# INVESTIGATIONS ON A COMPREHENSION AXIOM WITHOUT NEGATION IN THE DEFINING PROPOSITIONAL FUNCTIONS

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## Introduction

In the paper "Bemerkungen zum Komprehensionsaxiom" in *Zeitschr. f. math. Logik und Grundl.d. Math.*, Bd 3 (1957), p. 1-17, I showed that antinomies of the same kind as Russell's could be avoided in set theory, if this was based on a certain logic, due to Łukasiewicz, with infinitely many truth values. Indeed I proved the existence of domains such that the axiom of comprehension was satisfied for elementary propositional functions $\phi$, that is $\phi$ being built from atomic propositions $u \in v$ by use of conjunction, disjunction, implication and negation only. Later I proved the same for a certain 3-valued logic as shown in a paper which will appear in *Math. Scand.* Here I shall show in §1 and §2 that the same is true even for ordinary 2-valued logic, provided that only conjunction and disjunction are allowed in $\phi$. In §3 I prove that also the axiom of extensionality is valid for the domains constructed in §1 and §2. I call the $\phi$ constructed in this way positive propositions, abbreviated p. pr. The words "atomic propositions" are abbreviated to at. pr.

The two truth values can be $0$ (false) and $1$ (true). In the sequel I write the conjunction of $A$ and $B$ as $A \wedge B$ and their disjunction as $A \vee B$. Further $A(x)$ for all $x$ is written $\bigwedge xA(x)$ and $A(x)$ for some $x$ is written $\bigvee xA(x)$.

Now the p. pr. can be defined inductively as follows.

1. The truth constants $0$ and $1$ are p. pr.
2. Every at. pr. $x \in y$ is a p. pr. Here $x$ and $y$ are free variables.
3. If $A$ and $B$ are p. pr., so are $A \wedge B$ and $A \vee B$. The latter have the free and bound variables occurring in $A$ and $B$.
4. If $A(x, x_1, \ldots, x_n)$ is a p. pr. with $x, x_1, \ldots, x_n$ as free variables $\bigwedge xA(x, x_1, \ldots, x_u)$ and $\bigvee xA(x, x_1, \ldots, x_n)$ are p. pr. with $x$ as bound variable, $x_1, \ldots, x_n$ still as free variables, while the eventually occurring bound variables in $A(x, x_1, \ldots, x_n)$ remain bound in the latter expressions.

   If a set $y$ is such that

   $$\bigwedge x \quad ((x \in y) = U(x, x_1, \ldots, x_n))$$

   is true, where $x, x_1, \ldots, x_n$ are the set variables in the p.pr. $U$, then $y$ is a set function of $x_1, \ldots x_n$.

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Then the p. pr. are constructed by use of the rules 1) through 4). Of course we may use in our proofs the whole first order predicate calculus although I don't formalize the proofs in the sequel. I write $A = B$ instead of $A \leftrightarrow B$ (equivalence) and $A \leqq B$ instead of $A \rightarrow B$ (implication).

§1.

Let a p. pr. be called elementary if it can be constructed from the at. pr. by use of 1), 2), 3) alone so that quantifiers are not used. I shall then prove that a set theory with the axiom

(1) $\bigvee y \bigwedge x\ (x \in y = \phi(x))$, (Axiom of comprehension)

where $\phi$ denotes an arbitrary elementary p. pr., together with the axiom

(2) $\bigwedge x (x \in y = x \in z) \leqq \bigwedge u (y \in u = z \in u)$ (Axiom of extensionality)

is consistent. Such a rudimentary set theory is of course not at all sufficient for the development of mathematics. However this consistency proof shows that the usual deduction of Russell's antimony and similar ones is impossible when the $\phi$ in the axiom of comprehension is a p. pr. Indeed I shall give two different proofs.

A preliminary remark for the first proof is the following: Let $\mathfrak{F}(p)$ where p is a propositional variable, be an elementary expression, that is built up from $p$ by use of $\wedge$, conjunction and $\vee$ disjunction alone. Then either $0$ or $1$ is a fix-value for $\mathfrak{F}$. This means that the equation

$$p = \mathfrak{F}(p)$$

will be satisfied if either $0$ or $1$ is inserted instead of $p$. Indeed, $\mathfrak{F}(p)$ is either $0$ or $1$ or $p$, every function of p being either $= 0$, $= 1$ or $= p$. In the first case $0$ is a fix-point, in the second $1$ and in the third both $0$ and $1$.

We consider a domain $\mathfrak{D}$ of individuals. For an arbitrary non negative number $n$ let $x, x_1, \ldots, x_n$ denote individuals and let $U(x, x_1, \ldots, x_n)$ be any elementary p. pr. built up from the diverse atomic propositions $u \in v$, where $u$ and $v$ are among $x, x_1, \ldots, x_n$. Then the validity of the elementary axiom of comprehension in $\mathfrak{D}$ means that

(3) $\bigvee y \bigwedge x\ (x \in y = U(x, x_1, \ldots, x_n)$

is true. I shall show that it is possible to construct a domain $\mathfrak{D}$ of individuals together with a determination of the truth values of $u \in v$ for the diverse pairs $u, v$ in $\mathfrak{D}$ such that (3) is satisfied.

In order to do this we have to enumerate the functions $U$. Let us write them in the form

$$U_n\ (x, x_1, \ldots, x_{g(n)})$$

letting $g(n)$ denote the number of actually occurring variables apart from $x$ in the $n$th propositional function $U_n$ $U_n$ is then built up from $0, 1$ and the at. pr. $x \in x, x \in x_r, x_s \in x$, where $r$ and $s$ run through $1, 2, \ldots, g(n)$. We need not take into account at. pr. $x_r \in x_s$ because each of them is in every case either $0$ or $1$. It is clear that such an enumeration is possible. To begin with $U_0$ $(x)$ may be the truth constant $0, U_1\ (x)$ the constant $1$ and $U_1\ (x)$ the propositional function $x \in x$ so that $g(0) = g(1) = g(2) = 0$. Further $U_3(x, x_1)$ might be the propositional function $x \in x_1$ so that $g(3) = 1$ etc. We then construct individuals as follows. Let $f_n(x_1, \ldots, x_{g(n)})$ be just an expression corresponding to $U_n(x, x_1, \ldots, x_{g(n)})$. Then, starting with the symbols $0$, $V$, $W$ as the first individuals we let the expression we get from $f_n(x_1, \ldots, x_{g(n)})$ as often as earlier constructed individuals are inserted instead of $x_1, \ldots, x_{g(n)}$ be a new individual. Our domain $\mathfrak{D}$ shall consist of just the individuals generated in this way by use of all the $f_n(x_1, \ldots, x_{g(n)})$. In the special cases $n = 0, 1, 2$ when $g(n) = 0$ we may use the expressions $f_n(\ )$ with an empty parenthesis identifying them with $0, V, W$ respectively.

I introduce the notion height of individuals. The individuals $0$, $V$, $W$ shall have the height $0$, and if $h$ is the maximum of the heights of $x_1, \ldots, x_{g(n)}$, then the height of $f_n(x_1, \ldots, x_{g(n)})$ shall be h + 1. In the case $g(n) = 0$, the height shall be $0$.

Now the truth values of $u \in v$ for the diverse pairs $u$, $v$ of the individuals in $D$ shall be successively determined.

First $x \in 0$ shall be false for all $x$ or in other words $(x \in 0) = 0$. Similarly $(x \in V) = 1$ and $x \in W = x \in x$ for all $x$. Since there is no other iindividual than $0$, $V$, $W$ of height $0$, we have already determined $u \in v$ for all individuals of height $0$ if we choose for example $W \in W = 1$. Let us assume that we have already succeeded in determining the relation $\in$ for all pairs $u$, $v$, where $u$ and $v$ are of height $\leqq h$, in such a way that if $y = f_n(x_1, \ldots, x_{g(n)})$, then for all $x$ of height $\leqq h$ the truth value of $x \in y$ is that of $U_n(x, x_1, \ldots, x_{g(n)})$. As the reader easily verifies this is correct for $h = 0$, so that we may use complete induction. Let first $x$ still be of height $\leqq h$, whereas $y$ is of height $h + 1$, say $y = f_n(x_1, \ldots, x_{g(n)})$, all $x_r$ here being of height $\leqq h$. Now $U_n(x, x_1, \ldots, x_{g(n)})$ is built from $0$, $1$ and at. pr. $x \in x$, $x \in x_r$, $x_s \in x$, by conjunction and disjunction. In virtue of the hypothesis of induction the truth values of all these at. pr. are already determined. Then the value of $U_n(x, x_1, \ldots, x_{g(n)})$ is known for all $x$ of height $\leqq h$ and we choose this as the truth value of $x \in y$. Thus the validity of (3) is extended to all pairs $x$, $y$, where $x$ is of height $\leqq h$, $y$ of height $\leqq h + 1$. Now let $x$ be of height $h + 1$. Then I shall show, again by induction, that the value of $x \in z$ for every $z$ of height $\leqq h + 1$ can be preliminarily determined so as a function $f(z, p)$, where $p$ is $x \in x$, that the equation (3) is valid. Since $(x \in 0) = 0$, $x \in V = 1$, $x \in W = x \in x$, we already have such a determination for the $z$ of height $0$. Let us assume that this determination has been carried out up to the height $k$ of $z$, where $k \leqq h$, and let us then consider $x \in z$, $z$ of height $k + 1$, $z = f_m(x_1, \ldots, x_{g(m)})$. In $U_m(x, x_1, \ldots, x_{g(m)})$ we have the at. pr. $x \in x$, $x \in x_r$, $x_s \in x$. According to the hypothesis of induction every $x \in x_r$ is a known function $f(x_r, p)$, while the

$x_s \in x$ have already known values because of the first part of this proof since all $x_s$ have heights $\leqq h$. Then $U_m(x, x_1, \ldots, x_{g(m)})$ will be a known function $f(z,p)$ and putting $x \in z = f(z,p)$, the validity of (3) will be extended to $z$ of height $k + 1$. Hence by induction we have $x \in z = f(z,p)$ in accordance with (3) for all $z$ of height $\leqq h + 1$.

Hitherto we have only determined the $x \in z$ as functions of $x \in x$. Now we determine their values so. We have in particular $p = x \in x = f(x,p)$. In virtue of an above remark the equation $p = f(x,p)$ has a fixpoint. Taking for $x \in x$ this value, we obtain by inserting it in the diverse $f(z,p)$ for the $z$ of height $\leqq h + 1$ the values of the diverse $x \in z$. Thus we have obtained a continuation of the determination of $\in$ to all $x$ and $y$ of heights $\leqq h + 1$ such that (3) is valid.

This proves by induction that we can determine $\in$ so that the elementary axiom of comprehension is satisfied in our domain D.

## § 2.

There is however also another way of proving the existence of domains for which the axiom of comprehension is valid for elementary p. pr. This way will be more elucidating as it also shows the non-existence of finite domains with this property. First I shall prove the following theorem:

In order that the elementary axiom of comprehension be valid in a domain it is necessary and sufficient that six special cases of this axiom are valid, namely:

1) There is in the domain an individual $O$ such that $(x \in O) = 0$ for all $x$ in the domain.
2) There is in the domain an individual $W$ such that $(x \in W) = (x \in x)$ for all $x$ in the domain.
3) There is in the domain an individual $V$ such that $(x \in V) = 1$ for all $x$ in the domain.
4) For arbitrary individuals $a$ and $b$ there is an individual $c$ such that for all $x$
$$x \in a \wedge x \in b = x \in c.$$
5) For arbitrary $a$ and $b$ there is an individual $d$ such that for all $x$
$$x \in a \vee x \in b = x \in d.$$
6) For arbitrary $a$ there is an individual $b$ such that for all $x$
$$a \in x = x \in b.$$

Proof. The necessity is clear so that we have only to prove that the validity of 1) through 6) is sufficient. Let $U(x, x_1, \ldots, x_n)$ be an arbitrary elementary p. pr. It can be written as a conjunction of disjunctions between terms of the form $x \in x$, $x \in x_r$, $x_s \in x$, Let these disjunctions be $D_t$ for $t = 1, 2, \ldots, m$ say. Then each $D_t$ is a disjunction of terms each of which is either $x \in x$ or of the form $x \in x_r$ or $x_s \in x$. In virtue of 6) we may replace every term $x_s \in x$ by a term $x \in x_\sigma$. Then the modified expression $\overline{D}_t$ only contains terms of the forms $x \in x$ and $x \in x_r$. According to 2) and 5) there exists an individual $y$ such that for all $x$

$$x \epsilon y_t = \overline{\overline{D}}_t = D_t.$$

Finally there is in virtue of 4) an element $y$ such that for all $x$

$$x \epsilon y = x \epsilon y_1 \wedge x \epsilon y_2 \wedge \ldots\ldots \wedge x \epsilon y_m.$$

That is for all individuals $x$ in the domain

$$x \epsilon y = D_1 \wedge D_2 \wedge \ldots . \wedge D_m = U(x, x_1, \ldots, x_n).$$

Thus our theorem is proved.

Now let $\mathfrak{D}$ be a domain in which the elementary axiom of comprehension is valid. Then $\mathfrak{D}$ has the properties 1) through 6). In the first instance we have the individuals $O$, $W$, $V$. Since $0$ and $1$ are different truth values, the individuals $O$ and $V$ must be different. Further $W \neq O$ because $V \epsilon W = V \epsilon V = 1$ and $W \neq V$ because $O \epsilon W = O \epsilon O = 0$. As to the value of $W \epsilon W$ it can be either $0$ or $1$. Let us first assume $W \epsilon W = 1$. For the three elements $O$, $W$, $V$ we then have the following table of values for the relation $\epsilon$ :

| $\epsilon$ | $O$ | $W$ | $V$ |
|---|---|---|---|
| $O$ | $0$ | $0$ | $1$ |
| $W$ | $0$ | $1$ | $1$ |
| $V$ | $0$ | $1$ | $1$ |

Now an element $A_1$ shall exist such that $O \epsilon x = x \epsilon A_1$ for all $x$. Then one gets $O \epsilon A_1 = O \epsilon O = 0$, $W \epsilon A_1 = O \epsilon W = 0$, $V \epsilon A_1 = O \epsilon V = 1$, $A_1 \epsilon A_1 = O \epsilon A_1 = 0$

besides, according to the definitions of $O$, $W$, $V$,
$A_1 \epsilon O = 0$, $A_1 \epsilon W = A_1 \epsilon A_1 = 0$, $A_1 \epsilon V = 1$.

Thus we see that $A_1$ must be different from $O$, $W$, $V$ and we get the $\epsilon$-table for the 4 individuals $O$, $A_1$ $W$, $V$

| $\epsilon$ | $O$ | $A_1$ | $W$ | $V$ |
|---|---|---|---|---|
| $O$ | $0$ | $0$ | $0$ | $1$ |
| $A_1$ | $0$ | $0$ | $0$ | $1$ |
| $W$ | $0$ | $0$ | $1$ | $1$ |
| $V$ | $0$ | $1$ | $1$ | $1$ |

Further an individual $B_1$ must exist such that for all $x$ $V \epsilon x = x \epsilon B_1$. Again

$B_1$ *is* $\neq$ O, $A_1$, $W$, $V$. Indeed

$O \in B_1 = V \in O = 0$, $A_1 \in B_1 = V \in A_1 = 1$, $W \in B_1 = V \in W = 1$, $V \in B_1 = V \in V = 1$, $B_1 \in B_1 = V \in B_1 = 1$, $B_1 \in O = 0$, $B_1 \in A_1 = O \in B_1 = 0$, $B_1 \in W = B_1 \in B_1 = 1$.

So far we have proved that our domain must contain at least 5 elements $O$, $A_1$, $W$, $B_1, V$ with the $\epsilon$-values for their pairs given by the table

| $\epsilon$ | $O$ | $A_1$ | $W$ | $B_1$ | $V$ |
|---|---|---|---|---|---|
| $O$ | 0 | 0 | 0 | 0 | 1 |
| $A_1$ | 0 | 0 | 0 | 1 | 1 |
| $W$ | 0 | 0 | 1 | 1 | 1 |
| $B_1$ | 0 | 0 | 1 | 1 | 1 |
| $V$ | 0 | 1 | 1 | 1 | 1 |

This can be continued indefinitely. The next step is to take into account the existence of an element $A_2$ with the property $A_1 \in x = x \in A_2$ for all $x$. $A_2$ is again different from $O$, $A_1$, $W$, $B_1$, $V$. Indeed we now get the table

| $\epsilon$ | $O$ | $A_1$ | $A_2$ | $W$ | $B_1$ | $V$ |
|---|---|---|---|---|---|---|
| $O$ | 0 | 0 | 0 | 0 | 0 | 1 |
| $A_1$ | 0 | 0 | 0 | 0 | 1 | 1 |
| $A_2$ | 0 | 0 | 0 | 0 | 1 | 1 |
| $W$ | 0 | 0 | 0 | 1 | 1 | 1 |
| $B_1$ | 0 | 0 | 1 | 1 | 1 | 1 |
| $V$ | 0 | 1 | 1 | 1 | 1 | 1 |

Then an individual $B_2$ must exist so that $B_1 \in x = x \in B_2$ for all $x$. Again $B_2$ turns out to be different from $O$, $A_1$, $A_2$, $W$, $B_1$, $V$. One obtains the $\epsilon$-table

| $\epsilon$ | $O$ | $A_1$ | $A_2$ | $W$ | $B_2$ | $B_1$ | $V$ |
|---|---|---|---|---|---|---|---|
| $O$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| $A_1$ | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| $A_2$ | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| $W$ | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| $B_2$ | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| $B_1$ | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| $V$ | 0 | 1 | 1 | 1 | 1 | 1 | 1 |

Since this procedure can be continued into infinity we obtain the result that no finite domain with $W \in W = 1$ exists satisfying the elementary axiom of comprehension. On the other hand it is easily seen that the infinite domain constructed by the infinite process, consisting of the elements $O$, $A_1$, $A_2$, $\ldots, W, \ldots, B_2, B_1$, $V$, has the properties 1) through 6) so that it satisfies the elementary axiom of comprehension. Its $\epsilon$-table is

| $\epsilon$ | $O$ | $A_1$ | $A_2$ | $A_3$ | ..... | $W$ | ..... | $B_3$ | $B_2$ | $B_1$ | $V$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $O$ | 0 | 0 | 0 | 0 | ..... | 0 | ..... | 0 | 0 | 0 | 1 |
| $A_1$ | 0 | 0 | 0 | 0 | ..... | 0 | ..... | 0 | 0 | 1 | 1 |
| $A_2$ | 0 | 0 | 0 | 0 | ..... | 0 | ..... | 0 | 1 | 1 | 1 |
| $A_3$ | 0 | 0 | 0 | 0 | ..... | 0 | ..... | 1 | 1 | 1 | 1 |
| . | . | . | . | . | ..... | . | ..... | . | . | . | . |
| . | . | . | . | . | ..... | . | ..... | . | . | . | . |
| . | . | . | . | . | ..... | . | ..... | . | . | . | . |
| . | . | . | . | . | ..... | . | ..... | . | . | . | . |
| $W$ | 0 | 0 | 0 | 0 | ..... | 1 | ..... | 1 | 1 | 1 | 1 |
| . | . | . | . | . | ..... | . | ..... | . | . | . | . |
| . | . | . | . | . | ..... | . | ..... | . | . | . | . |
| . | . | . | . | . | ..... | . | ..... | . | . | . | . |
| . | . | . | . | . | ..... | . | ..... | . | . | . | . |
| $B_3$ | 0 | 0 | 0 | 0 | ..... | 1 | ..... | 1 | 1 | 1 | 1 |
| $B_2$ | 0 | 0 | 0 | 1 | ..... | 1 | ..... | 1 | 1 | 1 | 1 |
| $B_1$ | 0 | 0 | 1 | 1 | ..... | 1 | ..... | 1 | 1 | 1 | 1 |
| $V$ | 0 | 1 | 1 | 1 | ..... | 1 | ..... | 1 | 1 | 1 | 1 |

One observes that the sets $O, A_1, \ldots, V$ are linearly ordered by inclusion

$$O \subseteq A_1 \subseteq A_2 \subseteq \ldots W \ldots B_2 \subseteq B_1 \subseteq V.$$

A simple conclusion of this is that the domain has the properties 4) and 5). Indeed when set $A$ is contained in another set $B$, then $B$ is their union and $A$ their intersection. Further, the domain has also the property 6) because one notices that in the above table every horizontal row is equal to a vertical one. In virtue of the above theorem the axiom of comprehension for elementary p. pr. *is generally valid in the domain.*

All this was based on the assumption that $W \epsilon W = 1$. Let us now assume $W \epsilon W = 0$. Then the $\epsilon$-table for $O, W, V$ is

| $\epsilon$ | $O$ | $W$ | $V$ |
|---|---|---|---|
| $O$ | $0$ | $0$ | $1$ |
| $W$ | $0$ | $0$ | $1$ |
| $V$ | $0$ | $1$ | $1$ |

Now an element $B_1$ shall exist such that $V \epsilon x = x \epsilon B_1$ for all $x$. One finds $O \epsilon B_1 = 0$, $W \epsilon B_1 = 1$, $V \epsilon B_1 = 1$, $B_1 \epsilon B_1 = V \epsilon B_1 = 1$, $B_1 \epsilon W = B_1 \epsilon B_1 = 1$, so that for $O, W, B_1, V$ we get the $\epsilon$-table

| $\epsilon$ | $O$ | $W$ | $B_1$ | $V$ |
|---|---|---|---|---|
| $O$ | $0$ | $0$ | $0$ | $1$ |
| $W$ | $0$ | $0$ | $1$ | $1$ |
| $B_1$ | $0$ | $1$ | $1$ | $1$ |
| $V$ | $0$ | $1$ | $1$ | $1$ |

Further an element $A_1$ shall exist such that $O \epsilon x = x \epsilon A_1$ for all $x$. It is seen that

$$O \epsilon A_1 = 0, \quad W \epsilon A_1 = 0, \; B_1 \epsilon A_1 = 0, \; V \epsilon A_1 = 1. \quad A_1 \epsilon A_1 = O \epsilon A_1 = 0.$$

Hence we get the $\epsilon$-table

| $\epsilon$ | $O$ | $A_1$ | $W$ | $B_1$ | $V$ |
|---|---|---|---|---|---|
| $O$ | $0$ | $0$ | $0$ | $0$ | $1$ |
| $A_1$ | $0$ | $0$ | $0$ | $1$ | $1$ |
| $W$ | $0$ | $0$ | $0$ | $1$ | $1$ |
| $B_1$ | $0$ | $0$ | $1$ | $1$ | $1$ |
| $V$ | $0$ | $1$ | $1$ | $1$ | $1$ |

Here we may take into account an element $B_2$ such that $B_1 \in x = x \in B_2$ etc. The reader will easily convince himself that this again leads to an infinite process. The table we obtain after introduction of $A_1, \ldots, A_n, B_1, \ldots, B_n$ is just the same as that obtained in the first process except that $W \in W$ now is $0$, not $1$. Therefore we see that also if $W \in W = 0$, there is no finite domain satisfying the elementary axiom of comprehension. Thus the non-existence of a finite domain of this kind is proved completely. On the other hand the infinite process yields an infinite domain with the same $\in$-table as the infinite domain above, only with the difference that $W \in W$ now is $0$. It is seen just as before that the elementary axiom of comprehension is valid also for this latter domain.

The existence of more complicated domains satisfying the elementary axiom of comprehension will be proved elsewhere.

§ 3.

The successively performed definition of the relation $\in$ carried out in §1 is not unique because sometimes both $0$ and $1$ are fixpoints for a p. pr. of $p = x \in x$. However the procedure of successive determination of $\in$ can be made unique for example by making the convention that if $0$ is a fixpoint, it shall be taken as the value of $x \in x$, so that $1$ is only chosen when $1$, but not $0$, is a fixpoint. Of course one can make many different choices, but let us assume that we have in some way made the determination of $\in$ unique. Then I shall prove that the axiom of extensionality is valid for our domain $\mathfrak{D}$. That this axiom is fulfilled for the domains $\mathfrak{D}$ obtained in §2 is evident.

Let us assume that $x$ and $\xi$ have the property $\wedge u(u \in x = u \in \xi)$. Then I assert that $x \in x = \xi \in \xi$.

Proof. Let first $x$ be $O$. Then $(x \in x) = 0$ and $(\xi \in x) = 0$, whence $\xi \in x = \xi \in \xi = 0$. Similarly if $\xi = O$. Then let

$$x = f_n(x_1, \ldots, x_{g(n)}) \quad , \quad \xi = f_\nu(x_1, \ldots, x_{g(\nu)})$$

so that for arbitrary u

$$u \in x = U_n(u, x_1, \ldots, x_{g(n)})) \quad , \quad u \in \xi = U_\nu(u, x_1, \ldots, x_{g(\nu)}).$$

Here $U_n(u, x_1, \ldots, x_{g(n)})$ is a p. pr. of $u \in u$ with coefficients which are p. pr. of the other atomic propositions in $U_n$, $u \in x_r$, $x_s \in u$. Similarly for $U_\nu$. Now these coefficients must be such that both functions coincide, that is that $U_n$ and $U_\nu$ must be the same function of $u \in u$ Otherwise, either for $u \in u = 0$ or for $u \in u = 1$, which both are realized for some $u$, $U_n$ and $U_\nu$ would have different truthvalues so that we got $u \in x \neq u \in \xi$ contrary to our supposition. Hence we may write

$$u \in x = F(u \in u) \quad , \quad u \in \xi = F(u \in u),$$

whence

$$x \in x = F(x \in x) \quad , \quad \xi \in \xi = F(\xi \in \xi).$$

Because of the assumed unique choice of fixpoints we obtain $x \in x = \xi \in \xi$.

Further, since we have $x \in 0 = \xi \in 0$, namely both $= 0$, that means $x \in v = \xi \in v$ for all $v$ of height $0$. Now let us assume that we have $x \in v = \xi \in v$ for all $v$ of height $\leqq h$ and let us compare $x \in y$ with $\xi \in y$, $y$ being of height $h + 1$. Then $y$ has the form

$$y = f_n(x_1, \ldots, x_{g(n)})$$

where $x_1, \ldots, x_{g(n)}$ are all of height $\leqq$ h. Further for all $x$ and all $\xi$

$$x \in y = U_n(x, x_1, \ldots, x_{g(n)}) \quad , \quad \xi \in y = U_n(\xi, x_1, \ldots, x_{g(n)})$$

Here $U_n(x)$ resp. $U_n(\xi)$ is a p. pr. built up from the atomic propositions $x \in x$, $x \in x_r$, $x_s \in x$, resp. $\xi \in \xi$, $\xi \in x_r$, $x_s \in \xi$, $r$ and $s$ running through $1, 2, \ldots, g(n)$. According to a result above $x \in x = \xi \in \xi$ and according to the hypothesis of induction every $x \in x_r = \xi \in x_r$ while $x_r \in x = x_r \in \xi$ in virtue of the assumption we made that $x$ and $\xi$ contain just the same elements. Hence $x \in y = \xi \in y$. Thus we have proved the truth of the axiom of extensionality.

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