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A NEW CLASS OF MODAL SYSTEMS

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In [15][1], p. 354, Zeman considered a formula

Z1 $\mathfrak{C}LMpCLMqCMKpqLMKpq$ [In [15] formula (52)]

which, obviously, is a consequence of S5, and at the same time is verified by Lewis-Langford Group II. The addition of **Z1**, as a new axiom, to the systems S4.3.1, S4.3.2 and S4.4 generates the modal systems which Zeman calls S4.3.3, S4.3.4 and S4.6 respectively. And, as it has been proved by Zeman, the systems S4.3.3 and S4.3.4 are distinct, and each of them is a proper subsystem of S4.6 which in its turn is contained in his system S4.9, i.e., in Schumm's system S4.7, *cf.*, [5], [15], [8] and [9]. Hence, S4.6 is a consequence of my system V1 and, at the same time and independently, of S5. In [15] Zeman has remarked that formula **Z1** is clearly provable in the field of K3.1 or of K3.2, since each of those systems contains the formula

K2* $\mathfrak{C}LMpCLMqLMKpq$ [In [15], p. 354, formula (54)]

which according to Zeman, *cf.* [15], p. 349, formula (15), in the field of S4.4, is inferentially equivalent to

K1 $\mathfrak{C}LMpMLp$.

But, it is self-evident that in the field of S4 formula **K2*** is inferentially equivalent to McKinsey's formula, *cf.* [1], p. 92, formula (F),

K2 $\mathfrak{C}LMpCLMqMKpq$

which, as I have proved in [13], pp. 77-78, section **5**, in the field of S4 is inferentially equivalent to **K1**. Therefore, any system belonging to the family $\mathcal{K}$ of non-Lewis modal systems, *cf.* a definition of this family in [10], p. 313, contains formula **Z1**. Whence, besides Zeman's systems S4.3.3, S4.3.4 and S4.6 which are mentioned above, there can be other

1. An acquaintance with the papers which are cited in this note and, especially, with, *cf.* [8], pp. 347-350, the enumeration of the extensions of S4 and their proper axioms, is presupposed.

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systems obtained by the addition of **Z1**, as a new axiom, to S4 and its proper extensions up to S4.4 inclusively. In this note all systems which can be constructed now in such a way will be defined and their mutual connections will be investigated briefly.

1 Since the actual enumeration of the proper extensions of S4 is already cumbersome, I shall call any extension of S4 up to S4.4 inclusively, which contains **Z1** modal system, $\mathcal{Z}$. Because it has been proved by Schumm, *cf.* [6] and [9], that the systems S4.1 and S4.1.1 are inferentially equivalent and, therefore, also the systems S4.1.2 and S4.1.3, there are only ten generators which are known at present of the $\mathcal{Z}$ systems, viz. S4, S4.04, S4.1, S4.1.2, S4.2, S4.2.1, S4.3, S4.3.1, S4.3.2 and S4.4. But, we have only nine distinct $\mathcal{Z}$ systems obtained in the above mentioned way, since, as will be proved in **1.1**, the addition of **Z1** to S4.04 yields a system which contains {S4.1.2; **Z1**}.

1.1 From the definitions of the systems S4.4 and S4.1.2, *cf.* [8] and [12], it follows at once that in order to prove {S4.04; **Z1**} $\rightleftarrows$ {S4.1.2; **Z1**} it is sufficient to show that in the field of S4 the formulas **L1** and **Z1** imply **N1**. Hence, let us assume S4, **L1** and **Z1**. Then:

Z1 $\mathfrak{C}pMp$ [S1]

Z2 $\mathfrak{C}\mathfrak{C}LpMqLMCpq$ [S2°, *cf.* [13]]

Z3 $LMCpLp$ [*Z2*, q/Lp; *Z1*, p/Lp]

Z4 $LMCMpp$ [*Z3*, p/Np; S1°]

Z5 $\mathfrak{C}LMpCMLqMKpq$ [S4°]

Z6 $\mathfrak{C}LMKCpqCqrLMCpr$ [S2°]

Z7 $\mathfrak{C}Lp\mathfrak{C}Lq\mathfrak{C}\mathfrak{C}LpCvr\mathfrak{C}\mathfrak{C}LpCLqCrs\mathfrak{C}\mathfrak{C}st\mathfrak{C}vt$ [S4°]

Z8 $\mathfrak{C}MLCpLpLMCMpLp$ [*Z7*, $p/MCMpp$, $q/MCpLp$, $v/MLCpLp$, $r/MKCMppCpLp$, $s/LMKCMppCpLp$, $t/LMCMpLp$; *Z4*; *Z3*; *Z5*, $p/CMpp$, $q/CpLp$; **Z1**, $p/CMpp$, $q/CpLp$; *Z6*, p/Mp, q/p, r/Lp]

Z9 $\mathfrak{C}MLCpLpLMCMpLLp$ [*Z8*; S4°]

Z10 $\mathfrak{C}LMCMpLqLMLCpq$ [S2°, *cf.* [7]]

Z11 $\mathfrak{C}LMLp\mathfrak{C}pLp$ [**L1**; S4°]

Z12 $\mathfrak{C}\mathfrak{C}pq\mathfrak{C}\mathfrak{C}qr\mathfrak{C}\mathfrak{C}rs\mathfrak{C}ps$ [S3°]

Z13 $\mathfrak{C}MLCpLp\mathfrak{C}CpLp\mathfrak{C}pLp$ [*Z12*, $p/MLCpLp$, $q/LMCMpLLp$, $r/LMLCpLp$, $s/\mathfrak{C}CpLp\mathfrak{C}pLp$; *Z9*; *Z10*, q/Lp; *Z11*, $p/CpLp$]

Z14 $\mathfrak{C}\mathfrak{C}\mathfrak{C}CpLp\mathfrak{C}pLprCMLCpLpr$ [*Z13*; S2]

$\mathfrak{S}$II $\mathfrak{C}C\mathfrak{C}\mathfrak{C}CpLp\mathfrak{C}pLpCpLpCMLCpLpCpLpCC\mathfrak{C}\mathfrak{C}CNpLNp\mathfrak{C}NpLNpCNpLNp$ $CMLCNpLNpCNpLNpC\mathfrak{C}\mathfrak{C}pLpLpCMLpLp$ [S4]

$\mathfrak{S}$II is Schumm's formula which, as he has proved metalogically, holds in S4, *cf.* [6]. A logical proof that **$\mathfrak{S}$II** is a consequence of S4 is given in [9], section **1.2.2**, formula *S3*.

N2 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pLpLpCMLpLp$ [**$\mathfrak{S}$II**; S3°; *Z14*, $r/CpLp$; *Z14*, p/Np, $r/CNpLNp$]

N1 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pLppCMLpp$ [**N2**; S4]

Thus, it has been proved that {S4.04; **Z1**} $\rightleftarrows$ {S4; **L1**; **Z1**} $\rightleftarrows$ {S4; **L1**; **N1**; **Z1**} $\rightleftarrows$ {S4.04; **N1**; **Z1**} $\rightleftarrows$ {S4.1.2; **Z1**}.

1.2 Due to the result presented in **1.1** and in accordance with a definition of the $\mathcal{Z}$ systems, nine extensions of S4 containing **Z1** can be defined and, subsequently, proven to be distinct. Namely,

Z1 = {S4; **Z1**}
Z2 = {S4; **L1**; **Z1**} = {S4.04; **Z1**} = {Z1; **L1**}
Z3 = {S4; **N1**; **Z1**} = {S4.1; **Z1**} = {Z1; **N1**}
Z4 = {S4; **G1**; **Z1**} = {S4.2; **Z1**} = {Z1; **G1**}
Z5 = {S4; **G1**; **N1**; **Z1**} = {S4.2; **N1**; **Z1**} = {S4.2.1; **Z1**} = {Z4; **N1**}
Z6 = {S4; **D1**; **Z1**} = {S4.3; **Z1**} = {Z1; **D1**}
Z7 = {S4; **D1**; **N1**; **Z1**} = {S4.3; **N1**; **Z1**} = {S4.3.1; **Z1**} = {Z6; **N1**} = {S4.3.3} = {D; **Z1**}[2]
Z8 = {S4; **F1**; **Z1**} = {S4.3.2; **Z1**} = {Z1; **F1**} = {S4.3.4}
Z9 = {S4; **R1**; **Z1**} = {S4.4; **Z1**} = {Z1; **R1**} = {S4.6}

Obviously, the systems Z7, Z8 and Z9 are Zeman's systems S4.3.3, S4.3.4 and S4.6 respectively. It should be noticed that an addition of **L1**, as a new axiom, to each of the systems Z4-Z8 gives system Z9 at once, *cf.* [12].

2 In the following, I shall use the set of Matrices $\mathfrak{M}1$-$\mathfrak{M}12$ given in [8], p. 350. Due to a very bad misprint in [8], a wrong value for $M12$ appears in matrix $\mathfrak{M}10$. Namely, instead of $M12 = 1$, it obviously should be: $M12 = NLN12 = NL5 = N13 = 4$. Therefore, a correct version of $\mathfrak{M}10$ follows:

	p	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\mathfrak{M}10$	Mp	1	1	1	4	5	5	5	8	1	1	1	4	13	13	13	16
	Lp	1	4	4	4	13	16	16	16	9	12	12	12	13	16	16	16

3 It is easy to establish the relations existing from one side between the given system $\mathcal{Z}$ and the corresponding extension of S4, and from the other side between the same system $\mathcal{Z}$ and the corresponding $\mathcal{K}$ system. Namely,

3.1 Since $\mathfrak{M}8$ verifies S4.4, but, *cf.* [15], p. 354, falsifies **Z1**, it is clear that the systems S4, S4.04, S4.1, S4.2, S4.2.1, S4.3, S4.3.1, S4.3.2 and S4.4 are properly contained in the systems Z1-Z9 respectively.

3.2 Since $\mathfrak{M}2$, i.e., Lewis-Langford Group III, verifies S5, but falsifies every $\mathcal{K}$ system and since, in the field of S4, clearly, {**J1**} → {**N1**}, we see at once that the systems Z1, Z3-Z9 are the proper subsystems of K1, K1.1, K2, K2.1, K3, K3.1, K3.2 and K4 respectively. On the other hand, it has been shown, *cf.* [8], pp. 353-355, that formula **L1** is a consequence neither of K1 nor of K1.1 but, in the field of S4, **H1**, i.e., the proper axiom of K1.2 implies **L1**, and, therefore, due to the matrix $\mathfrak{M}2$ we know that Z2 is properly contained in K1.2.

2. My systems S4.3.1 and K3.1, *cf.* their definitions in [11], p. 306, and in [10], p. 316, are the systems D and D* of Prior and of Makinson respectively, *cf.* [4].

4 Since each of the matrices $\mathfrak{M}4$, $\mathfrak{M}5$, $\mathfrak{M}6$ and $\mathfrak{M}9$ verifies **Z1**, we are able to establish the following connections among the $\mathcal{Z}$ systems:

4.1 $\mathfrak{M}4$ verifies **N1**, **G1** and **D1**, but it falsifies **R1**, *cf.* [11], p. 310, and **F1**, *cf.* [16], p. 297. Moreover, $\mathfrak{M}4$ falsifies **L1** for $p/2$: $\mathfrak{C}LML2C2L2 = \mathfrak{C}LM6C26 = \mathfrak{C}L15 = LC15 = L5 = 5$. Hence, the systems Z1 and Z3, *cf.* section **1.1**, are properly contained in Z2; systems Z4-Z7 do not contain Z2; Z7 is a proper subsystem of Z9; Z7 does not contain Z8; and also, Z6 is a proper subsystem of Z8.

4.2 $\mathfrak{M}5$ verifies **L1** and **N1**, but falsifies **G1**, *cf.* [3] and [11], pp. 310-311, and, therefore, also **D1**, **F1** and **R1**. Hence, system Z1 is properly contained in Z4; Z2 is a proper subsystem of Z9 and it does not contain the systems Z4-Z8; Z5 is a proper extension of Z3; and Z3 does not contain Z4.

4.3 $\mathfrak{M}6$ verifies **F1**, but falsifies **N1** and **R1**, *cf.* [16], pp. 296-298, and [8], p. 352, section **2.4**. Moreover, $\mathfrak{M}6$ falsifies **L1** for $p/5$: $\mathfrak{C}LML5C5L5 = \mathfrak{C}LM7C57 = \mathfrak{C}L13 = LC13 = L3 = 7$. Hence, system Z4 is properly contained in Z5; Z8 is a proper subsystem of Z9; the systems Z4, Z6 and Z8 do not contain Z2; and moreover, Z8 does not contain Z7.

4.4 $\mathfrak{M}9$ verifies **N1** and **G1**, but falsifies **D1**, *cf.* [3] and [11], pp. 310-311, and **L1** for $p/2$: $\mathfrak{C}LML2C2L2 = \mathfrak{C}LM10C210 = \mathfrak{C}L19 = LC19 = L9 = 9$. Hence, system Z4 is properly contained in Z6 and Z5 is a proper subsystem of Z7.

5 It follows immediately from the definitions of systems Z1-Z9 and the connections which are established in sections **3** and **4** that

(1) System Z1 is a proper extension of S4, a proper subsystem of K1, and it is properly contained in each of the systems Z2, Z3 and Z4.
(2) System Z2 is a proper extension of S4.1.2, a proper subsystem of K1.1.2, and it is properly contained in Z9. Moreover, Z2 properly contains Z1 and Z3, *cf.* section **1.1**, and neither contains the systems Z4-Z8 nor is contained in any such system.
(3) System Z3 is a proper extension of S4.1, a proper subsystem of K1.1, and it is properly contained in Z2 and in Z5. Moreover, Z3 neither contains Z4 nor is contained in it.
(4) System Z4 is proper extension of S4.2, a proper subsystem of K2, and it is properly contained in Z5 and in Z6.
(5) System Z5 is a proper extension of S4.2.1, a proper subsystem of K2.1, and it is properly contained in Z7. Moreover, Z5 neither contains Z6 nor is contained in it.
(6) System Z6 is proper extension of S4.3, a proper subsystem of K3, and it is properly contained in Z7 and in Z8.
(7) System Z7 is a proper extension of S4.3.1, a proper subsystem of K3.1, and it is properly contained in Z9. Moreover, Z7 neither contains Z8 nor is contained in it.
(8) System Z8 is a proper extension of S4.3.2, a proper subsystem of K3.2, and is properly contained in Z9.

It should be noted here that the proofs of (7) and (8) are exactly the same as Zeman used in order to prove that the systems S4.3.3 (i.e., Z7) and S4.3.4 (i.e., Z8) are distinct and each of them is properly included in S4.6 (i.e., Z9), *cf.* [15], p. 354.

(9) System Z9 is a proper extension of S4.4, and it is a proper subsystem of K4 and of S5. Moreover, since, *cf.* [14] and [8], pp. 352-353, section **2.3**, system K4 properly contains V1 which, as Zeman has proved in [15], contains S4.9 and S4.6 (i.e., Z9) is contained in the latter system, and, since matrix $\mathfrak{M}3$ verifies system S4.9, but falsifies **V1**, *cf.* [11], p. 306, Z9 is properly contained in V1. A problem which Zeman did not solve in [15], namely, whether Z9 is a proper subsystem of S4.9, remains open.

Thus, it has been proved that the systems Z1-Z9 are distinct.

6 The following four diagrams

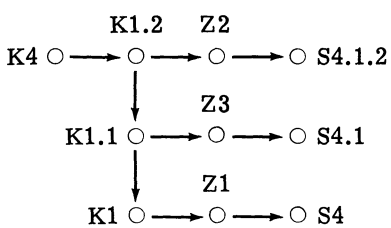

Diagram I

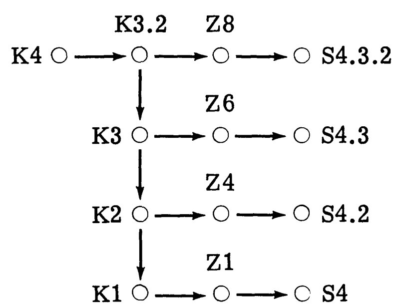

Diagram II

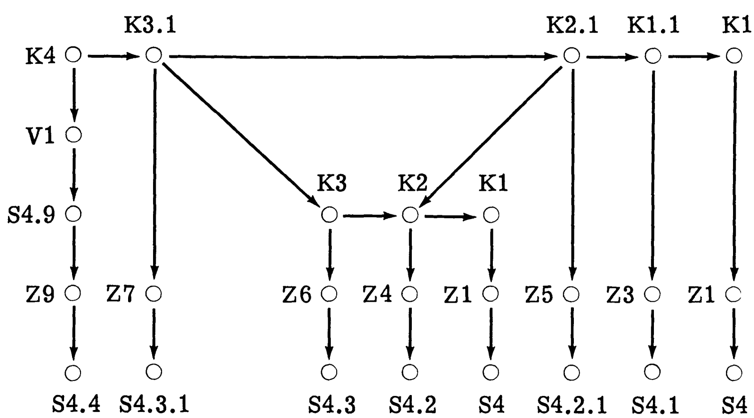

Diagram III

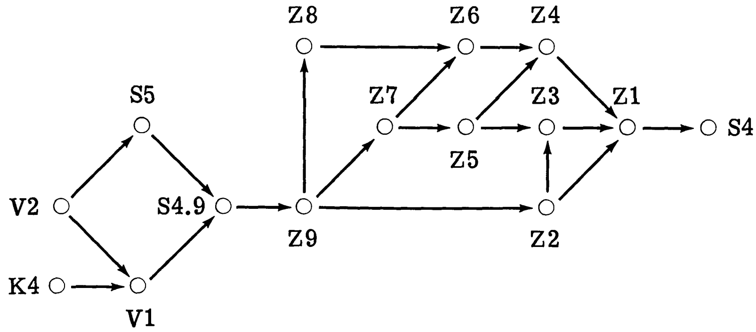

Diagram IV

visualize the relations among the systems under consideration. An arrow occurring between two systems indicates that a tail system is an extension of an edge system. With the exception of S4.9 it is proved that any other system represented in these diagrams is a proper extension of its edge system.

7 It should be remarked that in the axiom-systems which are given in section **1.2** of Z4, Z5 and Z9 an assumption S4 can be substituted by a weaker one, namely instead of S4 we can adopt S3. An easy proof can be obtained by an application of the same mode of reasonings which are presented in [13], pp. 75-76, section **3.6**, and in [2], p. 148.

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