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COMPLETE MODALIZATION IN S4.4 AND S4.0.4

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In [1] bases for S4 and S4.2 were offered which didn't require axioms beyond those of the **PC**; in [2] a version of the deduction theorem was presented giving a unified treatment of this metatheorem for S4, S4.2, and S5. The central idea in these papers was the notion of "complete modalization"; the three different concepts of complete modalization presented serve as characterizations of these three distinct systems.

Two other systems which lend themselves to the treatment of those papers are S4.4 [4] and S4.0.4 [3]. It is possible to find modifications of the notion of complete modalization chracteristic of each of these systems. We recall first of all that S4.4 and S4.0.4 result from the addition, respectively, of $CpCMLpLp$ and $CpCLMLpLp$ to S4 (we assume, of course, that the added axiom comes under the sway of a rule to infer $L\varphi$ from any theorem φ; if S4 is thought of as being in the original Lewis formulation rather than a Lemmon-type [5] base, we would add versions of these extra axioms with strict, rather than material, implication as the main connective).

We now state the rules of [1] for the introduction of L in antecedent and consequent of an implication:

RL1: $C\alpha\beta \rightarrow \vdash CL\alpha\beta$
RL2: $C\alpha\beta \rightarrow \vdash C\alpha L\beta$, *provided* α *is completely modalized.*

Complete modalization in S4 may be recursively defined as follows:

(a) If φ is an S4 theorem, φ is completely modalized in S4
(b) $L\varphi$ is completely modalized in S4.
(c) If φ and ψ are both completely modalized in S4, so too is $K\varphi\psi$.

We now extend the definition to S4.4 and S4.0.4:

(d) If φ is completely modalized in S4, it is completely modalized in both S4.4 and S4.0.4.
(e) $K\varphi ML\varphi$ is completely modalized in S4.4
(e') $K\varphi LML\varphi$ is completely modalized in S4.0.4

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(f) as clause (c) reading S4.4 for S4
(f') as clause (c) reading S4.0.4 for S4.

As shown in [1], S4 will be derivable in **PC** + **RL1** + **RL2** with the above definitions of complete modalization. With the S4.4 definition, we have

(1)	$CKpMLpp$	**PC**
(2)	$CKpMLpLp$	(1), **RL2** for S4.4
(3)	$CpCMLpLp$	(2), **PC**

Thus is S4.4 derivable in the above basis with the S4.4 version of complete modalization. If we assume the S4.0.4 version, we can obviously repeat steps (1)-(3) above with LML for ML, getting

(4) $CpCLMLpLp$

Going the other way, we assume first S4.4; **RL1** is obviously derivable therein. So far as **RL2** is concerned,

(5)	$\vdash C\alpha\beta$, α completely (S4.4) modalized	Hyp.
(6)	$\vdash CL\alpha L\beta$	(5), S4

Let α' be the formula exactly like α except for containing the subformula $L\gamma$ wherever α contains $K\gamma ML\gamma$, for each such γ in α. An S4.4 thesis is

(7) $LELpKpMLp$

thus

(8) $\vdash CL\alpha' L\beta$ (7), (6), S1°

By our definitions of complete modalization, α' is a formula which is of form $L\gamma$, or is a conjunction whose "ultimate conjuncts" all begin with L. But for such an α' in S4

(9) $LE\alpha' L\alpha'$ S4, Df. α'

Then

(10) $\vdash C\alpha' L\beta$ (8), (9), S1°

By reversing the procedure which took us from formula (6) to (8), we get

(11) $\vdash C\alpha L\beta$ (10), (7), S1°.

The movement from (5) to (11) shows that **RL2** is a rule of inference in S4.4. The base **PC** + **RL1** + **RL2** with S4.4 definition, then, is equivalent to S4.4. If the S4.0.4 definition of complete modalization is assumed, it is clear that we can repeat the above steps, using the S4.0.4 thesis

(12) $LELpKpLMLp$

in place of (7). With the S4.0.4 definition, then, our base is equivalent to S4.0.4.

We now turn to the question of the deduction theorem in these systems;

[2] stated and proved that theorem for S4, S4.2, and S5 in C-N-L Lemmon-style bases, so we shall do the same for S4.4 and S4.0.4. If a base like those earlier examined in this paper is assumed, the restriction on the use of **RL2** will be exactly the same as that given below for the use of the rule to infer $L\alpha$ from α. Assuming then, for S4.4, Lemmon S4 plus formula (3) above, we set down the following definition:

We say that β is deducible from the hypotheses $\alpha_1, \ldots, \alpha_n$ and we write $\alpha_1, \ldots, \alpha_n \vdash \beta$ provided there is a sequence of formulas $\beta_1, \ldots, \beta_m$ (called a deduction) such that $\beta_m = \beta$, and for all $i \leq m$:

(i) β_i is one of the hypotheses, or
(ii) β_i is an alphabetic variant of an axiom, or
(iii) β_i is the result of a detachment involving β_j and β_k, j, $k < i$ with $\beta_k = C\beta_j\beta_i$.
(iv) β_i results from a substitution in β_j, $j < i$ provided the variable substituted for does not occur in any of the hypotheses.
(v) β_i results from an application of the rule to infer $L\alpha$ from α, provided each of the hypotheses is *completely modalized* in the system at question.

The deduction theorem, of course, is:

If $\alpha_1, \ldots, \alpha_n \vdash \beta$, *then* $\alpha_1, \ldots, \alpha_{n-1} \vdash C\alpha_n\beta$.

It is clear that the proof of this metatheorem for the cases corresponding to clauses (i)-(iv) of the above definition of deducibility will follow exactly as for the **PC**. We thus extend the proof for S4.4 to the case covered by clause (v). This will be a case in the induction step of the proof; the induction is on the length of the deduction from the hypotheses to β, and the assumption is that the metatheorem holds when the deduction is no longer than k steps long. Supposing it is $k+1$ steps long, and the last step attained by clause (v) above, β_{k+1} $(= \beta)$ must be of form $L\beta_j$, where $j \leq k$. By the induction assumption, then,

(13) $$\alpha_1, \ldots, \alpha_{n-1} \vdash C\alpha_n\beta_j$$

Since the n - 1 remaining hypotheses still meet the proviso of (v), we get

(14) $$\alpha_1, \ldots, \alpha_{n-1} \vdash LC\alpha_n\beta_j$$

Distributing L, this yields

(15) $$\alpha_1, \ldots, \alpha_{n-1} \vdash CL\alpha_nL\beta_j.$$

But by **PC** and **RL2**, above shown to hold in S4.4, we have, for completely modalized α_n (which by hypothesis it is):

(16) $$\vdash C\alpha_nL\alpha_n$$

We thus may move from (15) to

(17) $$\alpha_1, \ldots, \alpha_{n-1} \vdash C\alpha_nL\beta_j$$

and the deduction theorem holds. It should be clear that the proof of this metatheorem for S4.0.4 will follow lines exactly parallel to those above indicated for S4.4; it therefore seems unnecessary to explicitly go through the proof for S4.0.4.

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