NOTES ON FOUNDATIONS

G. Y. RAINICH

Note I. Vectors and Axioms

Using vectors in formulating a system of axioms for Geometry has the advantage of permitting to obtain propositions by calculation (embodying thus Leibniz's idea of Characteristica Geometrica) without having the defect of saying too much, a defect that is inherent in attempts to introduce Geometry as Analytic Geometry; if we say (with Hardy,[1] for instance) that a point is a triple of numbers we imply (but we do not want to imply it) that a statement, for instance, that the first coordinate is a prime integer is a significant geometrical statement. Of course, the defect can be remedied by introducing transformations of coordinates and considering invariance under such transformations as a criterion for a statement of the theory to have geometrical meaning. However, esthetically at least–and who will deny that the esthetic point of view plays a role in Mathematics–it seems desirable to make deducibility from the axioms a criterion not only for "truth" but also for geometrical significance.

Using vectors satisfies this requirement but one must not simply identity vectors with points; if one does it the procedure still has, in a milder form, the same defect as basing Geometry on coordinates, the point corresponding to the vector zero appears as different from other points whereas, as Leibniz said, all points are equal. (This point was stressed by Burali-Forti[2]). If the points are not vectors we must introduce into our axiom system an axiom establishing a connection between points and vectors, and it is the purpose of this Note to propose such an axiom. The form in which it is presented here is due to a remark by Z. D. Fiegl, a student in the Geometry course I was teaching in the University of Notre Dame in the Fall of 1960.

The connection between points and vectors is given by considering vectors as differences between points. It may be interesting to note that Leibniz failed in his attempt to set up what he called Characteristica

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Geometrica because he wanted to base it on the idea of addition (of points). Of course, addition is usually considered as a direct operation (as opposed to the inverse operation of subtraction) and thus more fundamental; therefore, one would think, it should come first. But the example of the calculus of segments shows that there is something wrong in preferring a direct operation; in fact forming a ratio is a counterpart of division—and nobody after Euclid, or is it Eudoxos, doubts the fundamental rôle that the ratios play in the theory. In a way, division rather than multiplication also comes first in certain considerations of group theory[3]).

Be it as it may, just *saying* that a vector is the difference of two points does not exhaust the question of the connection between points and vectors. This connection is established by the following

Axiom.

$$(X - Y) + (Y - Z) + (Z - X) = 0 \tag{1}$$

Here the minus signs refer to subtraction of points, the difference of two points being a vector; and the plus signs refer to addition of vectors which is assumed to have been introduced in axioms dealing with vectors only (see below).

To bring out the significance of this axiom we first derive a few consequences. Setting $Y = Z = X$ we obtain

$$(X - X) + (X - X) + (X - X) = 0$$

from which we conclude (using axioms on vectors) that

$$X - X = 0. \tag{2}$$

This means that the zero vector is the difference of two points if they are coincident points.

We make now $Z = Y$ in (1) and get, $(X - Y) + (Y - Y) + (Y - X) = 0$ which in view of (2) gives

$$(X - Y) + (Y - X) = 0 \quad \text{or} \quad Y - X = -(X - Y). \tag{3}$$

This means that if we interchange the tip and the tail of a vector we obtain the negative of that vector.

We can discuss now free mobility of the space. By this we mean that given any vector we can find a vector equal to it with its tail at any given point. In fact, if the given vector is $u = X - Y$ and Z is a given point then u may be written as

$$u = X - Y = [Z + (X - Y)] - Z.$$

We do not use the full power of (1) here, but only the consequences (2) and (3).

Further, we can rewrite (1) as

$$X - Z = (X - Y) + (Y - Z)$$

and we have similarly

$$Y - T = (Y - Z) + (Z - T)$$

Using these identities we arrive at the

Theorem. If $X - Y = Z - T$, *then* $X - Z = Y - T$.

In terms of Euclidean Geometry this means that if two opposite sides of a quadrilateral are equal and parallel then the same thing can be said about the other pair of opposite sides—that is, that we have a paralellogram.

Another form that (1) can take (we write O for Z) is

$$X - Y = (X - O) - (Y - O). \tag{4}$$

This reduces operations on points to operations on vectors. Now we may, after having chosen an arbitrary but fixed point O, represent every point X by the vector $X - O$. If our vector space contains a basis so that every vector is a linear combination of basis vectors then the coefficients of that linear combination may be used as *coordinates* of the point. We have then all the advantages of analytic geometry.

Remark 1. Here O is a point—the origin of our coordinate system—not to be confused with the vector 0 (although in the above representation the vector 0 represents the point O).

Remark 2. What the relation (1) really does is to make the space *flat*. A different and more complicated connection between points and vectors may be used to introduce curved space. The separation of axioms into those dealing with vectors only and those dealing with the connection between points and vectors, has also the advantage of permitting to extend the classification of Geometries in terms of groups (the Erlangen Program) to curved spaces (the possibility of such an extension was first indicated in terms of tangent flats by E. Cartan[4]).

We assume in what precedes that vectors have been introduced by axioms. In most of the preceding discussion however the vector space need be nothing more than an abelian group of characteristic zero. If we want to use coordinates in our space we must have a field in conjunction with the vector space – possibly the vector space may be over a field (or something of that nature). By further axioms we may introduce a dot product into our vector space; then distance between points may be introduced into the Geometry by the formula

$$d = \sqrt{(X - Y) \cdot (X - Y)}.$$

But the main part of the paper does not depend on these additions, or to put it another way, applies to all cases.

Another point of view according to which vectors are considered as equivalence classes under a subgroup of the group of a Geometry will be discussed in one of the subsequent Notes of this series.

REFERENCES

[1] G. Hardy. What is Geometry? *Mathem. Gazette 12*, 1925, p. 312.

[2] C. Burali-Forti. *Introduction à la Géométrie Differentielle*, p. 9 Paris, 1897.

[3] H. Boggs and G. Y. Rainich. Note on Group Postulates. *Bullet. Amer. Math. Society*, 1937, p. 84.

[4] E. Cartan. *La méthode du Repère Mobile*. Paris, 1935.

To be continued.

University of Notre Dame
Notre Dame, Indiana