

7) $\mathbf{c}(CAB).$
8) $C\neq A.$
9) $C\neq B.$ } [*A6'*, 4, 5, 6]
10) $C\varepsilon\alpha.$ [*D3'*, 4, 5, 7]
11) $\mathbf{i}(Af).$ [*D7*, 4, 3, *D4*, *D3'*, 2, 1]
12) $\mathbf{i}(Bf).$ [*D7*, 5, 3, *D4*, *D3'*, 2, 1]
13) $\mathbf{i}(Cf)\therefore$ [*D7*, 10, 3, 7, 1]

$[\exists ABC].\mathbf{i}(Af).\mathbf{i}(Bf).\mathbf{i}(Cf).A\neq B.B\neq C.C\neq A$ [11, 12, 13, 6, 9, 8]

P8 $[ABC]:\mathbf{c}(ABC).\supset.\mathbf{k}(ABC)$

PF $[ABC]:\mathrm{Hp}(1).\supset.$

2) $\mathbf{r}(BC).$ [*D3'*, 1]
3) $\mathbf{R}\{BC\}\varepsilon\beta.$ [*D6*, *D4*, 2]
4) $A\varepsilon\alpha.$ [*D1*, *D3'*, *A3*, *D2*, 1]
5) $B\varepsilon\alpha.$ [*D1*, *L2*, *D2*, 2]
6) $C\varepsilon\alpha.$ [*D1*, *L1*, *L2*, *D2*, 2]
7) $A\varepsilon\mathbf{R}\{BC\}.$ [*D4*, 1]
8) $B\varepsilon\mathbf{R}\{BC\}.$ [*D4*, *D3'*, 2]
9) $C\varepsilon\mathbf{R}\{BC\}.$ [*D4*, *D3'*, 2]
10) $\mathbf{i}(A\mathbf{R}\{BC\}).$ [*D7*, 4, 7, 3]
11) $\mathbf{i}(B\mathbf{R}\{BC\}).$ [*D7*, 5, 8, 3]
12) $\mathbf{i}(C\mathbf{R}\{BC\}).$ [*D7*, 6, 9, 3]

$\mathbf{k}(ABC)$ [*D9*, 10, 11, 12]

$P9$ $[ABC]: B \neq C . \mathsf{k}(ABC) . \supset . \mathsf{c}(ABC)$

PF $[ABC] \because \mathrm{Hp}(1\text{-}2) . \supset ::$

$[\exists f] ::$

3) $\mathsf{i}(Af)$. [D9, 2]

4) $\mathsf{i}(Bf)$.

5) $\mathsf{i}(Cf) \therefore$

$[\exists DE] \therefore$

6) $\mathsf{r}(DE)$: [D6, D7, 3]

7) $[K]: \mathsf{c}(KDE) . \equiv . K \varepsilon f$:

8) $f = \mathbf{R}\{DE\}$. [P3, 6, 7]

9) $\mathsf{c}(ADE)$. [7, D7, 3]

10) $\mathsf{c}(BDE)$. [7, D7, 4]

11) $\mathsf{c}(CDE)$:: [7, D7, 5]

$\mathsf{c}(ABC)$ [A5, 1, 9, 10, 11]

$P10$ $[ABC]: B \neq C . \mathsf{k}(ABC) . \equiv . \mathsf{c}(ABC)$ [P8, A2, A3, D3', D2; P9]

$P11$ $[ABCDlp]: \mathsf{i}(Al) . \mathsf{i}(Bl) . \mathsf{i}(Cp) . \mathsf{i}(Dp) . A \neq B . C \neq D . \sim(\mathsf{i}(Ap))$.
$\sim(\mathsf{i}(Bp)) . \sim(\mathsf{i}(Cl)) . \sim(\mathsf{i}(Dl)) . \supset . \mathsf{t}(ABCD)$

PF $[ABCDlp]: \mathrm{Hp}(1\text{-}10) . \supset .$

11) $l = \mathbf{R}\{AB\}$. [AI, D7, 1, 2, 5]

12) $p = \mathbf{R}\{CD\}$. [AI, D7, 3, 4, 6]

13) $\mathsf{r}(AB)$. [L19, D7, 1, 2, 5]

14) $\mathsf{r}(CD)$. [L19, D7, 3, 4, 6]

15) $\sim(\mathsf{c}(ACD))$. [D4, D7, 7, 1, 12, 3]

16) $\sim(\mathsf{c}(BCD))$. [D4, D7, 8, 2, 12, 3]

17) $\sim(\mathsf{c}(CAB))$. [D4, D7, 9, 3, 11, 1]

18) $\sim(\mathsf{c}(DAB))$. [D4, D7, 10, 4, 11, 1]

$\mathsf{t}(ABCD)$ [A7, 13, 14, 15, 16, 17, 18]

$P12$ $[ABXfg]: X \neq A . \mathsf{i}(Xf) . \mathsf{i}(Af) . \mathsf{i}(Xg) . \mathsf{i}(Bg) . f \neq g . A = B . \supset . A \neq B$

PF $[ABXfg]: \mathrm{Hp}(1\text{-}7) . \supset .$

8) $X \varepsilon \alpha$. [D7, 2]

9) $A \varepsilon \alpha$. [D7, 3]

10) $f = g$. [AI, 8, 9, 1, 2, 3, 4, 5, 7]

$A \neq B$ [6, 10]

$P13$ $[ACDXfgp]: \mathsf{i}(Af) . \mathsf{i}(Cf) . \mathsf{i}(Xf) . \mathsf{i}(Xg) . \mathsf{i}(Dg) . \mathsf{i}(Cp)$.
$\mathsf{i}(Dp) . A \neq C . f \neq g . X \neq D . \mathsf{i}(Ap) . \supset . \sim(\mathsf{i}(Ap))$

PF $[ACDXfgp]: \mathrm{Hp}(1\text{-}11) . \supset .$

12) $p = f$. [AI, D7, 1, 2, 8, 11, 6]

13) $\mathsf{i}(Df)$. [7, 12]

14) $f = g$. [AI, D7, 4, 5, 10, 3, 13]

$\sim(\mathsf{i}(Ap))$ [9, 14]

AIV' $[ABCDEFGHIJKfghlpmnuv]: \mathsf{k}(XAC) . \mathsf{k}(XBD) . \mathsf{k}(XEF) . X \neq A$.
$X \neq C . A \neq C . X \neq B . X \neq D . B \neq D . X \neq E . X \neq F . E \neq F . \mathsf{i}(Xf) . \mathsf{i}(Af) . \mathsf{i}(Xg)$.
$\mathsf{i}(Bg) . \mathsf{i}(Xh) . \mathsf{i}(Eh) . f \neq g . f \neq h . g \neq h . \mathsf{i}(Al) . \mathsf{i}(Bl) . \mathsf{i}(Cp)$.
$\mathsf{i}(Dp) . \mathsf{i}(Il) . \mathsf{i}(Ip) . \mathsf{i}(Bm) . \mathsf{i}(Em) . \mathsf{i}(Dn) . \mathsf{i}(Fn) . \mathsf{i}(Jm)$.
$\mathsf{i}(Jn) . \mathsf{i}(Au) . \mathsf{i}(Eu) . \mathsf{i}(Cv) . \mathsf{i}(Fv) . \mathsf{i}(Ku) . \mathsf{i}(Kv) . \supset . \mathsf{k}(IJK)$

PF $[ABCDEFGHIJKfghlpmnuv] \therefore \mathrm{Hp}(1\text{-}39) . \supset :$

40) $A \neq B$. [P12, 4, 13, 14, 15, 16, 19]

41) $\mathsf{i}(Cf)$. [AI, D7, 13, 14, 4, D9, 1]

42) $\mathsf{i}(Dg)$. [$A1$, $D7$, 15, 16, 7, $D9$, 2]
43) $C \neq D$. [$P12$, 5, 13, 41, 15, 42, 19]
44) $\sim(\mathsf{i}(Ap))$. [$P13$, 14, 41, 13, 15, 42, 24, 25, 6, 19, 8]
45) $\sim(\mathsf{i}(Bp))$. [$P13$, 16, 42, 15, 13, 41, 25, 24, 9, 19, 6]
46) $\sim(\mathsf{i}(Cl))$. [$P13$, 41, 14, 13, 15, 16, 22, 23, 6, 19, 7]
47) $\sim(\mathsf{i}(Dl))$. [$P13$, 42, 16, 15, 13, 14, 22, 23, 9, 19, 4]
48) $\mathsf{t}(ABCD)$. [$P11$, 22, 23, 24, 25, 40, 43, 44, 45, 46, 47]
49) $B \neq E$. [$P12$, 7, 15, 16, 17, 18, 20]
50) $\mathsf{i}(Fh)$. [$A1$, $D7$, 17, 18, 10, $D9$, 3]
51) $D \neq F$. [$P12$, 8, 15, 42, 17, 50, 21]
52) $\sim(\mathsf{i}(Bn))$. [$P13$, 16, 42, 15, 17, 50, 30, 31, 9, 21, 11]
53) $\sim(\mathsf{i}(En))$. [$P13$, 18, 51, 17, 15, 42, 31, 30, 12, 21, 8]
54) $\sim(\mathsf{i}(Dm))$. [$P13$, 42, 16, 15, 17, 18, 28, 29, 9, 21, 10]
55) $\sim(\mathsf{i}(Fm))$. [$P13$, 50, 18, 17, 15, 16, 29, 28, 12, 21, 7]
56) $\mathsf{t}(BEDF)$. [$P11$, 28, 29, 30, 31, 49, 51, 52, 53, 54, 55]
57) $A \neq E$. [$P12$, 4, 13, 14, 17, 18, 21]
58) $C \neq F$. [$P12$, 5, 13, 41, 17, 50, 21]
59) $\sim(\mathsf{i}(Av))$. [$P13$, 14, 41, 13, 17, 51, 36, 37, 6, 20, 11]
60) $\sim(\mathsf{i}(Ev))$. [$P13$, 18, 50, 17, 13, 41, 37, 36, 12, 20, 5]
61) $\sim(\mathsf{i}(Cu))$. [$P13$, 41, 14, 13, 17, 18, 34, 35, 6, 20, 10]
62) $\sim(\mathsf{i}(Fu))$. [$P13$, 50, 18, 17, 13, 14, 34, 35, 12, 20, 4]
63) $\mathsf{t}(AECF)$. [$P11$, 34, 35, 36, 37, 57, 58, 59, 60, 61, 62]
64) $\mathsf{c}(XAC)$. [$P10$, 6, 1]
65) $\mathsf{c}(XBD)$. [$P10$, 9, 2]
66) $\mathsf{c}(XEF)$. [$P10$, 12, 3]
67) $\mathsf{c}(IAB)$. [$P10$, 40, $D9$, 26, 22, 23]
68) $\mathsf{c}(ICD)$. [$P10$, 43, $D9$, 27, 24, 25]
69) $\mathsf{c}(JBE)$. [$P10$, 49, $D9$, 32, 28, 29]
70) $\mathsf{c}(JDF)$. [$P10$, 51, $D9$, 33, 31, 30]
71) $\mathsf{c}(KAE)$. [$P10$, 57, $D9$, 38, 34, 35]
72) $\mathsf{c}(KCF)$: [$P10$, 58, $D9$, 39, 36, 37]
73) $\mathsf{c}(IJK) . \vee . J = K$:
[$A8'$, 4, 7, 5, 8, 10, 11, 48, 56, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72]
$\mathsf{k}(IJK)$ [73; $P10$; $D9$, $A1$, $D7$, 26, 32]

P14 $[ABCD] : \mathsf{t}(ABCD) . \supset . [\exists fg] . \mathsf{i}(Af) . \mathsf{i}(Bf) . \mathsf{i}(Cg) . \mathsf{i}(Dg) .$
$\sim(\mathsf{i}(Ag)) . \sim(\mathsf{i}(Bg)) . \sim(\mathsf{i}(Cf)) . \sim(\mathsf{i}(Df))$

PF $[ABCD] : \mathrm{Hp}(1) . \supset .$
2) $\mathsf{r}(AB)$. [$A7'$, 1]
3) $\mathsf{r}(CD)$. [$A7'$, 1]
4) $\sim(\mathsf{c}(ACD))$. [$A7'$, 1]
5) $\sim(\mathsf{c}(BCD))$. [$A7'$, 1]
6) $\sim(\mathsf{c}(CAB))$. [$A7'$, 1]
7) $\sim(\mathsf{c}(DAB))$. [$A7'$, 1]
8) $A \varepsilon \alpha$. [$D1$, 1]
9) $B \varepsilon \alpha$. [$D1$, $L1$, 1]
10) $C \varepsilon \alpha$. [$D1$, $L2$, 1]
11) $D \varepsilon \alpha$. [$D1$, $L3$, 1]

12) $A \varepsilon \mathbf{R}\{AB\}.$ [$D4$, $D3'$, 2]
13) $B \varepsilon \mathbf{R}\{AB\}.$ [$D4$, $D3'$, 2]
14) $C \varepsilon \mathbf{R}\{CD\}.$ [$D4$, $D3'$, 3]
15) $D \varepsilon \mathbf{R}\{CD\}.$ [$D4$, $D3'$, 3]
16) $\mathbf{R}\{AB\} \varepsilon \beta.$ [$D6$, $D4$, $D3'$, 2]
17) $\mathbf{R}\{CD\} \varepsilon \beta.$ [$D6$, $D4$, $D3'$, 3]
18) $\sim(\mathbf{i}(A\mathbf{R}\{CD\})).$ [$D7$, $D4$, 4]
19) $\sim(\mathbf{i}(B\mathbf{R}\{CD\})).$ [$D7$, $D4$, 5]
20) $\sim(\mathbf{i}(C\mathbf{R}\{AB\})).$ [$D7$, $D4$, 6]
21) $\sim(\mathbf{i}(D\mathbf{R}\{AB\})).$ [$D7$, $D4$, 7]
$[\exists fg].\mathbf{i}(Af).\mathbf{i}(Bf).\mathbf{i}(Cg).\mathbf{i}(Dg).\sim(\mathbf{i}(Ag)).$
$\sim(\mathbf{i}(Bg)).\sim(\mathbf{i}(Cf)).\sim(\mathbf{i}(Df))$
[$D7$, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]

$D13'$ $[ABCD]:\mathbf{t}(ABCD).\equiv.A \neq B.C \neq D.[\exists fg].\mathbf{i}(Af).\mathbf{i}(Bf).\mathbf{i}(Cg).\mathbf{i}(Dg).$
$\sim(\mathbf{i}(Ag)).\sim(\mathbf{i}(Bg)).\sim(\mathbf{i}(Cf)).\sim(\mathbf{i}(Df))$
[$A2$, $L5$, $P14$; $P11$]

$D10$ $[A]:A \varepsilon \alpha.\equiv.[\exists f].\mathbf{i}(Af)$ [$D13'$, $D1$; $D7$]
$D11$ $[f]:f \varepsilon \beta.\equiv.[\exists A].\mathbf{i}(Af)$ [$L58$; $D7$]
$D12$ $[ABC]:A \varepsilon \mathbf{R}\{BC\}.\equiv.B \neq C.\mathbf{k}(ABC)$ [$L59$, $L60$; $L61$]

2.4 SYSTEM (P) IMPLIES SYSTEM (P*)

In this section the analogs of *M14-M27* of section 1.4 will be used. The proofs are similar to those given in 1.4 if *D13'* is substituted for *D13*, except for *M13* which is an immediate consequence of *AIII'*, and *M16* and *M24-M26* which will be proved below.

$M16$ $[ABC]:A \varepsilon \alpha.B \varepsilon \alpha.C \varepsilon \alpha.A \neq B.A \neq C.B \neq C.\sim(\mathbf{k}(ABC)).\supset.[\exists D].$
$\mathbf{t}(ABCD)$

PF $[ABC]\therefore \mathrm{Hp}(1\text{-}7).\supset::$
$[\exists f]::$
8) $\mathbf{i}(Af).$
9) $\mathbf{i}(Bf).\therefore$ [AI, 1, 2, 4]
$[\exists E].\therefore$
10) $\mathbf{i}(Ef).$
11) $E \neq A.$
12) $E \neq B.$ [$AIII'$, $D11$, 8, 9, 4]
13) $C \neq E:$ [$D9$, 7, 8, 9, 10]
$[\exists g]:$
14) $\mathbf{i}(Cg).$
15) $\mathbf{i}(Eg).$ [AI, 3, $D10$, 10, 13]
16) $f \neq g.$ [$D9$, 8, 9, 14]
17) $\sim(\mathbf{i}(Ag)).$ [AI, 1, $D10$, 10, 11, 8, 10, 15, 16]
18) $\sim(\mathbf{i}(Bg)).$ [AI, 2, $D10$, 10, 12, 9, 10, 15, 16]
19) $\sim(\mathbf{i}(Cf)).$ [AI, 3, $D10$, 15, 13, 10, 14, 15, 16]
$[\exists D].$
20) $\mathbf{i}(Dg).$
21) $D \neq C.$
22) $D \neq E.$ [$AIII'$, $D11$, 14, 15, 13]

23) $\sim(\mathrm{i}(Df))::$ [*AI*, *D10*, 15, 20, 22, 10, 15, 20, 16]
$[\exists D].\mathrm{t}(ABCD)$ [*D13'*, 4, 21, 8, 9, 14, 20, 17, 18, 19, 23]

A1 $[\exists ABCD].\mathrm{t}(ABCD)$ [*M16*, *AIII*]
A2 $[ABCD]:\mathrm{t}(ABCD).\supset.A \neq B$ [*D13'*]
A3(a) $[ABCD]:\mathrm{t}(ABCD).\supset.\mathrm{t}(DCAB)$ [*D13'*]
A3(b) $[ABCD]:\mathrm{t}(ABCD).\supset.\mathrm{t}(ABDC)$ [*D13'*]

The analogs of *L1*-*L17* now hold in (**P**). The proofs are similar to those in **1.3** except for *L14* which follows immediately from *D13'*. Again for brevity *A3* and *A2* will be given as reasons for any of *L1*-*L12* and *L13*-*L17* respectively.

D7 $[Af]:\mathrm{i}(Af).\equiv.A\varepsilon\alpha.A\varepsilon f.f\varepsilon\beta$ [*D10*, *M14*, *D11*; *M15*]
D1 $[A]:A\varepsilon\alpha.\equiv.[\exists BCD].\mathrm{t}(ABCD)$ [*M20*; *M21*]
M24 $[ABf]:A \neq B.\mathrm{i}(Af).\mathrm{i}(Bf).\supset.[\exists CD].\mathrm{t}(ABCD)$
PF $[ABf]:\mathrm{Hp}(1\text{-}3).\supset.$
4) $A\varepsilon\alpha.$ [*D10*, 2]
5) $B\varepsilon\alpha.$ [*D10*, 3]
$[\exists C].$
6) $\sim(\mathrm{i}(Cf)).$
7) $C\varepsilon\alpha.$ [*M23*, 2, 3, 1]
8) $C \neq A.$ [2, 6]
9) $C \neq B.$ [3, 6]
10) $\sim(\mathrm{k}(ABC)).$ [*D9*, *AI*, 4, 5, 1, 2, 3, 6]
$[\exists CD].\mathrm{t}(ABCD)$ [*M16*, 4, 5, 7, 1, 8, 9, 10]

Q1 $[ABCX]:\mathrm{i}(Af).\mathrm{i}(Bf).\mathrm{i}(Cf).B \neq C.\mathrm{t}(AXBC).\supset.\sim(\mathrm{t}(AXBC))$
PF $[ABCX]:\mathrm{Hp}(1\text{-}5).\supset.$
$[\exists g].$
6) $\mathrm{i}(Bg).$
7) $\mathrm{i}(Cg).$ [*D13'*, 5]
8) $\sim(\mathrm{i}(Ag)).$
9) $g=f.$ [*AI*, *D7*, 2, 3, 4, 6, 7]
$\sim(\mathrm{t}(AXBC))$ [1, 8, 9]

M25 $[ABC]:A\varepsilon\mathbf{R}\{BC\}.\supset.\mathrm{c}(ABC)$
PF $[ABC]::\mathrm{Hp}(1).\supset\therefore$
2) $B \neq C.$
3) $\mathrm{k}(ABC)\therefore$ [*D12*, 1]
$[\exists f]\therefore$
4) $\mathrm{i}(Af).$
5) $\mathrm{i}(Bf).$ [*D9*, 3]
6) $\mathrm{i}(Cf).$
7) $A\varepsilon\alpha.$ [*D10*, 4]
8) $[X].\sim(\mathrm{t}(AXBC)).$ [*Q1*, 4, 5, 6, 2]
9) $\mathrm{r}(BC):$ [*D2*, *M24*, 2, 5, 6]
10) $A=B.\vee.A=C.\vee.[X].\sim(\mathrm{t}(AXBC)).A\varepsilon\alpha\therefore$ [8, 7]
$\mathrm{c}(ABC)$ [*D3'*, 9, 10]

Q2 $[ABC]:A\varepsilon\alpha.B\varepsilon\alpha.C\varepsilon\alpha.A \neq B.A \neq C.B \neq C.\sim(\mathrm{k}(ABC)).\supset.[\exists D].$
$\mathrm{t}(ADBC)$
[*M16*, *D9*, *L5*]

Q3		$[ABC]:\mathbf{c}(ABC).\sim(A\,\varepsilon\,\mathbf{R}\{BC\}).\supset.A\,\varepsilon\,\mathbf{R}\{BC\}$	
PF		$[ABC]\therefore\mathrm{Hp}(1\text{-}2).\supset:$	
	3)	$\mathbf{r}(BC):$	$[D3', 1]$
	4)	$A=B.\lor.A=C.\lor.[X].\sim(\mathbf{t}(AXBC).A\,\varepsilon\,\alpha:$	$[D3', 1]$
	5)	$B\neq C.$	$[A2, A3, D2, 3]$
	6)	$B\,\varepsilon\,\alpha.$	$[D1, A3, D2, 3]$
	7)	$C\,\varepsilon\,\alpha.$	$[D1, A3, D2, 3]$
	8)	$\sim(\mathbf{k}(ABC)):$	$[D12, 2]$
	9)	$A=B.\lor.A=C:$	$[4, Q2, 6, 7, 5, 8]$
	10)	$\mathbf{i}(B\mathbf{R}\{BC\}).$	$[AI, 6, 7, 5]$
	11)	$\mathbf{i}(C\mathbf{R}\{BC\}).$	$[AI, 6, 7, 5]$
	12)	$\mathbf{i}(A\mathbf{R}\{BC\}).$	$[9, 10, 11]$
	13)	$\mathbf{k}(ABC).$	$[D9, 10, 11, 12]$
		$A\,\varepsilon\,\mathbf{R}\{BC\}$	$[D12, 5, 13]$
M26		$[ABC]:\mathbf{c}(ABC).\supset.A\,\varepsilon\,\mathbf{R}\{BC\}$	$[Q3]$
D4		$[ABC]:A\,\varepsilon\,\mathbf{R}\{BC\}.\equiv.\mathbf{c}(ABC)$	$[M25; M26]$
Q4		$[f]::f\,\varepsilon\,\beta.\supset\therefore[\exists AB]\therefore\mathbf{r}(AB):[C]:\mathbf{c}(CAB).\equiv.C\,\varepsilon\,f$	
PF		$[f]::\mathrm{Hp}(1).\supset\therefore$	
		$[\exists AB]\therefore$	
	2)	$\mathbf{i}(Af).$	
	3)	$\mathbf{i}(Bf).$	$[AIII', 1]$
	4)	$A\neq B.$	
	5)	$A\,\varepsilon\,\alpha.$	$[D7, 2]$
	6)	$B\,\varepsilon\,\alpha.$	$[D7, 3]$
	7)	$f=\mathbf{R}\{AB\}:$	$[AI, 5, 6, 4, 2, 3]$
	8)	$[C]:\mathbf{c}(CAB).\equiv.C\,\varepsilon\,f:$	$[D4, 7]$
	9)	$\mathbf{r}(AB)\therefore$	$[D3', 8, D7, 2]$
		$[\exists AB]\therefore\mathbf{r}(AB):[C]:\mathbf{c}(CAB).\equiv.C\,\varepsilon\,f$	$[9, 8]$
D6		$[f]::f\,\varepsilon\,\beta.\equiv\therefore[\exists AB]\therefore\mathbf{r}(AB):[C]:\mathbf{c}(CAB).\equiv.C\,\varepsilon\,f$	$[M27; Q4]$
A4'		$[ABCD]:\mathbf{t}(ABCD).\supset.[\exists M].\mathbf{c}(MAB).\mathbf{c}(MCD)$	
PF		$[ABCD]\therefore\mathrm{Hp}(1).\supset:$	
	2)	$A\neq B.$	$[D13', 1]$
	3)	$C\neq D:$	$[D13', 1]$
		$[\exists fg]:$	
	4)	$\mathbf{i}(Af).$	
	5)	$\mathbf{i}(Bf).$	
	6)	$\mathbf{i}(Cg).$	$[D13', 1]$
	7)	$\mathbf{i}(Dg).$	
	8)	$\sim(\mathbf{i}(Ag)).$	
	9)	$f\neq g.$	$[4, 8]$
	10)	$f=\mathbf{R}\{AB\}.$	$[AI, D7, 4, 5, 2]$
	11)	$g=\mathbf{R}\{CD\}.$	$[AI, D7, 6, 7, 3]$
		$[\exists M].$	
	12)	$\mathbf{i}(Mf).$	$[AII', D7, 4, 6, 9]$
	13)	$\mathbf{i}(Mg).$	
	14)	$M\,\varepsilon\,\mathbf{R}\{AB\}.$	$[D7, 12, 10]$
	15)	$M\,\varepsilon\,\mathbf{R}\{CD).$	$[D7, 13, 11]$
	16)	$\mathbf{c}(MAB).$	$[D4, 14]$
	17)	$\mathbf{c}(MCD):$	$[D4, 15]$
		$[\exists M].\mathbf{c}(MAB).\mathbf{c}(MCD)$	$[16, 17]$

The analogs of *L20, A5* and *P9* now follow. *L20* as in **1.3**, *A5* as in **1.4**, and *P9* as in **2.3**.

A6'	$[AB]:A\varepsilon\alpha\,.\,B\varepsilon\alpha\,.\,A\neq B\,.\supset.\,[\exists C]\,.\,\mathsf{c}(CAB)\,.\,C\neq A\,.\,C\neq B$	
PF	$[AB]\therefore \mathrm{Hp}(1\text{-}3)\,.\supset:$	
	$[\exists f]:$	
4)	$\mathsf{i}(Af)\,.$	$[AI,\ 1,\ 2,\ 3]$
5)	$\mathsf{i}(Bf)\,.$	
6)	$f\varepsilon\beta.$	$[D7,\ 4]$
	$[\exists C]\,.$	
7)	$\mathsf{i}(Cf)\,.$	
8)	$C\neq A.$	$[AIII',\ 6,\ 4,\ 5,\ 3]$
9)	$C\neq B.$	
10)	$\mathsf{k}(CAB)\,.$	$[D9,\ 4,\ 5,\ 7]$
11)	$\mathsf{c}(CAB)\,.$	$[P9,\ 3,\ 10]$
	$[\exists C]\,.\,\mathsf{c}(CAB)\,.\,C\neq A\,.\,C\neq B$	$[11,\ 8,\ 9]$
Q5	$[ABCD]:\mathsf{r}(AB)\,.\,\mathsf{r}(CD)\,.\sim(\mathsf{c}(ACD))\,.\sim(\mathsf{c}(BCD))\,.\sim(\mathsf{c}(CAB))\,.$ $\sim(\mathsf{c}(DAB))\,.\supset.\,\mathsf{t}(ABCD)$	
PF	$[ABCD]:\mathrm{Hp}(1\text{-}6)\,.\supset.$	
7)	$A\neq B.$	$[A2,\ D2,\ 1]$
8)	$C\neq D.$	$[A2,\ D2,\ 2]$
9)	$A\varepsilon\alpha\,.$	$[D1,\ A3,\ D2,\ 1]$
10)	$B\varepsilon\alpha\,.$	$[D1,\ A3,\ D2,\ 1]$
11)	$C\varepsilon\alpha\,.$	$[D1,\ A3,\ D2,\ 2]$
12)	$D\varepsilon\alpha\,.$	$[D1,\ A3,\ D2,\ 2]$
13)	$\mathsf{i}(A\mathsf{R}\{AB\})\,.$	$[AI,\ 9,\ 10,\ 7]$
14)	$\mathsf{i}(B\mathsf{R}\{AB\})\,.$	$[AI,\ 9,\ 10,\ 7]$
15)	$\mathsf{i}(C\mathsf{R}\{CD\})\,.$	$[AI,\ 11,\ 12,\ 8]$
16)	$\mathsf{i}(D\mathsf{R}\{CD\})\,.$	$[AI,\ 11,\ 12,\ 8]$
17)	$\sim(\mathsf{i}(A\mathsf{R}\{CD\}))\,.$	$[D7,\ D4,\ 3]$
18)	$\sim(\mathsf{i}(B\mathsf{R}\{CD\}))\,.$	$[D7,\ D4,\ 4]$
19)	$\sim(\mathsf{i}(C\mathsf{R}\{AB\}))\,.$	$[D7,\ D4,\ 5]$
20)	$\sim(\mathsf{i}(D\mathsf{R}\{AB\})\,.$	$[D7,\ D4,\ 6]$
	$\mathsf{t}(ABCD)$	$[D13',\ 7,\ 8,\ 13,\ 14,\ 15,\ 16,\ 17,\ 18,\ 19,\ 20]$
Q6	$[ABCD]:\mathsf{t}(ABCD)\,.\supset.\,\mathsf{r}(AB)\,.\,\mathsf{r}(CD)\,.\sim(\mathsf{c}(ACD))\,.\sim(\mathsf{c}(BCD))\,.$ $\sim(\mathsf{c}(CAB))\,.\sim(\mathsf{c}(DAB))$	
PF	$[ABCD]\therefore \mathrm{Hp}(1)\,.\supset:$	
2)	$A\neq B.$	$[D13',\ 1]$
3)	$C\neq D.$	$[D13',\ 1]$
	$[\exists fg]:$	
4)	$\mathsf{i}(Af)\,.$	
5)	$\mathsf{i}(Bf)\,.$	
6)	$\mathsf{i}(Cg)\,.$	
7)	$\mathsf{i}(Dg)\,.$	$[D13',\ 1]$
8)	$\sim(\mathsf{i}(Ag))\,.$	
9)	$\sim(\mathsf{i}(Bg))\,.$	
10)	$\sim(\mathsf{i}(Cf))\,.$	
11)	$\sim(\mathsf{i}(Df))\,.$	

12) $f = \mathbf{R}\{AB\}$. [AI, D7, 4, 5, 2]
13) $g = \mathbf{R}\{CD\}$: [AI, D7, 6, 7, 3]
14) $\sim(A \varepsilon \mathbf{R}\{CD\})$. [D7, 8, D7, 4, D7, 6, 13]
15) $\sim(B \varepsilon \mathbf{R}\{CD\})$. [D7, 9, D7, 5, D7, 6, 13]
16) $\sim(C \varepsilon \mathbf{R}\{AB\})$. [D7, 10, D7, 6, D7, 4, 12]
17) $\sim(D \varepsilon \mathbf{R}\{AB\})$. [D7, 11, D7, 7, D7, 4, 12]
18) $\mathbf{r}(AB)$. [D2, 1]
19) $\mathbf{r}(CD)$. [D2, A3, 1]
$\mathbf{r}(AB).\mathbf{r}(CD).\sim(\mathbf{c}(ACD)).\sim(\mathbf{c}(BCD)).\sim(\mathbf{c}(CAB)).$
$\sim(\mathbf{c}(DAB))$ [18, 19, D4, 14, 15, 16, 17]

A7' $[ABCD]\therefore \mathbf{t}(ABCD). \equiv . \mathbf{r}(AB).\mathbf{r}(CD).\sim(\mathbf{c}(ACD)).\sim(\mathbf{c}(BCD)).$
$\sim(\mathbf{c}(CAB)).\sim(\mathbf{c}(DAB))$ [Q5; Q6]

The analogs of *L22, L23* and *P8* follow as in **2.3.**

Q7 $[ABCD]: \mathbf{t}(ABCD). \mathbf{R}\{AC\} = \mathbf{R}\{BD\}. \supset . \mathbf{R}\{AC\} \neq \mathbf{R}\{BD\}$

PF $[ABCD]$: Hp(1-2) . ⊃ .
3) $\mathbf{r}(BD)$. [D2, A3, 1]
4) $C \neq D$. [D13', 1]
5) $\mathbf{c}(DBD)$. [D3', 3]
6) $\mathbf{c}(DAC)$. [D0, 2, D4, 5]
7) $D \neq A$. [D13', 1]
8) $\mathbf{c}(ACD)$. [L22, L23, 4, 7, 6]
9) $A \varepsilon \alpha$. [D1, 1]
$[\exists g]$.
10) $\mathbf{i}(Cg)$.
11) $\mathbf{i}(Dg)$.
12) $\sim(\mathbf{i}(Ag))$. [D13', 1] (10–12)
13) $g = \mathbf{R}\{CD\}$. [AI, D7, 10, 11, 4]
14) $\mathbf{R}\{CD\} \varepsilon \beta$. [D7, 10, 13]
15) $\sim(A \varepsilon \mathbf{R}\{CD\})$. [D7, 12, 13, 9, 14]
16) $\sim(\mathbf{c}(ACD))$. [D4, 15]
$\mathbf{R}\{AB\} \neq \mathbf{R}\{CD\}$ [16, 8]

A8' $[ABCDEFGHIJKX]: X \neq A . X \neq B . X \neq C . X \neq D . X \neq E . X \neq F . \mathbf{t}(ABCD).$
$\mathbf{t}(BEDF).\mathbf{t}(AECF).\mathbf{c}(XAC).\mathbf{c}(XBD).\mathbf{c}(XEF).\mathbf{c}(IAB).\mathbf{c}(ICD).$
$\mathbf{c}(JBE).\mathbf{c}(JDE).\mathbf{c}(KAE).\mathbf{c}(KCF). \supset . \mathbf{c}(IJK). \vee . J = K$

PF $[ABCDEFGHIJKX]\therefore$ Hp(1-18) . ⊃ :
19) $\mathbf{k}(XAC)$. [P8, 10]
20) $\mathbf{k}(XBD)$. [P8, 11]
21) $\mathbf{k}(XEF)$. [P8, 12]
22) $A \neq C$. [D13', 7]
23) $B \neq D$. [D13', 8]
24) $E \neq F$. [D13', 9]
25) $\mathbf{i}(X\mathbf{R}\{AC\})$. [AI, D7, D9, 19, 22]
26) $\mathbf{i}(A\mathbf{R}\{AC\})$. [AI, D7, D9, 19, 22]
27) $\mathbf{i}(X\mathbf{R}\{BD\})$. [AI, D7, D9, 20, 23]
28) $\mathbf{i}(B\mathbf{R}\{BD\})$. [AI, D7, D9, 20, 23]
29) $\mathbf{i}(X\mathbf{R}\{EF\})$. [AI, D7, D9, 21, 24]

30)	$\mathsf{i}(E\mathsf{R}\{EF\})$.	[$A1$, $D7$, $D9$, 21, 24]
31)	$\mathsf{R}\{AC\} \neq \mathsf{R}\{BD\}$.	[$Q7$, 7]
32)	$\mathsf{R}\{BD\} \neq \mathsf{R}\{EF\}$.	[$Q7$, 8]
33)	$\mathsf{R}\{EF\} \neq \mathsf{R}\{AC\}$.	[$Q7$, 9]
34)	$\mathsf{i}(A\mathsf{R}\{AB\})$.	[$A1$, $D7$, 26, 28, $A2$, 7]
35)	$\mathsf{i}(B\mathsf{R}\{AB\})$.	[$A1$, $D7$, 26, 28, $A2$, 7]
36)	$\mathsf{i}(C\mathsf{R}\{CD\})$.	[$A1$, $D1$, $A3$, 7, $A2$, 7]
37)	$\mathsf{i}(D\mathsf{R}\{CD\})$.	[$A1$, $D1$, $A3$, 7, $A2$, 7]
38)	$\mathsf{i}(I\mathsf{R}\{AB\})$.	[$A1$, $D4$, $D12$, $D9$, $D7$, 13]
39)	$\mathsf{i}(I\mathsf{R}\{CD\})$.	[$A1$, $D4$, $D12$, $D9$, $D7$, 14]
40)	$\mathsf{i}(B\mathsf{R}\{BE\})$.	[$A1$, $D7$, 28, 30, $A2$, 8]
41)	$\mathsf{i}(E\mathsf{R}\{BE\})$.	[$A1$, $D7$, 28, 30, $A2$, 8]
42)	$\mathsf{i}(D\mathsf{R}\{DF\})$.	[$A1$, $D7$, $D9$, $P8$, 16, $A2$, 8]
43)	$\mathsf{i}(F\mathsf{R}\{DF\})$.	[$A1$, $D7$, $D9$, $P8$, 16, $A2$, 8]
44)	$\mathsf{i}(J\mathsf{R}\{BE\})$.	[$A1$, $D4$, $D12$, $D9$, $D7$, 15]
45)	$\mathsf{i}(J\mathsf{R}\{DF\})$.	[$A1$, $D4$, $D12$, $D9$, $D7$, 16]
46)	$\mathsf{i}(A\mathsf{R}\{AE\})$.	[$A1$, $D7$, 26, 30, $A2$, 9]
47)	$\mathsf{i}(E\mathsf{R}\{AE\})$.	[$A1$, $D7$, 26, 30, $A2$, 9]
48)	$\mathsf{i}(C\mathsf{R}\{CF\})$.	[$A1$, $D7$, 36, 43, $A2$, 9]
49)	$\mathsf{i}(F\mathsf{R}\{CF\})$.	[$A1$, $D7$, 36, 43, $A2$, 9]
50)	$\mathsf{i}(K\mathsf{R}\{AE\})$.	[$A1$, $D7$, $D9$, $P8$, 17, $A2$, 9]
51)	$\mathsf{i}(K\mathsf{R}\{CF\})$.	[$A1$, $D7$, $D9$, $P8$, 18, $A2$, 9]
52)	$\mathsf{k}(IJK)$:	[$A1V'$, 19, 20, 21, 1, 3, 22, 2, 4, 23, 5, 6, 24-51]
	$\mathsf{c}(IJK) . \vee . J = K$	[$P9$, 52]

CHAPTER III: HYPERBOLIC GEOMETRY

In this chapter two systems for hyperbolic geometry, (H^*) a point geometry and (H) a point-line geometry, are formalized and shown to be equivalent.

System (H^*) assumes one set of objects, points. This set is again denoted by α and its members by $A, B, C, \ldots$. One quaternary relation is used as the primitive notion. Because of its many common properties with the primitives of Chapters I and II it will be denoted by $\mathsf{t}(ABCD)$. Again it must be remembered that this is *not* the same relation as the one in (A^*) or in (P^*). Intuitively $\mathsf{t}(ABCD)$ in (H^*) means that the points A, B, C, and D are the vertices of a hyperbolic trapezoid, i.e. if the segment AB is one base of the trapezoid it is possible to have two segments CD and CE with C, D, and E non-collinear and such that either CD or CE can act as the other base. $\mathsf{r}(AB)$ and $\mathsf{c}(CAB)$ are defined as before and a system of axioms for hyperbolic geometry is given. Several auxiliary definitions are added to establish the equivalence with (H).

System (H) is a formalization of the axioms of DeBaggis [5] for Bolyai-Lobachevsky hyperbolic geometry. Two sets of objects, points and lines, are assumed, here represented by α and β respectively. Incidence is the primitive notion. Intersecting lines and collinear points are defined in the usual manner. The following axioms are used.

Axiom 1. *If A and B are two distinct points there is exactly one line containing both A and B.*

Axiom 2. *Each line contains at least one point.*

Axiom 3. *There are at least three collinear and three non-collinear points.*

Axiom 4. *If a and b are distinct lines, and P is a point not on a or b, then through P there exists a line which intersects a but not b.*

Axiom 5. *If P, Q, R are three points such that through each pair there exist two intersecting lines which fail to intersect a line through the third point, then P, Q, R are non-collinear.*

Axiom 6. *If a is a given line, then through each point not on a, there are two distinct lines which do not intersect a.*

In the formalization of (**H**) a line is logically identified with all the points lying on it in order to establish an equivalence with the point system (**H***).

The analog of *D5* of Chapter I appears in both (**H***) and (**H**). Although it never appears explicitly in the proofs it is included because of its use in the proofs quoted as similar to those in Chapter I.

It should be mentioned that (**H***) and (**H**) are not systems for a complete hyperbolic geometry, but only contain the properties of incidence, betweenness, and hyperbolic parallelism. Considerable work has been done by Menger [9], [10], [11] on hyperbolic geometry. The most significant result for this work is that Menger has shown that all of hyperbolic geometry can be built with incidence as the single primitive notion in a point-line system. The actual axiom systems were developed by his students. Jenks [6] formulated the incidence, betweenness, and parallel portions, Abbott [1] the congruence and perpendicular parts. DeBaggis [5] improved upon Jenks' work. Since incidence can be defined in terms of $\bar{\mathbf{t}}(ABCD)$ it follows that **t** can be used as a primitive for all of hyperbolic geometry although the congruence and perpendicular portions have not been developed here.

DeBaggis' system was chosen for comparison rather than an existing point system as Szmielew [20] or Tarski [23] in order to take advantage of the results of Chapters I and II.

3.1 SYSTEM (H*)

Definitions

D1 $[A]:A\varepsilon\alpha.\equiv.[\exists BCD].\mathbf{t}(ABCD)$

D2 $[AB]\therefore\mathbf{r}(AB).\equiv:[\exists CD]:\mathbf{t}(ABCD).\vee.\mathbf{t}(ACBD).\vee.\mathbf{t}(CBAD)$

D3' $[ABC]\therefore\mathbf{c}(ABC).\equiv:\mathbf{r}(BC):A=B.\vee.A=C.\vee.[X].\sim(\mathbf{t}(AXBC))$.
$A\varepsilon\alpha$

Axioms

A1 $[\exists ABCD].\mathbf{t}(ABCD)$

A2 $[ABCD]:\mathbf{t}(ABCD).\supset.A\neq B$

A3(a) $[ABCD]:\mathbf{t}(ABCD).\supset.\mathbf{t}(DCAB)$

A3(b) $[ABCD]:\mathbf{t}(ABCD).\supset.\mathbf{t}(ABDC)$

A4'' $[ABC]::A\varepsilon\alpha.B\varepsilon\alpha.C\varepsilon\alpha.\supset\therefore\sim(\mathbf{c}(CAB)).\equiv.[\exists DE].\mathbf{t}(ABCE).$
$\mathbf{t}(ABCD).\sim(\mathbf{c}(DCE)).\lor.A=B$

A5 $[ABCMN]:A\neq B.\mathbf{c}(AMN).\mathbf{c}(BMN).\mathbf{c}(CMN).\supset.\mathbf{c}(CAB)$

A6'' $[\exists ABC].\mathbf{c}(CAB).C\neq A.C\neq B$

A7 $[ABCD]\therefore\mathbf{t}(ABCD).\equiv:\mathbf{r}(AB).\mathbf{r}(CD):[M]:\mathbf{c}(MAB).\supset.\sim(\mathbf{c}(MCD))$

A8'' $[ABDE]:\sim(\mathbf{c}(DEA)).\mathbf{r}(EA).\sim(\mathbf{c}(BEA)).B\varepsilon\alpha.\sim(\mathbf{c}(BDA)).$
$\mathbf{r}(DA).\supset.[\exists FC].\mathbf{t}(BFDA).\mathbf{c}(CEA).\mathbf{c}(CBF)$

A9 $[ABCDEFJKLMNOXYZ]:\mathbf{t}(CEJA).\mathbf{t}(CEKB).\mathbf{c}(DJA).\mathbf{c}(DKB).$
$\mathbf{c}(XJA).\sim(\mathbf{c}(XKB)).\mathbf{t}(BGLC).\mathbf{t}(BGMA).\mathbf{c}(FLC).\mathbf{c}(FMA).$
$\mathbf{c}(YLC).\sim(\mathbf{c}(YMA)).\mathbf{t}(AINB).\mathbf{t}(AIOC).\mathbf{c}(HNB).\mathbf{c}(HOC).$
$\mathbf{c}(ZNB).\sim(\mathbf{c}(ZOC)).\supset.\sim(\mathbf{c}(ABC))$

Auxiliary Definitions

D4 $[ABC]:A\varepsilon\mathbf{R}\{BC\}.\equiv.\mathbf{c}(ABC)$

D5 $[ABCD]:\mathbf{s}(ABCD).\equiv.\mathbf{c}(ACD).\mathbf{c}(BCD).A\neq B$

D6 $[f]::f\varepsilon\beta.\equiv\therefore[\exists AB]\therefore\mathbf{r}(AB):[C]:\mathbf{c}(CAB).\equiv.C\varepsilon f$

D7 $[Af]:\mathbf{i}(Af).\equiv.A\varepsilon\alpha.A\varepsilon f.f\varepsilon\beta$

D8 $[fg]\therefore\mathbf{p}(fg).\equiv:f\varepsilon\beta.g\varepsilon\beta:g=f.\lor.\sim([\exists A].\mathbf{i}(Ag).\mathbf{i}(Af))$

D9 $[ABC]:\mathbf{k}(ABC).\equiv.[\exists g].\mathbf{i}(Ag).\mathbf{i}(Bg).\mathbf{i}(Cg)$

3.2 SYSTEM (H)

Definitions

D10 $[A]:A\varepsilon\alpha.\equiv.[\exists f].\mathbf{i}(Af)$

D11 $[f]:f\varepsilon\beta.\equiv.[\exists A].\mathbf{i}(Af)$

D9 $[ABC]:\mathbf{k}(ABC).\equiv.[\exists g].\mathbf{i}(Ag).\mathbf{i}(Bg).\mathbf{i}(Cg)$

D12 $[ABC]:A\varepsilon\mathbf{R}\{BC\}.\equiv.B\neq C.\mathbf{k}(ABC)$

Axioms

BI $[AB]\therefore A\varepsilon\alpha.B\varepsilon\alpha.A\neq B.\supset.[\exists f].\mathbf{i}(Af).\mathbf{i}(Bf).f=\mathbf{R}\{AB\}:[gh]:$
$\mathbf{i}(Ag).\mathbf{i}(Bg).\mathbf{i}(Ah).\mathbf{i}(Bh).\supset.g=h$

BII $[f]:f\varepsilon\beta.\supset.[\exists A].\mathbf{i}(Af)$

BIII(a) $[\exists ABC].A\varepsilon\alpha.B\varepsilon\alpha.C\varepsilon\alpha.A\neq B.B\neq C.C\neq A.\mathbf{k}(ABC)$

BIII(b) $[\exists ABC].A\varepsilon\alpha.B\varepsilon\alpha.C\varepsilon\alpha.A\neq B.B\neq C.C\neq A.\sim(\mathbf{k}(ABC))$

BIV $[ABfg]::\mathbf{i}(Af).\mathbf{i}(Ag).f\neq g.B\varepsilon\alpha.\sim(\mathbf{i}(Bf)).\sim(\mathbf{i}(Bg)).\supset\therefore$
$[\exists Ch]\therefore\mathbf{i}(Bh).\mathbf{i}(Ch).\mathbf{i}(Cf):[D]:\mathbf{i}(Dh).\supset.\sim(\mathbf{i}(Dg))$

BV $[ABCDFHfghjlmnoq]\therefore\mathbf{i}(Cf).\mathbf{i}(Ag).\mathbf{i}(Bh).\mathbf{i}(Dg).\mathbf{i}(Dh).$
$g\neq h:[M]:\mathbf{i}(Mf).\supset.\sim(\mathbf{i}(Mg)).\sim(\mathbf{i}(Mh)):\mathbf{i}(Aj).\mathbf{i}(Bl).$
$\mathbf{i}(Cm).\mathbf{i}(Fl).\mathbf{i}(Fm).l\neq m:[N]:\mathbf{i}(Nj).\supset.\sim(\mathbf{i}(Nl)).$
$\sim(\mathbf{i}(Nm)):\mathbf{i}(Bn).\mathbf{i}(Co).\mathbf{i}(Aq).\mathbf{i}(Ho).\mathbf{i}(Hq).o\neq q:[K]:$
$\mathbf{i}(Kn).\supset.\sim(\mathbf{i}(Ko)).\sim(\mathbf{i}(Kq)):\supset.\sim(\mathbf{k}(ABC))$

BVI $[Af]::f\varepsilon\beta.A\varepsilon\alpha.\sim(\mathbf{i}(Af)).\supset\therefore[\exists gh]\therefore g\neq h.\mathbf{i}(Ag).\mathbf{i}(Ah):$
$[B]:\mathbf{i}(Bg).\supset.\sim(\mathbf{i}(Bf)):[C]:\mathbf{i}(Ch).\supset.\sim(\mathbf{i}(Cf))$

Auxiliary Definitions

D8 $[fg] .\because. \mathsf{p}(fg) . \equiv : f \varepsilon \beta . g \varepsilon \beta : g = f . \mathrm{v} . \sim ([\exists A] . \mathsf{i}(Ag) . \mathsf{i}(Af))$

D13 $[ABCD] : \mathsf{t}(ABCD) . \equiv . A \neq B . C \neq D . [\exists fg] . \mathsf{p}(fg) . f \neq g . \mathsf{i}(Af) .$
$\mathsf{i}(Bf) . \mathsf{i}(Cg) . \mathsf{i}(Dg)$

D2 $[AB] .\because. \mathsf{r}(AB) . \equiv : [\exists CD] : \mathsf{t}(ABCD) . \mathrm{v} . \mathsf{t}(ACBD) . \mathrm{v} . \mathsf{t}(CBAD)$

D3' $[ABC] .\because. \mathsf{c}(ABC) . \equiv : \mathsf{r}(BC) : A = B . \mathrm{v} . A = C . \mathrm{v} . [X] . \sim (\mathsf{t}(AXBC)) .$
$A \varepsilon \alpha$

D5 $[ABCD] : \mathsf{s}(ABCD) . \equiv . \mathsf{c}(ACD) . \mathsf{c}(BCD) . A \neq B$

3.3 SYSTEM (H*) IMPLIES SYSTEM (H)

In the following the analogs of *L1-L20, L23, L27, L29, L31, L32, L34, L35, L44-L46* and *L54-L61* will be used. The proofs are similar to those in **1.3** if *D3'* is substituted for *D3* except for *L14* and *L18* which are proved below. For brevity *A3* will be used to designate any of *L1-L12* and *A2* for any of *L13-L17*.

L14 $[ABCD] : \mathsf{t}(ABCD) . \supset . A \neq C$

PF $[ABCD] .\because. \mathrm{Hp}(1) . \supset :$

2)	$\mathsf{t}(CDAB)$.	[*L5*, 1]
3)	$\mathsf{r}(CD)$.	[*A7*, 2]
4)	$\mathsf{r}(AB)$:	[*A7*, 2]
5)	$[M] : \mathsf{c}(MCD) . \supset . \sim (\mathsf{c}(MAB))$:	[*A7*, 2]
6)	$\mathsf{c}(CCD)$.	[*D3'*, 3]
7)	$\sim (\mathsf{c}(CAB))$.	[5, 6]
	$C \neq A$	[*D3'*, 4, 7]

L18 $[AB] : A \varepsilon \alpha . B \varepsilon \alpha . A \neq B . \sim (\mathsf{r}(AB)) . \supset . \mathsf{r}(AB)$

PF $[AB] .\because. \mathrm{Hp}(1\text{-}4) . \supset :$

5)	$\sim (\mathsf{c}(AAB))$:	[*D3'*, 4]
	$[\exists C]$:	
6)	$\mathsf{t}(ABAC) . \mathrm{v} . A = B$:	[*A4"*, 1, 2, 1, 5]
	$\mathsf{r}(AB)$	[6; *L14*; 3]

BI $[AB] .\because. A \varepsilon \alpha . B \varepsilon \alpha . A \neq B . \supset . [\exists f] . \mathsf{i}(Af) . \mathsf{i}(Bf) . f = \mathbf{R}\{AB\} : [gh] :$
$\mathsf{i}(Ag) . \mathsf{i}(Bg) . \mathsf{i}(Ah) . \mathsf{i}(Bh) . \supset . g = h$ [*L34, L35, D0*]

BII $[f] : f \varepsilon \beta . \supset . [\exists A] . \mathsf{i}(Af)$ [*L58*]

BIII(a) $[\exists ABC] . A \varepsilon \alpha . B \varepsilon \alpha . C \varepsilon \alpha . A \neq B . A \neq C . B \neq C . \mathsf{k}(ABC)$

PF $[\exists ABC]$.

1)	$\mathsf{c}(CAB)$.	[*A6''*] (1–3)
2)	$C \neq A$.	
3)	$C \neq B$.	
4)	$A \varepsilon \alpha$.	[*D1*, *L20*, 1]
5)	$B \varepsilon \alpha$.	[*D1*, *L20*, 1]
6)	$C \varepsilon \alpha$.	[*D1*, *L20*, 1]
7)	$\mathsf{r}(AB)$.	[*D3'*, 1]
8)	$A \neq B$.	[*L19*, 7]
9)	$\mathbf{R}\{AB\} \varepsilon \beta$.	[*D6*, 7, *D4*]
10)	$A \varepsilon \mathbf{R}\{AB\}$.	[*D4*, *D3'*, 7]
11)	$B \varepsilon \mathbf{R}\{AB\}$.	[*D4*, *D3'*, 7]

12) $C\varepsilon\mathbf{R}\{AB\}$. [$D4$, 1]

13) $\mathsf{k}(ABC)$. [$D9$, $D7$, 4, 5, 6, 10, 11, 12, 9]

$[\exists ABC].A\varepsilon\alpha.B\varepsilon\alpha.C\varepsilon\alpha.A\neq B.A\neq C.B\neq C.\mathsf{k}(ABC)$ [4, 5, 6, 8, 2, 3, 13]

BIII(b) $[\exists ABC].A\varepsilon\alpha.B\varepsilon\alpha.C\varepsilon\alpha.A\neq B.B\neq C.C\neq A.\sim(\mathsf{k}(ABC))$ [$L44$, $A1$, $D1$, $L1$, $L2$, $A2$, $L14$, $L15$]

H1 $[Af]:\mathsf{i}(Af).\supset.[\exists B].A\neq B.\mathsf{i}(Bf).\mathbf{R}\{AB\}=f$

PF $[Af]::\mathrm{Hp}(1).\supset.\because$

2) $f\varepsilon\beta.\because$ [$D7$, 1]

$[\exists CD].\because$

3) $\mathsf{r}(CD):$ [$D6$, 2]

4) $[X]:\mathsf{c}(XCD).\equiv.X\varepsilon f:$ [$D6$, 2]

5) $C\neq D:$ [$L19$, 3]

6) $A\neq C.\vee.A\neq D:$ [5]

7) $\mathsf{c}(CCD)$. [$D3'$, 3]

8) $\mathsf{c}(DCD)$. [$D3'$, 3]

9) $C\varepsilon\alpha$. [$D1$, $A3$, $D2$, 3]

10) $D\varepsilon\alpha$. [$D1$, $A3$, $D2$, 3]

11) $C\varepsilon f$. [4, 7]

12) $D\varepsilon f$. [4, 8]

13) $\mathsf{i}(Cf)$. [$D7$, 9, 11, 2]

14) $\mathsf{i}(Df)$. [$D7$, 10, 12, 2]

$[\exists B].A\neq B.\mathsf{i}(Bf).\mathbf{R}\{AB\}=f$ [6, 13, 14, BI, $D7$, 1, 13, 14]

H2 $[ADE]:\mathsf{r}(EA).\mathsf{r}(DA).\mathbf{R}\{EA\}\neq\mathbf{R}\{DA\}.\mathsf{c}(DEA).\supset.\sim(\mathsf{c}(DEA))$

PF $[ADE]:\mathrm{Hp}(1\text{-}4).\supset.$

5) $D\varepsilon\alpha$. [$D1$, $A3$, $D2$, 2]

6) $A\varepsilon\alpha$. [$D1$, $A3$, $D2$, 2]

7) $\mathbf{R}\{EA\}\varepsilon\beta$. [$D6$, 1, $D4$]

8) $\mathsf{c}(AEA)$. [$D3'$, 1]

9) $\mathsf{i}(D\mathbf{R}\{EA\})$. [$D7$, 5, $D4$, 4, 7]

10) $\mathsf{i}(A\mathbf{R}\{EA\})$. [$D7$, 6, $D4$, 8, 7]

11) $D\neq A$. [$A2$, $D2$, 2]

12) $\mathbf{R}\{DA\}=\mathbf{R}\{EA\}$. [$BI$, 5, 6, 11, 9, 10]

$\sim(\mathsf{c}(DEA))$ [3, 12]

H3 $[ABfg]:\mathsf{i}(Af).\mathsf{i}(Ag).f\neq g.\supset.[\exists DE].\mathbf{R}\{EA\}=f.\mathbf{R}\{DA\}=g.$
$\sim(\mathsf{c}(DEA))$

PF $[ABfg].\because.\mathrm{Hp}(1\text{-}3).\supset:$

$[\exists D]:$

4) $A\neq D$. [$H1$, 2]

5) $\mathsf{i}(Dg)$. [$H1$, 2]

6) $\mathbf{R}\{AD\}=g$. [$H1$, 2]

7) $\mathsf{r}(DA)$. [$L19$, $D7$, 2, 5, 4]

8) $\mathbf{R}\{DA\}=g$. [$L27$, $L32$, 7, 6]

$[\exists E]$.

9) $A\neq E$. [$H1$, 1]

10) $\mathsf{i}(Ef)$. [$H1$, 1]

11) $\mathbf{R}\{AE\}=f$. [$H1$, 1]

12) $\mathsf{r}(EA)$. [$L19$, $D7$, 1, 10, 9]

13) $\mathbf{R}\{EA\}=f.$ [$L32$, $L27$, 12, 11]
14) $\sim(\mathbf{c}(DEA))$: [$H2$, 12, 7, 3, 8, 13]
$[\exists DE].\mathbf{R}\{EA\}=f.\mathbf{R}\{DA\}=g.\sim(\mathbf{c}(DEA))$ [13, 8, 14]

BIV $[ABfg]::\mathbf{i}(Af).\mathbf{i}(Ag).f\neq g.B\varepsilon\alpha.\sim(\mathbf{i}(Bf)).\sim(\mathbf{i}(Bg)).\supset.\because$
$[\exists Ch].\because.\mathbf{i}(Bh).\mathbf{i}(Ch).\mathbf{i}(Cf):[D]:\mathbf{i}(Dh).\supset.\sim(\mathbf{i}(Dg))$

PF $[ABfg]\because$ Hp(1-6)$.\supset::$
$[\exists KE]::$
7) $\mathbf{R}\{EA\}=f.$
8) $\mathbf{R}\{KA\}=g.$ [$H3$, 1, 2, 3]
9) $\sim(\mathbf{c}(KEA)).$
10) $\mathbf{r}(EA).$ [$D3'$, $D4$, $D7$, 1, 7]
11) $\mathbf{r}(KA).$ [$D3'$, $D4$, $D7$, 2, 8]
12) $\sim(B\varepsilon\mathbf{R}\{EA\}).$ [$D7$, 5, 4, $D7$, 1, 7]
13) $\sim(B\varepsilon\mathbf{R}\{KA\}).$ [$D7$, 6, 4, $D7$, 2, 8]
14) $\sim(\mathbf{c}(BEA)).$ [$D4$, 12]
15) $\sim(\mathbf{c}(BKA)).\because$ [$D4$, 13]
$[\exists FC].\because$
16) $\mathbf{t}(BFKA).$
17) $\mathbf{c}(CEA).$ [$A8''$, 9, 10, 14, 4, 15, 11]
18) $\mathbf{c}(CBF).$
19) $\mathbf{r}(BF)$:
20) $[D]:\mathbf{c}(DBF).\supset.\sim(\mathbf{c}(DKA))$: [$A7$, 16]
21) $\mathbf{R}\{BF\}\,\varepsilon\beta.$ [$D6$, 19, $D4$]
22) $\mathbf{i}(B\mathbf{R}\{BF\}).$ [$D7$, 4, $D4$, $D3'$, 19, 21]
23) $C\varepsilon\alpha.$ [$D1$, $A3$, $D2$, $D3'$, 17]
24) $\mathbf{i}(C\mathbf{R}\{BF\}).$ [$D7$, 23, $D4$, 18, 21]
25) $\mathbf{i}(Cf)$: [$D7$, 23, $D4$, 17, 7, $D7$, 1]
26) $[D]:\mathbf{i}(D\mathbf{R}\{BF\}).\supset.\sim(\mathbf{i}(Dg))::$
[$D7$, $D0$, 8, $D4$, 20, $D7$]
$[\exists Ch].\because.\mathbf{i}(Bh).\mathbf{i}(Ch).\mathbf{i}(Cf):[D]:\mathbf{i}(Dh).\supset.\sim(\mathbf{i}(Dg))$
[22, 24, 25, 26]

H4 $[ABCDfgh].\because.\mathbf{i}(Cf).\mathbf{i}(Ag).\mathbf{i}(Bh).\mathbf{i}(Dg).\mathbf{i}(Dh).g\neq h:[M]:$
$\mathbf{i}(Mf).\supset.\sim(\mathbf{i}(Mg)).\sim(\mathbf{i}(Mh)):\supset.[\exists EJKX].\mathbf{t}(CEJA).$
$\mathbf{t}(CEKB).\mathbf{c}(DJA).\mathbf{c}(DKB).\mathbf{c}(XJA).\sim(\mathbf{c}(XKB))$

PF $[ABCDfgh]::$ Hp(1-7)$:\supset\because$
$[\exists E]\because$
8) $\mathbf{i}(Ef).$
9) $\mathbf{R}\{CE\}=f.$ [$H1$, 1]
10) $\mathbf{r}(CE)::$ [$D3'$, $D4$, $D7$, 8, 9]
$[\exists J]::$
11) $\mathbf{i}(Jg).$
12) $\mathbf{R}\{AJ\}=g.$ [$H1$, 2]
13) $D\varepsilon\mathbf{R}\{AJ\}.$ [$D7$, 4, 12]
14) $\mathbf{c}(DJA).$ [$L23$, $D4$, 13]
15) $\mathbf{r}(JA)$: [$D3'$, 14]
16) $[M]:\mathbf{c}(MCE).\supset.\sim(\mathbf{c}(MJA))$:
[$D4$, $D7$, $D1$, $L20$, $L23$, 11, 12, 7, $D7$, 8, 9]

17) $\mathbf{t}(CEJA) .\because$ [$A7$, 10, 15, 16]

$[\exists K] .\because$

18) $\mathbf{i}(Kh)$.

19) $\mathbf{R}\{BK\} = h$. [$H1$, 3]

20) $D \varepsilon \mathbf{R}\{BK\}$. [$D7$, $D4$, 18]

21) $\mathbf{c}(DKB)$. [$L23$, $D4$, 20]

22) $\mathbf{r}(KB)$: [$D3'$, 21]

23) $[M] : \mathbf{c}(MCE) . \supset . \sim(\mathbf{c}(MKB))$:

[$D4$, $D7$, $D1$, $L20$, $L23$, 18, 19, 7, $D7$, 8, 9]

24) $\mathbf{t}(CEKB)$. [$A7$, 10, 22, 23]

$[\exists X]$.

25) $X \varepsilon \mathbf{R}\{AJ\}$.

26) $\sim(X \varepsilon \mathbf{R}\{BK\}$. [$D0$, 6, 12, 19]

27) $\mathbf{c}(XJA)$. [$L23$, $D4$, 25]

28) $\sim(\mathbf{c}(XKB) ::$ [$L23$, $D4$, 26]

$[\exists EJKX] . \mathbf{t}(CEJA) . \mathbf{t}(CEKB) . \mathbf{c}(DJA) . \mathbf{c}(DKB) .$
$\mathbf{c}(XJA) . \sim(\mathbf{c}(XKB))$ [17, 24, 14, 21, 27, 28]

BV $[ABCDFHfghjlmnoq] .\because \mathbf{i}(Cf) . \mathbf{i}(Ag) . \mathbf{i}(Bh) . \mathbf{i}(Dg) . \mathbf{i}(Dh) .$
$g \neq h : [M] : \mathbf{i}(Mf) . \supset . \sim(\mathbf{i}(Mg)) . \sim(\mathbf{i}(Mh)) : \mathbf{i}(Aj) . \mathbf{i}(Bl) .$
$\mathbf{i}(Cm) . \mathbf{i}(Fl) . \mathbf{i}(Fm) . l \neq m : [N] : \mathbf{i}(Nj) . \supset . \sim(\mathbf{i}(Nl)).$
$\sim(\mathbf{i}(Nm)) : \mathbf{i}(Bn) . \mathbf{i}(Co) . \mathbf{i}(Aq) . \mathbf{i}(Ho) . \mathbf{i}(Hq) . o \neq q : [K] :$
$\mathbf{i}(Kn) . \supset . \sim(\mathbf{i}(Ko)) . \sim (\mathbf{i}(Kq)) : \supset . \sim (\mathbf{k}(ABC))$

PF $[ABCDFHfghjlmnoq] :: \mathrm{Hp}(1\text{-}21) : \supset .\because$

22) $B \neq C .\because$ [21, 15, 16]

$[\exists EJKX] .\because$

23) $\mathbf{t}(CEJA)$.

24) $\mathbf{t}(CEKB)$.

25) $\mathbf{c}(DJA)$.

26) $\mathbf{c}(DKB)$.

27) $\mathbf{c}(XJA)$.

28) $\sim(\mathbf{c}(XKB))$: [$H4$, 1-7]

$[\exists GLMY]$:

29) $\mathbf{t}(BGLC)$.

30) $\mathbf{t}(BGMA)$.

31) $\mathbf{c}(HLC)$.

32) $\mathbf{c}(HMA)$.

33) $\mathbf{c}(YCL)$.

34) $\sim(\mathbf{c}(YMA))$. [$H4$, 15-21]

$[\exists INOZ]$.

35) $\mathbf{t}(AINB)$.

36) $\mathbf{t}(AIOC)$.

37) $\mathbf{c}(FNB)$.

38) $\mathbf{c}(FOC)$.

39) $\mathbf{c}(ZNB)$.

40) $\sim(\mathbf{c}(ZOC)) .\because$ [$H4$, 8-14]

41) $\sim(\mathbf{c}(ABC))$. [$A9$, 23-40]

$\sim(\mathbf{k}(ABC))$ [$L45$, 41, 22]

H5 $[ADE]:\mathbf{r}(EA).\mathbf{r}(DA).\sim(\mathbf{c}(DAE).\supset.\mathbf{R}\{EA\}\neq\mathbf{R}\{DA\}$

PF $[ADE]:\mathrm{Hp}(1\text{-}3).\supset.$

4) $\sim(D\varepsilon\mathbf{R}\{EA\}).$ [*D4*, *L23*, 3]

5) $\mathbf{c}(DDA).$ [*D3'*, 2]

6) $D\varepsilon\mathbf{R}\{DA\}.$ [*D4*, 5]

$\mathbf{R}\{EA\}\neq\mathbf{R}\{DA\}$ [*D0*, 4, 6]

BVI $[Af]::f\varepsilon\beta.A\varepsilon\alpha.\sim(\mathbf{i}(Af)).\supset.\because.[\exists gh].\because.g\neq h.\mathbf{i}(Ag).\mathbf{i}(Ah):$
$[B]:\mathbf{i}(Bg).\supset.\sim(\mathbf{i}(Bf)):[C]:\mathbf{i}(Ch).\supset.\sim(\mathbf{i}(Cf))$

PF $[Af]::\mathrm{Hp}(1\text{-}3).\supset\because$

$[\exists B]\because$

4) $\mathbf{i}(Bf)::$ [*BII*, 1]

$[\exists C]::$

5) $\mathbf{i}(Cf).$

6) $C\neq B.$

7) $f=\mathbf{R}\{BC\}.$ [*H1*, 4] (5–7)

8) $\sim(\mathbf{c}(ABC)).\because.$ [*D4*, *D7*, 3, 7]

$[\exists DE].\because.$

9) $\mathbf{t}(BCAE).$

10) $\mathbf{t}(BCAD).$

11) $\sim(\mathbf{c}(DAE)).$ [*A4''*, *D7*, 4, 5, 2, 8, 6] (9–11)

12) $\mathbf{t}(EABC).$ [*A3*, 9]

13) $\mathbf{t}(DABC):$ [*A3*, 10]

14) $[K]:\mathbf{c}(KEA).\supset.\sim(\mathbf{c}(KBC)):$ [*A7*, 12]

15) $[K]:\mathbf{c}(KDA).\supset.\sim(\mathbf{c}(KBC)):$ [*A7*, 13]

16) $[K]:\mathbf{i}(K\mathbf{R}\{EA\}).\supset.\sim(\mathbf{i}(Kf)):$ [*D7*, *D4*, 14, 7]

17) $[K]:\mathbf{i}(K\mathbf{R}\{DA\}).\supset.\sim(\mathbf{i}(Kf)):$ [*D7*, *D4*, 15, 7]

18) $\mathbf{r}(EA).$ [*D2*, 12]

19) $\mathbf{r}(DA).$ [*D2*, 13]

20) $\mathbf{R}\{EA\}\neq\mathbf{R}\{DA\}.$ [*H5*, 18, 19, 11]

21) $\mathbf{i}(A\mathbf{R}\{EA\}).$ [*D7*, *L19*, 18, *D6*, 18, *D4*, *D3'*]

22) $\mathbf{i}(A\mathbf{R}\{DA\})\because$ [*D7*, *L19*, 19, *D6*, 19, *D4*, *D3'*]

$[\exists gh].\because.g\neq h.\mathbf{i}(Ag).\mathbf{i}(Ah):[B]:\mathbf{i}(Bg).\supset.\sim(\mathbf{i}(Bf)):$
$[C]:\mathbf{i}(Ch).\supset.\sim(\mathbf{i}(Cf))$ [20, 21, 22, 16, 17]

D13 $[ABCD]:\mathbf{t}(ABCD).\equiv.A\neq B.C\neq D.[\exists fg].\mathbf{p}(fg).f\neq g.\mathbf{i}(Af).$
$\mathbf{i}(Bf).\mathbf{i}(Cg).\mathbf{i}(Dg)$ [*L56*, *D8*, *D7*]

D10 $[A]:A\varepsilon\alpha.\equiv.[\exists f].\mathbf{i}(Af)$ [*L57*; *D7*]

D11 $[f]:f\varepsilon\beta.\equiv.[\exists A].\mathbf{i}(Af)$ [*BII*; *D7*]

D12 $[ABC]:A\varepsilon\mathbf{R}\{BC\}.\equiv.B\neq C.\mathbf{k}(ABC)$ [*L59*, *L60*; *L61*]

3.4 SYSTEM (H) IMPLIES SYSTEM (H*)

In the following the analogs of *L1*-*L20*, *M1*, *M2*, *M6*-*M28*, *M31*, *M33*-*M35*, *Q2* and *Q3* will be used. The proofs are similar to those in **1.3**, **1.4** and **2.4** except for *M7*, *M10*, and *M16* which are given below. In *L18*, *A4''* must be substituted for *A4*; *D3'* must always be substituted for *D3*; *BI* and *BIII*(*a*) are the analogs of *AI* and *AIII* respectively; in *M24*, *K5* which is proved below must be used in place of *AII*; *M26* is proved as in **2.4** using *Q2* and *Q3*.

$A2$ $[ABCD]:\mathsf{t}(ABCD).\supset.A\neq B$ [$D13$]

$A3(a)$ $[ABCD]:\mathsf{t}(ABCD).\supset.\mathsf{t}(DCAB)$ [$D13$, $M2$]

$A3(b)$ $[ABCD]:\mathsf{t}(ABCD).\supset.\mathsf{t}(ABDC)$ [$D13$]

$M7$ $[ABCf]:\mathsf{i}(Af).B\varepsilon\alpha.C\varepsilon\alpha.\sim(\mathsf{k}(ABC)).\mathsf{p}(f\mathsf{R}\{BC\}).B\neq C.A\neq B.$
$\supset.[\exists D].A\neq D.\mathsf{i}(Df)$

PF $[ABCf]\therefore\mathrm{Hp}(1\text{-}7).\supset::$

8) $\mathsf{i}(B\mathsf{R}\{BC\}).$ [BI, 2, 3, 6]

9) $\mathsf{i}(C\mathsf{R}\{BC\}).$ [BI, 2, 3, 6]

10) $f\neq\mathsf{R}\{BC\}.$ [$D9$, 4, 1, 8, 9]

11) $\sim(\mathsf{i}(Cf)).$ [$D8$, 5, 10, 9]

12) $\sim(\mathsf{i}(Bf))::$ [$D8$, 5, 10, 8]

$[\exists g]::$

13) $\mathsf{i}(Ag).$

14) $\mathsf{i}(Bg).$ [BI, $D7$, 1, 2, 7]

15) $f\neq g.$ [12, 14]

16) $\sim(\mathsf{i}(Cg))\therefore$ [$D9$, 4, 13, 14]

$[\exists Dh]\therefore$

17) $\mathsf{i}(Dh).$

18) $\mathsf{i}(Df):$ [BIV, 1, 13, 15, 3, 11, 16]

19) $[K]:\mathsf{i}(Kh).\supset.\sim(\mathsf{i}(Kg)):$

20) $A\neq D::$ [19, 13, 17]

$[\exists D].A\neq D.\mathsf{i}(Df)$ [20, 18]

$K1$ $[ABCf]:\mathsf{p}(f\mathsf{R}\{BC\}).f\neq\mathsf{R}\{BC\}.\mathsf{i}(Af).B\neq C.\mathsf{k}(ABC).\supset.\sim(\mathsf{k}(ABC))$

PF $[ABCf]:\mathrm{Hp}(1\text{-}5).\supset.$

$[\exists g].$

6) $\mathsf{i}(Ag).$

7) $\mathsf{i}(Bg).$ [$D9$, 5]

8) $\mathsf{i}(Cg).$

9) $g=\mathsf{R}\{BC\}.$ [BI, $D7$, 7, 8, 4]

10) $\mathsf{i}(A\mathsf{R}\{BC\}).$ [6, 9]

11) $\sim(\mathsf{i}(Af)).$ [$D8$, 1, 2, 10]

$\sim(\mathsf{k}(ABC))$ [3, 11]

$K2$ $[ABC]:\mathsf{i}(Af).B\neq C.\mathsf{p}(f\mathsf{R}\{BC\}).B\varepsilon\alpha.C\varepsilon\alpha.\supset.[\exists E].A\neq E.$
$\mathsf{i}(Ef)$

PF $[ABC]\therefore\mathrm{Hp}(1\text{-}5).\supset:$

6) $\mathsf{i}(B\mathsf{R}\{BC\}).$ [BI, 4, 5, 2]

7) $\mathsf{i}(C\mathsf{R}\{BC\}):$ [BI, 4, 5, 2]

8) $f=\mathsf{R}\{BC\}.\vee.f\neq\mathsf{R}\{BC\}:$ [*Logic*]

$[\exists E].A\neq E.\mathsf{i}(Ef)$

[8; 2, 6, 7; $M7$, 1, 4, 5, $K1$, 3, 1, 2, 3, 2, $D8$, 3, 6, 1]

$M10$ $[ABCD]:\mathsf{i}(Af).B\neq C.C\neq D.D\neq B.\sim(\mathsf{k}(BCD)).\mathsf{p}(f\mathsf{R}\{BC\}).$
$\sim(\mathsf{i}(Df)).B\varepsilon\alpha.C\varepsilon\alpha.D\varepsilon\alpha.\supset.[\exists E].A\neq E.\mathsf{i}(Ef)$ [$K2$]

$D7$ $[Af]:\mathsf{i}(Af).\equiv.A\varepsilon\alpha.A\varepsilon f.f\varepsilon\beta$ [$D10$, $M14$, $D11$; $M15$]

$K3$ $[Af]:A\varepsilon\alpha.f\varepsilon\beta.\sim(\mathsf{i}(Af)).\supset.[\exists g].\mathsf{i}(Ag).\mathsf{p}(fg)$

PF $[Af]::\mathrm{Hp}(1\text{-}3).\supset\therefore$

$[\exists g]\therefore$

4) $\mathsf{i}(Ag):$

5) $[B]:i(Bg).\supset.\sim(\mathsf{i}(Bf)):$ [BVI, 1, 2, 3]

6) $g \varepsilon \beta.$ [$D11$, 4]
7) $\mathbf{p}(fg).\therefore$ [$D8$, 2, 6. 5]
$[\exists g].\mathbf{i}(Ag).\mathbf{p}(fg)$ [4, 7]

$K4$ $[Af]:\mathbf{i}(Af).\supset.\mathbf{p}(ff)$ [$D11$, $M1$]

$K5$ $[Af]:A\varepsilon\alpha.f\varepsilon\beta.\supset.[\exists g].\mathbf{i}(Ag).\mathbf{p}(fg)$ [$K3$, $K4$]

$K6$ $[ABC]:A\varepsilon\alpha.B\varepsilon\alpha.C\varepsilon\alpha.A\neq B.\sim(\mathbf{k}(ABC)).\supset.[\exists D].\mathbf{t}(ABCD)$

PF $[ABC]::\mathrm{Hp}(1\text{-}5).\supset.\therefore$
$[\exists f].\therefore$
6) $\mathbf{i}(Af).$
7) $\mathbf{i}(Bf).$ [BI, 1, 2, 4]
8) $f=\mathsf{R}\{AB\}.$
9) $\sim(\mathbf{i}(Cf)):$ [$D9$, 5, 6, 7]
$[\exists g]:$
10) $\mathbf{i}(Cg).$ [$K5$, 3, $D7$, 6]
11) $\mathbf{p}(fg).$
12) $f\neq g.$ [9, 10]
$[\exists D].$
13) $C\neq D.$ [$M13$, $D7$, 10]
14) $\mathbf{i}(Dg).\therefore$
$\mathbf{t}(ABCD)$ [$D13$, 4, 13, 11, 12, 6, 7, 10, 14]

$M16$ $[ABC]:A\varepsilon\alpha.B\varepsilon\alpha.C\varepsilon\alpha.A\neq B.A\neq C.B\neq C.\sim(\mathbf{k}(ABC)).\supset.[\exists D].$
$\mathbf{t}(ABCD)$ [$K6$]

$D1$ $[ABCD]:A\varepsilon\alpha.\equiv.[\exists BCD].\mathbf{t}(ABCD)$ [$M20$, $M21$]

$A1$ $[\exists ABCD].\mathbf{t}(ABCD)$ [$M16$, $BIII(b)$]

$D4$ $[ABC]:A\varepsilon\mathsf{R}\{BC\}.\equiv.\mathbf{c}(ABC)$ [$M25$, $M26$]

$D6$ $[f]::f\varepsilon\beta.\equiv.\therefore[\exists AB].\therefore\mathbf{r}(AB):[C]:\mathbf{c}(CAB).\equiv.C\varepsilon f$ [$M28$, $M13$; $M27$]

$K7$ $[ABC]:A\varepsilon\alpha.B\varepsilon\alpha.C\varepsilon\alpha.\sim(\mathbf{c}(CAB)).A\neq B.\supset.[\exists DE].\mathbf{t}(ABCE).$
$\mathbf{t}(ABCD).\sim(\mathbf{c}(DCE))$

PF $[ABC]::\mathrm{Hp}(1\text{-}5).\supset.\therefore$
6) $\mathbf{i}(A\mathsf{R}\{AB\}).$ [BI, 1, 2, 5]
7) $\mathbf{i}(B\mathsf{R}\{AB\}).$ [BI, 1, 2, 5]
8) $\mathsf{R}\{AB\}\varepsilon\beta.$ [$D7$, 6]
9) $\sim(C\varepsilon\mathsf{R}\{AB\}).\therefore$ [$D4$, 4]
$[\exists gh].\therefore$
10) $g\neq h.$
11) $\mathbf{i}(Cg).$
12) $\mathbf{i}(Ch):$ [BVI, 8, 3, $D7$, 9]
13) $[K]:\mathbf{i}(Kg).\supset.\sim(\mathbf{i}(K\mathsf{R}\{AB\}):$
14) $[M]:\mathbf{i}(Mh).\supset.\sim(\mathbf{i}(M\mathsf{R}\{AB\}):$
15) $\mathbf{p}(\mathsf{R}\{AB\}g).$ [$D8$, 8, $D7$, 11, 13]
16) $\mathbf{p}(\mathsf{R}\{AB\}h).$ [$D8$, 8, $D7$, 12, 14]
17) $\mathsf{R}\{AB\}\neq g.$ [9, 11]
18) $\mathsf{R}\{AB\}\neq h:$ [9, 12]
$[\exists E]:$
19) $C\neq E.$ [$M13$, $D7$, 11]
20) $\mathbf{i}(Eg).$
21) $\mathbf{t}(ABCE).$ [$D13$, 5, 19, 15, 17, 6, 7, 11, 20]

22) $g=\mathsf{R}\{CE\}$. [*BI*, *D7*, 11, 20, 19]
$[\exists D]$.
23) $C \neq D$. [*M13*, *D7*, 12]
24) $\mathsf{i}(Dh)$.
25) $\mathsf{t}(ABCD)$. [*D13*, 5, 23, 16, 18, 6, 7, 12, 24]
26) $\sim(\mathsf{i}(Dg))$. [*BI*, *D7*, 12, 24, 23, 11, 12, 24, 10]
27) $\sim(D\varepsilon\mathsf{R}\{CE\})$. [*D7*, 26, 22, *D7*, 24, 20, 22]
28) $\sim(\mathsf{c}(DCE))\therefore$ [*D4*, 27]
$[\exists DE].\mathsf{t}(ABCE).\mathsf{t}(ABCD).\sim(\mathsf{c}(DCE))$ [21, 25, 28]

A4'' $[ABC]::A\varepsilon\alpha.B\varepsilon\alpha.C\varepsilon\alpha.\supset\therefore\sim(\mathsf{c}(CAB)).\equiv.[\exists DE].\mathsf{t}(ABCE).$
$\mathsf{t}(ABCD).\sim(\mathsf{c}(DCE)).\lor.A=B$ [*K7*; *M31*, *D3'*, *D2*, *A3*, *A2*]

A5 $[ABCMN]:A\neq B.\mathsf{c}(AMN).\mathsf{c}(BMN).\mathsf{c}(CMN).\supset.\mathsf{c}(CAB)$

This is the analog of *A5* in **1.4**, hence it follows from *D0*, *D1*, *D3'*, *D4*, *D6*, *D7*, *L20*, and *BI*.

A6'' $[\exists ABC].\mathsf{c}(CAB).C\neq A.C\neq B$
PF $[\exists ABC]\therefore$
1) $A\varepsilon\alpha$.
2) $B\varepsilon\alpha$.
3) $A\neq B$.
4) $B\neq C$.
5) $C\neq A$.
6) $\mathsf{k}(ABC)$: [*BIII*(a)]
$[\exists g]$:
7) $\mathsf{i}(Ag)$.
8) $\mathsf{i}(Bg)$.
9) $\mathsf{i}(Cg)$. [*D9*, 6]
10) $g=\mathsf{R}\{AB\}$: [*BI*, 1, 2, 3, 7, 8]
11) $C\varepsilon\mathsf{R}\{AB\}$. [*D7*, 9, 10]
12) $\mathsf{c}(CAB)$. [*D4*, 11]
$[\exists ABC].\mathsf{c}(CAB).C\neq A.C\neq B$ [12, 5, 4]

A7 $[ABCD]\therefore\mathsf{t}(ABCD).\equiv:\mathsf{r}(AB).\mathsf{r}(CD):[M]:\mathsf{c}(MAB).\supset.$
$\sim(\mathsf{c}(MCD))$ [*M33*; *M34*]

A8'' $[ABDE]:\sim(\mathsf{c}(DEA)).\mathsf{r}(EA).\sim(\mathsf{c}(BEA)).B\varepsilon\alpha.\sim(\mathsf{c}(BDA)).$
$\mathsf{r}(DA).\supset.[\exists FC].\mathsf{t}(BFDA).\mathsf{c}(CEA).\mathsf{c}(CBF)$
PF $[ABDE]::\mathrm{Hp}(1\text{-}6).\supset.$
7) $\mathsf{i}(A\mathsf{R}\{EA\})$. [*BI*, *L19*, 2]
8) $\mathsf{i}(A\mathsf{R}\{DA\})$. [*BI*, *L19*, 6]
9) $\mathsf{i}(D\mathsf{R}\{DA\})$. [*BI*, *L19*, 6]
10) $\sim(D\varepsilon\mathsf{R}\{EA\})$. [*D4*, 1]
11) $D\varepsilon\mathsf{R}\{DA\}$. [*D7*, 9]
12) $\mathsf{R}\{EA\}\neq\mathsf{R}\{DA\}$. [10, 11]
13) $\sim(\mathsf{i}(B\mathsf{R}\{EA\}))$. [*D7*, *D4*, 3]
14) $\sim(\mathsf{i}(B\mathsf{R}\{DA\}))\therefore$ [*D7*, *D4*, 5]
$[\exists Ch]\therefore$
15) $\mathsf{i}(Bh)$.
16) $\mathsf{i}(Ch)$.
17) $\mathsf{i}(C\mathsf{R}\{EA\})$: [*BIV*, 7, 8, 12, 4, 13, 14]
18) $[K]:\mathsf{i}(Kh).\supset.\sim(\mathsf{i}(K\mathsf{R}\{DA\}))$:

19) $\mathsf{c}(CEA)$. [*D4*, *D7*, 17]

20) $\mathsf{p}(h\mathbf{R}\{DA\})$. [*D8*, *D7*, 15, 8, 18]

21) $h \neq \mathbf{R}\{DA\}$. [18]

$[\exists F]$.

22) $B \neq F$.

23) $\mathsf{i}(Fh)$. [*M13*, *D7*, 15]

24) $\mathsf{t}(BFDA)$. [*D13*, 22, *L19*, 6, 20, 21, 15, 23, 9, 8]

25) $h = \mathbf{R}\{BF\}$. [*B1*, *D7*, 15, 23, 22]

26) $C \varepsilon \mathbf{R}\{BF\}$. [*D7*, 16, 25]

27) $\mathsf{c}(CBF) .\because$ [*D4*, 26]

$[\exists FC] . \mathsf{t}(BFDA) . \mathsf{c}(CEA) . \mathsf{c}(CBF)$ [24, 19, 27]

A9 $[ABCDEFJKLMNOXYZ] : \mathsf{t}(CEJA) . \mathsf{t}(CEKB) . \mathsf{c}(DJA) . \mathsf{c}(DKB) .$
$\mathsf{c}(XJA) . \sim(\mathsf{c}(XKB)) . \mathsf{t}(BGLC) . \mathsf{t}(BGMA) . \mathsf{c}(FLC) . \mathsf{c}(FMA) .$
$\mathsf{c}(YLC) . \sim(\mathsf{c}(YMA)) . \mathsf{t}(AINB) . \mathsf{t}(AIOC) . \mathsf{c}(HNB) . \mathsf{c}(HOC) .$
$\mathsf{c}(ZNB) . \sim(\mathsf{c}(ZOC)) . \supset . \sim(\mathsf{c}(ABC))$

PF $[ABCDEFJKLMNOXYZ] :: \mathrm{Hp}(1\text{-}18) . \supset .\because$

19) $\mathsf{i}(C\mathbf{R}\{CE\})$. [*M35*, 1]

20) $\mathsf{i}(A\mathbf{R}\{JA\})$. [*M35*, 1]

21) $\mathsf{i}(B\mathbf{R}\{KB\})$. [*M35*, 2]

22) $\mathsf{i}(D\mathbf{R}\{JA\})$.
[*D7*, *D1*, *L20*, 3, *D4*, 3, *D6*, *D3'*, 3, *D4*]

23) $\mathsf{i}(D\mathbf{R}\{KB\})$.
[*D7*, *D1*, *L20*, 3, *D4*, 4, *D6*, *D3'*, 4, *D4*]

24) $\mathbf{R}\{JA\} \neq \mathbf{R}\{KB\}$: [*D0*, *D4*, 5, 6]

25) $[M] : \mathsf{i}(M\mathbf{R}\{CE\}) . \supset . \sim(\mathsf{i}(M\mathbf{R}\{JA\})) .$
$\sim(\mathsf{i}(M\mathbf{R}\{KB\}))$:
[*D7*, *A7*, 1, *D4*, *D7*; *D7*, *A7*, 2, *D4*, *D7*]

26) $\mathsf{i}(A\mathbf{R}\{AI\})$. [*M35*, 13]

27) $\mathsf{i}(B\mathbf{R}\{NB\})$. [*M35*, 13]

28) $\mathsf{i}(C\mathbf{R}\{OC\})$. [*M35*, 14]

29) $\mathsf{i}(H\mathbf{R}\{NB\})$.
[*D7*, *D1*, *L20*, 15, *D4*, 15, *D6*, *D3'*, 15, *D4*]

30) $\mathsf{i}(H\mathbf{R}\{OC\})$.
[*D7*, *D1*, *L20*, 16, *D4*, 16, *D6*, *D3'*, 16, *D4*]

31) $\mathbf{R}\{NB\} \neq \mathbf{R}\{OC\}$: [*D0*, *D4*, 17, 18]

32) $[M] : \mathsf{i}(M\mathbf{R}\{AI\}) . \supset . \sim(\mathsf{i}(M\mathbf{R}\{NB\})) .$
$\sim(\mathsf{i}(M\mathbf{R}\{OC\}))$:
[*D7*, *A7*, 13, *D4*, *D7*; *D7*, *A7*, 14, *D4*, *D7*]

33) $\mathsf{i}(B\mathbf{R}\{BG\})$. [*M35*, 7]

34) $\mathsf{i}(C\mathbf{R}\{LC\})$. [*M35*, 7]

35) $\mathsf{i}(A\mathbf{R}\{MA\})$. [*M35*, 8]

36) $\mathsf{i}(F\mathbf{R}\{LC\})$.
[*D7*, *D1*, *L20*, 9, *D4*, 9, *D6*, *D3'*, 9, *D4*]

37) $\mathsf{i}(F\mathbf{R}\{MA\})$.
[*D7*, *D1*, *L20*, 10, *D4*, 10, *D6*, *D3'*, 9, *D4*]

38) $\mathbf{R}\{LC\} \neq \mathbf{R}\{MA\}$: [*D0*, *D4*, 11, 12]

39) $[M] : \mathsf{i}(M\mathbf{R}\{BG\}) . \supset . \sim(\mathsf{i}(M\mathbf{R}\{LC\})) .$
$\sim(\mathsf{i}(M\mathbf{R}\{MA\}))$:
[*D7*, *A7*, 7, *D4*, *D7*; *D7*, *A7*, 8, *D4*, *D7*]

40)	$\sim(\mathsf{k}(ABC))$.	[BV, 19-39]
41)	$\sim(A \varepsilon \mathsf{R}\{BC\})$.·.	[$D12$, 40]
	$\sim(\mathsf{c}(ABC))$	[$D4$, 41]

CONCLUSION

Some similarities among affine, projective, and hyperbolic geometries are immediately evident from this development. *A1, A2, A3*(a), *A3*(b), and *A5* are common to all systems. It should be noted that *A3*(b) is not needed in the axioms for affine and hyperbolic geometries since it can be proved from *A7* and *A3*(a). It has been included however to point out the similarity of the systems.

Further common properties, which certainly exist, will be left as an open question. It seems that in order to find these it would be best to first simplify (**A***), (**P***), and (**H***), finding a formulation in terms of **t**($ABCD$) which is free of defined terms and if possible consisting of independent axioms. This should now be easier since pure point systems are established in this work.

The relation of Euclidean geometry to the primitive **t**($ABCD$) arises quite naturally. Some general work on primitives for Euclidean geometry has been done by Royden [18], Robinson [17], and Tarski [22]. Tarski makes use of Padoa's method [13] (also see Beth [3]) to obtain the following criterion for a primitive notion for Euclidean geometry.

> If a relation R between points of the n-dimensional space S can serve as the only primitive notion for n-dimension Euclidean geometry, then the set of all 1-1 transformations of S onto itself under which R is invariant coincides with the set of all similarity transformations of S.

Since **t**($ABCD$) in (**A***) is invariant under shearings it is immediately evident that it is insufficient as a sole primitive for Euclidean plane geometry. The problem of augmenting system (**A***) with another primitive to obtain Euclidean geometry will be left open as well as the problem of using **t**($ABCD$) to develop elliptic geometry.

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