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MEASURABLE CARDINALS AND CONSTRUCTIBILITY WITHOUT REGULARITY

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It has been shown (see Dana Scott [5]) that the axiom of constructibility ($\mathsf{V} = \mathsf{L}$) is incompatible with the existence of a measurable cardinal number. In [4] we gave a decomposition of $\mathsf{V} = \mathsf{L}$, over set theory without the axiom of regularity, into the axiom of regularity and the proposition:

$$\mathsf{P}\colon\ \forall x(x \in \mathsf{V} \wedge x \subset \mathsf{L} \rightarrow x \in \mathsf{L}).$$

In this paper we will show that even without the axiom of regularity P is sufficient to insure that there are no measurable cardinals. We shall work within the system of [1] but use the notation of [5]. Our result is thus formulated as follows:

Theorem I. *In* **GB** *set theory with* **AC** *but without the axiom of regularity, if* P *holds, then there does not exist a measurable cardinal.*

Our proof will follow that of Scott [5], who assumed $\mathsf{V} = \mathsf{L}$ in the following form:

(*) *If M is a class such that*

(i) $M \subset \mathcal{P}(M) \subset \bigcup_{x \in M} \mathcal{P}(x)$

(ii) $x - y, \bigcup x, \breve{x}, x|y, E|x \in M$, *for all* $x, y \in M$;

then $\mathsf{V} = M$.

(In the above, $\mathcal{P}$ denotes the power set operation so $\mathcal{P}(M)$ is the class of all subsets of M; $\bigcup x = \bigcup_{y \in x} y$; $\breve{x}$ denotes the operation of forming the converse of the relational part of x; $x|y$ denotes the operation of forming the relative product of the relational parts of x and y; $E|y = \{\langle u, v\rangle : u \in v \in x\}$.)

We shall formulate P in a similar way. We first note that a set x is called *grounded* if there does not exist an infinite descending $\in$-chain beginning with x.

Proposition II. *In the field of* **GB** *set theory with* **AC** *but without the axiom of regularity the following statements are equivalent:*

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(**P**) $\forall x(x \in \mathsf{V} \wedge x \subset \mathsf{L} \rightarrow x \in \mathsf{L})$

(**) *If M is a class such that* (i) *and* (ii) *of* (*) *hold, then $x \in \mathsf{V} - M \rightarrow x$ is not grounded.*

Proof: We will need the following lemma:

Lemma II.1. *In the field of set theory without the axiom of regularity, if a class K satisfies* (i) *and* (ii) *of* (*) *and $K \subset \mathsf{L}$, then $K = \mathsf{L}$.*

Proof: Suppose we have a class K such that K satisfies (i), (ii) and

(1) $K \subset \mathsf{L}$.

Since K satisfies (i) and (ii), it is a model of **GB** (without the axiom of regularity). By (1) and Lemma III.3.15 of [4],

(2) every element of K is grounded.

If we let $\Psi(\alpha) = \bigcup_{\beta<\alpha} \mathcal{P}(\Psi(\beta))$, $\Psi(0) = \phi$, then it is clear by (1) and the definition of constructibility that

(3) $x \in K \rightarrow \exists\alpha(x \in \Psi(\alpha))$.

By the results of Mendelson [2], (2) and (3) we have

(4) the axiom of regularity holds in K.

Statement (4) allows us to use the equivalence of (*) and $\mathsf{V} = \mathsf{L}$. Since K and L both satisfy (i) and (ii), we have $\mathsf{V} = K$ and $\mathsf{V} = \mathsf{L}$; hence, $K = \mathsf{L}$ and our lemma is proved.

We now return to the proof of Proposition II. First we will show that (**) $\Rightarrow$ **P**. Suppose we have a set x such that

(1) $\sim(x \in \mathsf{L})$.

But L, the class of constructible sets, satisfies (i) and (ii) of (*). Therefore, by (1) and (**),

(2) x is not grounded.

By (2) and Lemma III.3.2 of [4], there is some $y \in x$ such that

(3) y is not grounded.

Since every constructible set is grounded, (3) gives us

(4) $\sim(y \in \mathsf{L})$.

By (4) we know that $\sim(x \subset \mathsf{L})$. Therefore, we have shown that, under the assumption of (**), $(x \in \mathsf{V} \wedge \sim(x \in \mathsf{L}) \rightarrow \sim(x \subset \mathsf{L}))$. Hence, (**) $\Rightarrow$ **P**.

Now suppose that **P** is true and we have a class M that satisfies (i) and (ii). Let us also suppose that we have some set x such that

(5) $x \in \mathsf{V} - M$.

Since, by Lemma II.1, L is the smallest class satisfying (i) and (ii), we know that

(6) $\mathsf{L} \subset M$.

By (5) and (6), we have

(7) $x \in \mathsf{V} - \mathsf{L}$.

By (7), we have

(8) $\sim(x \in \mathsf{L})$.

By (5), (8), and **P**, we have

(9) $\sim(x \subset \mathsf{L})$,

hence, by (9), there is some x_1 such that

(10) $x_1 \in x$ and $\sim(x_1 \in \mathsf{L})$.

By the same reasoning, we obtain an x_2 such that

(11) $x_2 \in x_1$ and $\sim(x_2 \in \mathsf{L})$.

Proceeding in the same way as (10) and (11), we obtain an infinite descending $\in$-chain beginning with x. Therefore x is not grounded. Thus we have shown that $\mathbf{P} \Rightarrow (**)$ and Proposition II has been proved.

By looking at Proposition II and Theorems III.1 and III.3 of [4], we obtain the following diagram:

$$\begin{tikzcd} \mathsf{V} = \mathsf{L} \arrow[r, leftrightarrow] \arrow[d, shift right] & (*) \arrow[d, shift right] \\ \mathbf{P} \arrow[r, leftrightarrow] \arrow[u, shift right, "/" marking] & (**) \arrow[u, shift right, "/" marking] \end{tikzcd}$$

where the bottom entries represent the analogues of the upper entries without the axiom of regularity.

We now continue the proof of Theorem I. Since, by [4], **GCH** follows from **AC** + **P**, we can assume, as in [5], that ω_κ is the least measurable cardinal and that $\omega_\kappa = \kappa$. We pattern the rest of our proof after Scott's and use the same notation wherever possible. Let $\mu \in \{0, 1\}^{\mathcal{P}(\kappa)}$ be a 2-valued, non-trivial, countably additive measure defined on all subsets of κ. (If A is a class and b is a set, A^b denotes the class of all functions with domain b and range contained in A.) We now define relations Q_μ and E_μ over V^κ exactly as Scott does.

Definition I.1.

(i) $Q_\mu = \{\langle f, g\rangle : f, g \in \mathsf{V}^\kappa \wedge \mu(\{\xi < \kappa : f(\xi) = g(\xi)\}) = 1\}$;
(ii) $E_\mu = \{\langle f, g\rangle : f, g \in \mathsf{V}^\kappa \wedge \mu(\{\xi < \kappa : f(\xi) \in g(\xi)\}) = 1\}$.

The following lemma is proved exactly as in [5]:

Lemma I.1. *Q_μ is a congruence relation for E_μ over* V^κ.

The next lemma is different from Scott's in that we cannot prove his point (iii) since we do not have the axiom of regularity.

Lemma I.2.

(i) *If* $\{h \in V^{\kappa}: hE_{\mu}f\} = \{h \in V^{\kappa}: hE_{\mu}g\}$, *then* $fQ_{\mu}g$;

(ii) $\{h \in V^{\kappa}: hE_{\mu}f\} = \{h \in V^{\kappa}: \exists k[k \in \left(\bigcup_{\xi<k} f(\xi) \cup \{\phi\}\right)^{\kappa} \wedge kE_{\mu}f \wedge hQ_{\mu}k]\}$.

For the next lemma we need some notation for the functions which map onto grounded sets "almost everywhere." Therefore we let

$$G = \{f \in V^{\kappa}: \mu\{\xi: f(\xi) \text{ is grounded}\} = 1\}.$$

We then have:

Lemma I.3. *There is a function* σ *with domain* G *such that for* $f, g \in G$.

(i) $\sigma(f) = \{\sigma(h): h \in V^{\kappa} \wedge hE_{\mu}f\}$;
(ii) $\sigma(f) = \sigma(g)$ *if and only if* $fQ_{\mu}g$;
(iii) $\sigma(f) \in \sigma(g)$ *if and only if* $fE_{\mu}g$.

Proof: We follow the technique of Mostowski, [3], Theorem 3. We define the sets m_{γ} by induction:

$$m_0 = \{g: g \in G \wedge \mu\{\xi: g(\xi) = \phi\} = 1\};$$
$$m_{\gamma} = \left\{f: f \in G - \bigcup_{\alpha<\gamma} m_{\alpha} \text{ such that } \forall h\left(hE_{\mu}f \to h \in \bigcup_{\alpha<\gamma} m_{\alpha}\right)\right\}.$$

We then define the function σ by:

$$f \in m_0 \to \sigma(f) = \phi;$$
$$f \in m_{\gamma},\ \gamma > 0 \to \sigma(f) = \{\sigma(h): hE_{\mu}f\}.$$

It is clear that σ is the desired function.

Because of our definition of σ, the following definition is slightly different from Scott's:

Definition I.2. $M = \{\sigma(f): f \in G\}$.

The next lemma is proved in precisely the same way as Scott's:

Lemma I.4. $M \subset \mathcal{P}(M) \subset \bigcup_{x \in M} \mathcal{P}(x)$.

In the following, $\Phi(v_1, \ldots, v_k)$ will stand for any formula of set theory with free variables $v_1, \ldots, v_k$ and with all quantifiers restricted to V (i.e., no bound class variables). Further, $\Phi^{(M)}(v_1, \ldots, v_k)$ is the result of relativizing all the quantifiers of $\Phi(v_1, \ldots, v_k)$ to the class M.

Lemma I.5. If $f_1, \ldots, f_k \in G$, *then* $\Phi^{(M)}(\sigma(f_1), \ldots, \sigma(f_k))$ *if and only if* $\mu(\{\xi < \kappa: \Phi(f_1(\xi), \ldots, f_k(\xi))\}) = 1$.

As in [5], we can show that M satisfies (i) and (ii) of (*) and thus we have:

Corollary I.5.1. $x \in V - M \to x$ *is not grounded.*

We introduce another definition that is similar to Scott's:

Definition I.3. If x is grounded, then $x^* = \sigma(\{\langle \xi, x\rangle: \xi < \kappa\})$.

We then obtain the following corollary as a special case of Lemma I.5.

Corollary I.5.2. *If* $x_1, \ldots, x_k$ *are grounded, then* $\Phi^{(M)}(x_1^*, \ldots, x_k^*)$ *if and only if* $\Phi(x_1, \ldots, x_k)$.

If we now combine I.5.1 and I.5.2 and use the formula $\Phi(\kappa)$ that expresses in formal terms that κ is the least 2-valued measurable cardinal and note that since κ is an ordinal number it is grounded, we prove:

Corollary I.5.3. $\kappa = \kappa^*$.

The rest of the proof follows [5] exactly. We sketch the remainder of Scott's proof for the sake of completeness.

Definition I.4. $\delta = \sigma(\{\langle\xi, \xi\rangle : \xi < \kappa\})$.

Lemma I.6. *If* $\lambda < \kappa$, *then* $\lambda^* < \delta < \kappa^*$.

From I.5.2 it follows at once that the mapping from grounded sets x to sets x^* is one-one; hence, the set $\{\lambda^* : \lambda < \kappa\}$ must have cardinality κ. By Lemma I.6 it follows that δ must have cardinality at least that of κ. But I.5.3 and I.6 imply that $\delta < \kappa$, which contradicts the choice of κ as an initial ordinal, and thus no measurable cardinals exist.

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