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AN ELEMENTARY COMPLETENESS PROOF FOR A SYSTEM OF NATURAL DEDUCTION

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The system of deduction of this paper is based on three operations: negation, conjunction, and universal quantification. Let formulas be formed for these operations in the usual way, except that distinct styles of letter are to be used for free and bound variables. A finite string of formulas, each preceded by one of the three lables '[' (assume), '/' (infer), or ']/' (discharge an assumption and infer), is called a *deduction* if the brackets are appropriately mated. This will be the case if mated pairs of left and right brackets can be successively eliminated from the inside out to leave only left brackets or no brackets at all. By the *scope* of a labelled formula in a deduction is meant the string of formulas which precede it, exclusive of those already enclosed in brackets.

Let the following *Rules of Inference* be given:

Simplification: From a conjunction infer either conjunct.
Conjunction: From two formulas infer their conjunction.
Instantiation: From a universal formula infer any of its instances.
Generalisation: From a formula α infer a universal generalisation of α.

A deduction is *valid* if it satisfies the following *Rules of Deduction:*

Direct Proof: A formula labelled '/' must follow from other formulas in its scope by the Rules of Inference.
Indirect Proof: A formula labelled '[', called an *assumption,* must be a negation, and, in fact, the negation of the formula with the mated label ']/', provided such a formula occurs. This will be the case only if the assumption appears in the scope of some formula and its negation.
Special Rule: In any application of the Rule of Generalisation the instantial variable must have no occurrence in any assumption in the scope of the derived formula.

A valid deduction will be *canonical* if it conforms to the following *Rules of Introduction:*

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Initiation: Any negation may be introduced as the initial assumption of a canonical deduction.
Assumption: Each assumption after the first is a negation $-\alpha$ which has in its scope a formula of one of the following forms: $--\alpha$, $-(\alpha \,\&\, \beta)$, $-(\beta \,\&\, \alpha)$, $-\forall\, v\, \beta$. In the case of $-(\beta \,\&\, \alpha)$ the assumption $-\alpha$ is introduced only when and if the assumption $-\beta$ has been previously introduced and discharged. In the case of $-\forall\, v\, \beta$ the formula α must be an instance of $\forall\, v\, \beta$, where the instantial letter is alphabetically the first free variable not appearing in the deduction prior to the assumption of $-\alpha$.
Instantiation: An inference by the Rule of Instantiation occurs only with a free variable previously introduced into the deduction, if any, or with the alphabetically first free variable.
Conjunction: An inference of a formula $\alpha \,\&\, \beta$ by the Rule of Conjunction occurs only after $-\alpha$ has been assumed from $-(\alpha \,\&\, \beta)$, then discharged, then $-\beta$ assumed and discharged, in accordance with the Rule of Assumption.
Generalisation: An inference of a formula $\forall\, v\, \beta$ by the Rule of Generalisation occurs only after $-\alpha$ has been assumed from $-\forall\, v\, \beta$ and then discharged, in accordance with the Rule of Assumption.
Priority: After an arbitrary initial assumption, further steps of a canonical deduction are introduced according to the following order of priority: discharge of an assumption, conjunction, generalisation, simplification, instantiation, introduction of an assumption. Within each category introduction occurs on the first available formula. Simplification occurs on the first conjunct first. Instantiation occurs on the first available free variable.
Non-Repetition: No assumption may be introduced more than once from the same occurrence of the negative formula from which it is derived. No inferred formula may be introduced more than once unless all of its earlier occurrences have been bracketed-out.

We now easily prove completeness of canonical deduction as follows: Let some arbitrary formula α be given. Consider the class C consisting of every formula which has a labelled occurrence in some canonical deduction beginning with $-\alpha$ which cannot be bracketed-out by the introduction of further canonical steps of deduction. Let free variables in alphabetical order designate natural numbers. Let the universe consist of all numbers designated by variables occurring in the formulas of C, if any, or of the number 1. Let atomic formulas be interpreted as true if and only if they are in C. Under this interpretation the other formulas of C will be true as well. Assume not, and let β be a shortest false formula of C. The following cases, all impossible, arise:

1. β has the form $\gamma \,\&\, \delta$. Then, by the Rule of Priority and the Rule of Non-Repetition, simplification yields γ and δ, one of which must be a shorter false formula.
2. β has the form $\forall\, v\, \gamma$. Then instantiation yields a shorter false formula.
3. β has the form $-\gamma$. Then four subcases arise:

3a. γ is an atomic formula. Then β and γ are contradictory formulas in C, which is impossible.

3b. γ has the form $-\delta$. Then from $--\delta$ introduce $-\delta$, discharge this assumption, and infer δ, a shorter false formula.

3c. γ has the form δ & ϵ. Then from $-(\delta \,\&\, \epsilon)$ introduce the assumption $-\delta$. This formula, if it remains undischarged, is shorter than β. If discharged, introduce $-\epsilon$. If undischarged this is a shorter formula. If discharged then introduce δ & ϵ by conjunction and obtain a contradiction.

3d. γ has the form $\forall\, v\, \delta$. Then from $-\forall\, v\, \delta$ introduce the assumption $-\epsilon$. If undischarged this is a shorter false formula. If discharged a contradiction is reached by introducing $\forall\, v\, \delta$ by generalisation.

If the initial assumption $-\alpha$ can be discharged, then α is proved; if not, then α is not universally valid, since $-\alpha$ is in C, and therefore true. Q.E.D.

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