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ELEMENTARY LOGIC WITHOUT REFERENTIAL QUANTIFICATION

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Introduction This paper suggests an alternative way of presenting elementary logic with identity. The paper is preliminary to a program of axiomatizing a system, providing adequate semantics, and establishing the relation of these enterprises to the classical predicate calculus. In section 1 the classical semantics for elementary theories is recalled. Section 2 presents a few theorems which suggest a non-referential view of quantification while section 3 provides a semantical basis for the plausibility of these suggestions.

1. *Referential quantification and semantics* Elementary theories are considered here as systems which incorporate all truth-functional tautologies as theorems, employ classical principles of quantification with respect to one sort of variable, and contain primitive functors which form statements when applied to variables, that is, primitive predicates subject to given axioms. The predicates may be thought of as divided into two classes: logical (for example, identity) and non-logical (none of which will be treated in this paper). Besides the above, elementary theories may contain constants of the same sort as the variables, that is, substitutable for variables.

In order to provide classical models for elementary logical theories, an arbitrary non-empty set of elements is specified, called the domain of discourse or universe. This set is taken to be the range of the variables of the theory and then, any association or interpretation of predicates with respect to relations over the universe (constants with respect to elements of the universe) which satisfies the theorems of the theory, is a model for the theory.

From the point of view of this (classical) semantics, all consistent theories have *ontological commitment.* Since the range of values of the variables of quantification is identified with the domain of discourse, it follows that "...entities of a given sort are assumed by a theory if and only if some of them must be counted among the values of the variables in order that the statements affirmed in the theory be true" (see [1], p. 103).

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And there are numerous examples of existentially quantified statements affirmed in elementary logical theories.

One also has a criterion for determining which constants of the theory function as names: "... an expression is used in a theory as naming if and only if the existentially quantified identity built on that expression is true according to the theory" (see [2], p. 152). Now given the point of view of classical semantics where the range of variables is identified with the universe, all *expressions are names*: at least all expressions substitutable for variables in the theory will function as names (that is, as singular terms or proper nouns). In fact, one customarily calls the variables and constants of the theory individual symbols, because the elements of the universe are called individuals.

2. *Non-referential quantification* All of the above is familiar and presented only to provide a basis for comparison. The main task at hand is to put forth an alternative to the program: one which considers not just singular terms, but terms in general, that is, terms true of many individuals, or of one, or of none. It is hoped that it is clear in what follows that the status of *names of classes* is withheld from general terms, without denying, of course, that certain classes are often associated with general terms—otherwise than in the fashion of being named. "Whereas a singular term purports to name an entity, abstract or concrete, a general term does not; but a general term is *true of* an entity, or of each of many, or of none. The class of all entities of which a general term is true is called the *extension* of the term" (see [1], p. 21).

Consider the following elementary theory T containing only one primitive predicate for which the following are theorems.

$$[ab]:a\,\varepsilon\,b\,.\equiv.\,[\exists c].\,a\,\varepsilon\,c\,.\,c\,\varepsilon\,b$$
$$[ab]:a\,\varepsilon\,b\,.\supset.\,a\,\varepsilon\,a$$
$$[ab]:a\,\varepsilon\,b\,.\supset.\,[\exists c].\,c\,\varepsilon\,a$$
$$[abc]:a\,\varepsilon\,b\,.\,b\,\varepsilon\,c\,.\supset.\,a\,\varepsilon\,c$$

No other predicates are taken as primitive in T and one may justify calling T a logical theory for the same reasons that one has for calling the pure theory of identity a logical theory (see section **3**, below).

Within T two types of standard definition are allowed (see [3], p. 47): those for predicates and those for terms, of which the following theorems are examples.

$$[ab]:a\,\varepsilon\,b\,.\,b\,\varepsilon\,a\,.\equiv.\,a=b$$
$$[a]:a\,\varepsilon\,a\,.\,\sim(a\,\varepsilon\,a)\,.\equiv.\,a\,\varepsilon\,\wedge$$

The predicate defined here is intended to serve the purposes of identity within T.

Now if one uses identity as defined above to determine which expressions of T function as names, it is clear that the constant term defined above does not so function: the following is a theorem of T.

$$\sim[\exists b].\wedge=b$$

In other words, this constant serves as an example of an expression substitutable for the variables of the theory which fails to meet the criterion for names.

In fact, without the addition of special axioms, it is impossible to obtain the following as a theorem.

$$[\exists b] . b = b$$

This being the case, theory T lacks ontological commitment. The point here is not that T affirms no existentially quantified statements, but that it affirms none of the above type—and in T, questions of ontological commitment pivot on the affirmation of this type of quantified statement.

Finally, the quantification of T can be seen to be non-referential by reflecting on the following theorem.

$$\sim [a] . a = a$$

If this statement were interpreted in the classical way, it would mean that it is not the case that every individual is identical to itself: which is false. But the statement is a theorem of T and hence true (again, see section **3**, below).

It has been claimed that T lacks singular terms, ontological commitment and referential quantification. Up to this point the claim that theory T is non-commital with respect to existence has been merely suggestive. Proceeding piecemeal we have been evolving a plausible system of non-referential logic. However, by providing a semantics for T in the following section, a better perspective for evaluation of the theory will be given.

3. *Non-referential semantics* Any interpretation of T is to be based on some given set (of individuals), called the domain of discourse. The value of the variables of T (and of constants substitutable for the variables) are taken to be elements of the *power set* of the domain of discourse. For any given assignment of values to variables (and constants), an atomic formula of T is said to be satisfied just in case the value assigned to the first argument is a unit set which is a subset of the value assigned to the second argument. Satisfaction for non-atomic formulas is then construed in the usual manner.

Return now to the claim that the predicate defined above serves the purposes of identity in T. From the semantics given here it is clear that any statement of the form

$$a = b$$

is true in T just in case the individual of which the first term is true is identical to the individual of which the second term is true.

Moreover, none of the terms of the theory are proper nouns or singular terms. Under any given interpretation, the values of the variables in T are always classes of entities from the domain of discourse. That is to say, the value of any term in T is simply the extension of that term.

Recall the theorem:

$$\sim[a].a=a$$

which must be true under any interpretation of T.

With respect to ontological commitment, notice that any set of elements can be the basis for an interpretation of T—even the empty set. In fact, such an interpretation provides a rather simple model for T and shows that the theory requires the existence of no individual for its consistency.

Finally, the quantification of T is clearly non-referential: at least in the sense that the domain of discourse and the range of values of the variables of quantification are not identical. Indeed, there are no interpretations of T in which they can be identified, since the two are always of different cardinality.

4. *Concluding remarks* In closing, there is one more point of comparison to be made. One can identify individuals with their unit classes (see [4], p. 32). For the semantics envisaged for T this would have the effect of making the domain of discourse a subset of the range of the values of the variables. Nevertheless, the domain of discourse would be a proper subset of that range. In order to obtain the classical semantics it would also be necessary to ignore all the other elements in the range of values of the variables. And if that is to be done, one might as well reverse the process and identify unit classes with their elements obtaining interpretations based solely on given domains of discourse. Though now, since so much has been ignored, one pays the piper in the need of assuming existence: since ranges of variables cannot be empty without abandoning classical principles of quantification, domains of discourse must now be unempty.

With this last comparison, I wish to draw attention away from classes; in the course of the paper I have often mentioned them, but my position is not to be construed as committed to their existence. Though I have been translating theory T into the generally more familiar class theory, I have offered what I take to be a proper translation (if translation we must have) in hopes of forestalling improper ones. As an example of an improper translation, consider the following.

> The constants substitutable for variables in T are names of classes.
> The range of the variables and the domain of discourse of T are identical.
> T is merely class theory with a primitive constant differing from the usual 'element of'.

It may well be that theory T can be systematically (mis-)construed in this manner. But although the theory may have more than one interpretation, this should not argue against the interest of the (proper) interpretation I have given in section **3.** In particular, theory T *may* be used to talk about individuals without commitment to their existence (be they individual classes or urelements). Moreover T carries this advantage without abandoning any of the classical principles of quantification.

Finally, although I have said little about the power of this theory, briefly, the theory is at least as powerful as second order monadic predicate calculus (with identity). More precisely, it is possible to obtain translations of all the true statements of the latter in the former. This can be done by relativizing the quantifiers for individual variables so that the following, for instance:

$$[xy] .\because. [f] : f(x) .\equiv. f(y) :\equiv. x = y$$

is translated as:

$$[xy] .\because. x \varepsilon x . y \varepsilon y .\supset: [f] : x \varepsilon f .\equiv. y \varepsilon f :\equiv. x = y$$

Notice that in passing from the classical theory to T we no longer require a type distinction among the quantified variables: predicate and individual variables of the classical theory are handled by the sole type of variable available in T.

In sum, I am arguing that we can employ a logical theory to talk about individuals which is far stronger than what Quine calls "the virtual theory of classes" [4], and while employing this theory not only can we avoid commitment to the existence of classes (as the virtual theory does), but we can avoid commitment to the existence of individuals. The beauty of the theory is that it retains all the classical principles of quantification and all the theorems of a fairly powerful logic, while reducing ontological commitment and type distinction among variables. The price one pays for beauty is abandonment of proper nouns as constants substitutable for variables and acceptance of general terms only for such work: these are terms which do not name individuals but are merely *true of* many, or one, or none.

For the record, the first formula given above can serve as the axiom for St. Leśniewski's system of ontology: the theory described here is fully contained in ontology. (For a comparison of ontology with *Principia Mathematica* see [5].) Further, I conjecture that the intended interpretation given for T can be adequately extended for the full system of ontology, thus providing a logical theory at least as powerful as the full omega order predicate calculus, without ontological commitment to individuals. Here at last we have a powerful logical theory with classical principles of quantification which tells us nothing about the number of individuals in the world.

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