Notre Dame Journal of Formal Logic
Volume XIII, Number 4, October 1972
NDJFAM

RECOGNIZABLE ALGEBRAS OF FORMULAS

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In this paper we consider various algebras formed out of the formulas of first-order languages. We rely mostly on [1] for our notations and terminology. We deal with structures, $\mathfrak{A} = \langle A, R_\theta\rangle_{\theta<\xi}$, where each R_θ is an n_θ-ary relation on A; and with algebras, $\mathfrak{A} = \langle A, F_\theta\rangle_{\theta<\xi}$, where each F_θ is an n_θ-ary function on A (in both cases $0 \leq n_\theta < \omega$). If R_θ (resp. F_θ) is a 0-ary relation (resp. function) it is a distinguished constant and we write it as a_θ. The type of $\mathfrak{A}$ is $\mu = \langle n_\theta\rangle_{\theta<\xi}$. $\mathcal{L}_\mu$ is the appropriate language for $\mathfrak{A}$; usually we just write $\mathcal{L}$. For $S \subseteq A$, $\mathcal{L}(S)$ is the language $\mathcal{L}$ with a symbol added for each element of S. Thus $\mathcal{L}(\phi) = \mathcal{L}$ and $\mathcal{L}(A)$ is the diagram language. When we write definable we mean definable in the diagram language (i.e. definable by parameters).

We use α, β, γ for cardinals and assume that α is regular, $\beta \leq \alpha$ and $\gamma < \alpha$. We use ϕ, ψ, χ for formulas. When we write a formula ϕ as $\phi(x_0, \ldots, x_\iota, \ldots, a_0, \ldots a_\eta, \ldots)$, it is understood that $x_0, \ldots, x_\iota, \ldots$ are all the free variables of ϕ and $a_0, \ldots, a_\eta, \ldots$ are all the parameters of A in ϕ. The cardinal of $\mathfrak{A}$ is $\overline{\overline{A}}$ and the cardinal of μ is $\overline{\overline{\xi}}$; we denote it by $\overline{\overline{\mu}}$. Given a formula ϕ, $|\phi|_{\mathfrak{A}} = \{\psi \mid \mathfrak{A} \models \phi \leftrightarrow \psi\}$; usually we just write $|\phi|$.

In general we present our results for a collection of languages at a time; in particular, $\mathcal{N} = \{\mathcal{L}_{\alpha\beta}\}$ and $\mathcal{M} = \{\mathcal{L}_{\alpha\alpha}\}$. Note that $\mathcal{M} \subseteq \mathcal{N}$ and $\mathcal{L}_{\omega\omega}$ is the usual finitary first-order language with equality. Unless otherwise specified we assume that $\mathcal{L} \in \mathcal{N}$. The notions of elementary equivalence, elementary extension, and elementary embedding can be extended to the infinitary languages and we write $\mathcal{L}$ - $\equiv$, $\mathcal{L}$ - $\prec$, and $\mathcal{L}$-embedding respectively. We also deal with second-order languages $\mathcal{L}^2$; $\mathcal{L}^2$ contains all the symbols of $\mathcal{L}$ and variables of every degree γ, $0 < \gamma < \alpha$, which we write as V_i^γ. In the model under consideration the V_i^γ are interpreted as variable γ-ary relations. In $\mathcal{L}^2$ we may quantify over such relations. Individual variables have degree 0 and are denoted by x, y, and z.

1. Definitions and Examples. We assume a type μ and a language $\mathcal{L}$ appropriate for μ as given.

Received July 1, 1971

Definition 1. A recognizable algebra of formulas of a structure $\mathfrak{A}$ is an algebra $\mathrm{R}(\mathfrak{A}) = \langle \|\mathrm{T}(V^{\gamma})\|, F_0, \ldots, F_{\iota}, \ldots \rangle_{\iota<\delta}$ where

1) $\mathrm{T}(V^{\gamma})$ is either
a) a formula of $\mathcal{L}^2$ which contains one free variable, namely V^{γ}, and no bound variables of degree >0; or
b) the symbol E,
2) In case a) $\|\mathrm{T}(V^{\gamma})\|$ is the set of equivalence classes $|\phi|$ of formulas of $\mathcal{L}(A)$ with free variables $x_0, \ldots, x_{\iota}, \ldots (\iota < \gamma)$ such that $\mathfrak{A} \models \mathrm{T}(\phi)$; in case b) $\|\mathrm{E}\|$ is the set of equivalence classes of formulas of $\mathcal{L}(A)$,
3) Each F_{ι} is an operation on $\|\mathrm{T}(V^{\gamma})\|$ defined by an operation F_{ι}^* as follows:

$$F_{\iota}(|\phi_1|, \ldots, |\phi_n|) = |F_{\iota}^*(\phi_1, \ldots, \phi_n)|.$$

F_{ι}^* may be defined inductively as consisting of a finite number of applications of projections, connectives, quantifiers, and substitutions of variables for variables (when allowable by the usual rules). A 0-ary operation is a $|\phi|$ where ϕ is a formula of $\mathcal{L}$ and $|\phi| \in \|\mathrm{T}(V^{\gamma})\|$.

We will write F instead of F^* if this causes no confusion. We use $\mathfrak{R}$ to stand for a recognizable algebra of formulas and R for its domain when $\mathfrak{A}$ is not specified.

Definition 2. Given structures $\mathfrak{A}$ and $\mathfrak{B}$ of the same type, $\mathrm{R}(\mathfrak{A})$ and $\mathrm{R}(\mathfrak{B})$ are said to be a pair of corresponding recognizable algebras of formulas if the definition of $\mathrm{R}(\mathfrak{B})$ is the definition of $\mathrm{R}(\mathfrak{A})$ with $\mathfrak{A}$ replaced by $\mathfrak{B}$.

Now we give examples of recognizable algebras of formulas. In each case we deal with formulas of the diagram language of a structure.

Example 1. The Lindenbaum algebra of formulas: $\langle \|\mathrm{E}\|, \vee, \wedge, \sim \rangle$;
Example 2. The cylindric algebra of formulas;
Example 3. The Boolean algebra of formulas of one free variable: $\langle \|(\forall x)(V^1(x) \leftrightarrow V^1(x))\|, \vee, \wedge, \sim, |x \neq x|, |x = x| \rangle$;
Example 4. The lattice of formulas of one free variable;
Example 5. The Boolean algebra of formulas of γ free variables;
Example 6. The relation algebra of formulas of two free variables;
Example 7. The semigroup of definable unary functions: $\langle \|(\forall x)(\exists y)[V^2(x, y) \wedge (\forall z)(V^2(x, z) \rightarrow y = z)]\|, * \rangle$ where if f_1 is $|\phi_1(x, y)|$ and f_2 is $|\phi_2(x, y)|$, then $f_2 * f_1$ is $|(\exists z)(\phi_1(x, z) \wedge \phi_2(z, y))|$ where z is the first variable free for y in $\phi_1(x, y)$ and free for x in $\phi_2(x, y)$;
Example 8. The group of definable permutations;
Example 9. If $\overline{\overline{\mu}} < \alpha$ the semigroup of definable endomorphisms;
Example 10. If $\overline{\overline{\mu}} < \alpha$ the group of definable automorphisms: $\langle \|(\forall x)(\exists y)[V^2(x, y) \wedge (\forall z)(V^2(x, z) \rightarrow y = z)] \wedge (\forall y)(\exists x)[V^2(x, y) \wedge (\forall z)(V^2(z, y) \rightarrow x = z)] \wedge P_0 \ldots \wedge P_{\theta} \wedge \ldots (\theta < \xi), *, {}^{-1} \rangle$ where $*$ is defined as in Example 7, if f is $|\phi(x, y)|$ then f^{-1} is $|\phi(y, x)|$, and if $n_{\theta} > 0$, P_{θ} is $(\forall x_1, \ldots, x_{n_{\theta}}, y_1, \ldots, y_{n_{\theta}}) \{[V^2(x_1, y_1) \wedge \ldots \wedge V^2(x_{n_{\theta}}, y_{n_{\theta}})] \rightarrow [R_{\theta}(x_1, \ldots, x_{n_{\theta}}) \leftrightarrow R_{\theta}(y_1, \ldots, y_{n_{\theta}})]\}$, finally if $n_{\theta} = 0$, P_{θ} is $V^2(a_{\bar{\theta}}, a_{\theta})$;
Example 11. The subalgebra of a recognizable algebra obtained by

restricting $\|\mathrm{T}(V^\gamma)\|$ to formulas of $\mathcal{L}(S)$ where S is defined by a formula of $\mathcal{L}$ of one free variable.

2. The Main Theorems If $d : A \to B$ and $\phi(x_0, \ldots, x_\iota, \ldots, a_0, \ldots, a_\eta, \ldots)$ is a formula of $\mathcal{L}(A)$ then we write $d\phi$ for the formula $\phi(x_0, \ldots, x_\iota, \ldots, d(a_0), \ldots, d(a_\eta), \ldots)$.

Proposition 1. *$\mathfrak{A}$ is $\mathcal{L}$-embeddable in $\mathfrak{B}$ iff there is an embedding $d : A \to B$ such that for every pair of corresponding recognizable algebras* $\mathrm{R}(\mathfrak{A})$ *and* $\mathrm{R}(\mathfrak{B})$, *the map $|d| : |\phi|_{\mathfrak{A}} \to |d\phi|_{\mathfrak{B}}$ is an embedding of* $\mathrm{R}(\mathfrak{A})$ *into* $\mathrm{R}(\mathfrak{B})$.

Proof: ($\Longrightarrow$) If $\mathfrak{A}$ is $\mathcal{L}$-embeddable in $\mathfrak{B}$, let d be an $\mathcal{L}$-embedding of $\mathfrak{A}$ into $\mathfrak{B}$. Then $|d|$ has the required property.

($\Longleftarrow$) If $\mathfrak{A}$ is not $\mathcal{L}$-embeddable in $\mathfrak{B}$, let $d : A \to B$ be any embedding (if there is one). There is a sentence ψ of $\mathcal{L}(A)$ such that $\mathfrak{A} \models \psi$ and $\mathfrak{B} \models \sim d\psi$. Now let $\mathfrak{R} = \langle \|(\forall x)\ (V^1(x)\|\rangle$, and let ϕ be $\psi \vee x \neq x$. Then $|\phi|_{\mathfrak{A}} \in \mathrm{R}(\mathfrak{A})$ but $|d\phi|_{\mathfrak{B}} \notin \mathrm{R}(\mathfrak{B})$. Thus $|d|$ is not an embedding.

It follows that for every recognizable algebra R, the map $r : \mathfrak{A} \to \mathrm{R}(\mathfrak{A})$ is a functor from a category of models (the maps being $\mathcal{L}$-embeddings) to a category of algebras (the maps being embeddings).

Lemma 1. *Suppose that $\mathfrak{A}\ \mathcal{L}\text{-}\equiv \mathfrak{B}$. Then*

(i) *there is a ρ-tuple of A, $\langle a_0, \ldots, a_\eta, \ldots\rangle_{\eta<\rho}$, such that $|\phi(x_0, \ldots, x_\iota, \ldots, a_0, \ldots, a_\eta, \ldots)| \in \mathrm{R}(\mathfrak{A})$ iff there is a ρ-tuple of B, $\langle b_0, \ldots, b_\eta, \ldots\rangle_{\eta<\rho}$, such that $|\phi(x_0, \ldots, x_\iota, \ldots, b_0, \ldots, b_\eta, \ldots)| \in \mathrm{R}(\mathfrak{B})$.*

(ii) Replace the phrase "there is" by "for every" in (i).

For the rest of this section we assume that $\mathcal{L} \in \mathcal{M}$.

Theorem 1. *$\mathfrak{A}\ \mathcal{L}\text{-}\equiv \mathfrak{B}$ iff for every pair of corresponding recognizable algebras,* $\mathrm{R}(\mathfrak{A})$ *and* $\mathrm{R}(\mathfrak{B})$, $\mathrm{R}(\mathfrak{A})\ \mathcal{L}\text{-}\equiv \mathrm{R}(\mathfrak{B})$.

Proof: ($\Longrightarrow$) If $\mathfrak{A}\ \mathcal{L}\text{-}\equiv \mathfrak{B}$ and R is given, we translate each sentence J of the language of R to a set of sentences of $\mathcal{L}$ whose truth or falsity determines the truth or falsity of J. The translation is done by induction on the formulas of $\mathcal{L}$. We use y with subscripts for the variables in J and assume that each such variable is quantified only once. We do the case where $\mathrm{T}(V^\gamma) \neq \mathrm{E}$; if $\mathrm{T}(V^\gamma) = \mathrm{E}$ the proof goes through with a few modifications.

Denote by $\mathcal{L}^+$ the language $\mathcal{L}$ with the symbols Y_η added to it. Now the terms of the language of R are translated to terms of $\mathcal{L}^+$ by induction. A variable y_η is translated to $Y_\eta = Y_\eta(x_0, \ldots, x_\iota, \ldots)_{\iota<\gamma}$. A constant c_η in the language of R stands for an equivalence class of formulas $|\phi|$, where $\phi = \phi(x_0, \ldots, x_\iota, \ldots)_{\iota<\gamma}$ is a formula of $\mathcal{L}$. Then c_η is translated to $\mathrm{C}_\eta = \phi$. In the induction step if $t_1, \ldots, t_k$ are terms translated to $T_1, \ldots, T_k$ respectively, and if F is a k-ary operation of R, then $F(t_1, \ldots, t_k)$ is translated to $F(T_1, \ldots, T_k)$.

If t_i and t_j are translated to T_i and T_j respectively, then $t_i = t_j$ is translated to $\left(\bigforall_{\iota<\gamma} x_\iota\right)(T_i \leftrightarrow T_j)$. If J_1 is translated to J_1^* then $\sim J_1$ is translated to $\sim J_1^*$. If each J_ι, $\iota < \tau$, is translated to J_ι^*, then $\bigwedge_{\iota<\tau} J_\iota$ is translated to $\bigwedge_{\iota<\tau} J^*_\iota$. Now suppose that $J(y_0, \ldots, y_\zeta, \ldots)$ is translated to $J^*(Y_0, \ldots, Y_\zeta, \ldots)$. Then $\left(\exists_{\zeta<\rho} y_\zeta\right) J(y_0, \ldots, y_\zeta, \ldots)$ is translated to

$$\left(\exists_{\lambda<\eta_0} u_\lambda^0\right) \ldots \left(\exists_{\lambda<\eta_\zeta} u_\lambda^\zeta\right) \ldots_{\zeta<\rho} \left\{ \bigwedge_{\zeta<\rho} T(Y_\zeta(x_0, \ldots, x_\iota, \ldots, u_0^\zeta, \ldots, u_\lambda^\zeta, \ldots)) \wedge J^*(Y_0, \ldots, Y_\zeta, \ldots) \right\}.$$

Eventually J is translated to $J^*(Y_0, \ldots, Y_\zeta, \ldots)_{\zeta<\rho}$ where the Y_ζ are obtained from the bound variables of J. Now we treat each Y_ζ as a syntactical variable ranging over the formulas of $\mathcal{L}$ which have at least x_ι, $\iota < \gamma$, as free variables. In $J^*(Y_0, \ldots, Y_\zeta, \ldots)_{\zeta<\rho}$ substitute simultaneously a sequence of ρ formulas for the Y_ζ. This way for each sequence of ρ allowable formulas, say $\psi_0, \ldots, \psi_\zeta, \ldots, (\zeta < \rho)$ we obtain $J^*(\psi_0, \ldots, \psi_\zeta, \ldots)_{\zeta<\rho}$, a sentence of $\mathcal{L}$.

Let Q_ζ be the quantifier applied to y_ζ in J. By the lemma, $\mathrm{R}(\mathfrak{A}) \models J$ iff Q_0' allowable $\psi_0, \ldots, Q_\zeta'$ allowable $\psi_\zeta, \ldots, (\zeta < \rho)$ such that $\mathfrak{A} \models J^*(\psi_0, \ldots, \psi_\zeta, \ldots)_{\zeta<\rho}$. We use Q' to abbreviate the appropriate phrase "there exists an" or "for all". Similarly $\mathrm{R}(\mathfrak{B}) \models J$ under the same conditions. Since $\mathfrak{A}\ \mathcal{L}\text{-}\equiv \mathfrak{B}$, $\mathfrak{A} \models J^*(\psi_0, \ldots, \psi_\zeta, \ldots)$ iff $\mathfrak{B} \models J^*(\psi_0, \ldots, \psi_\zeta, \ldots)$. So $\mathrm{R}(\mathfrak{A})\ \mathcal{L}\text{-}\equiv \mathrm{R}(\mathfrak{B})$.

($\Leftarrow$) If $\mathfrak{A}\ \mathcal{L}\text{-}\not\equiv \mathfrak{B}$ then there is a sentence χ of $\mathcal{L}$ such that $\mathfrak{A} \models \chi$ and $\mathfrak{B} \models \sim\chi$. Let $\mathrm{R} = \langle \|\chi \vee (\forall x)\ (V^1(x) \leftrightarrow x = x)\|, \vee, \wedge \rangle$. Then $\mathrm{R}(\mathfrak{B})$ is the trivial lattice of one element, while $\mathrm{R}(\mathfrak{A})$ has at least two elements: $|x = x|$ and $|x \neq x|$. Thus $\mathrm{R}(\mathfrak{A})\ \mathcal{L}\text{-}\not\equiv \mathrm{R}(\mathfrak{B})$.

The next theorem is an improvement over Proposition 1 (for the case $\mathcal{L} \in \mathcal{M}$).

Theorem 2. *$\mathfrak{A}$ is $\mathcal{L}$-embeddable in $\mathfrak{B}$ iff there is an embedding $d : A \to B$ such that for every pair of corresponding recognizable algebras $\mathrm{R}(\mathfrak{A})$ and $\mathrm{R}(\mathfrak{B})$, the map $|d| : |\phi|_{\mathfrak{A}} \to |d\phi|_{\mathfrak{B}}$ is an $\mathcal{L}$-embedding of $\mathrm{R}(\mathfrak{A})$ into $\mathrm{R}(\mathfrak{B})$.*

Proof: ($\Rightarrow$) If $\mathfrak{A}$ is $\mathcal{L}$-embeddable in $\mathfrak{B}$ then the $|d|$ of Proposition 1 is an embedding. To show that $|d|$ is an $\mathcal{L}$-embedding we repeat the procedure used in the ($\Rightarrow$)-proof of Theorem 1. However now J may contain parameters of $\mathrm{R}(\mathfrak{A})$. Such a parameter p stands for an equivalence class of formulas $|\phi|$, where $\phi = \phi(x_0, \ldots, x_\iota, \ldots, a_0, \ldots, a_\eta, \ldots)$ is a formula of $\mathcal{L}(A)$. Then when terms are translated, p is translated to $P = \phi$. The rest of the proof is done as in Theorem 1. However now $J^*(\psi_0, \ldots, \psi_\zeta, \ldots)$ is a sentence of $\mathcal{L}(A)$. Since d is an $\mathcal{L}$-embedding, $\mathfrak{A} \models J^*(\psi_0, \ldots, \psi_\zeta, \ldots)$ iff $\mathfrak{B} \models dJ^*(\psi_0, \ldots, \psi_\zeta, \ldots)$. So $\mathrm{R}(\mathfrak{A}) \models J$ iff $\mathrm{R}(\mathfrak{B}) \models |d|J$. This shows that $|d|$ is an $\mathcal{L}$-embedding.

($\Leftarrow$) If $\mathfrak{A}$ is not $\mathcal{L}$-embeddable in $\mathfrak{B}$ repeat the procedure used in the ($\Leftarrow$)-proof of Proposition 1. Since $|d|$ is not an embedding, it is not an $\mathcal{L}$-embedding.

It follows that for every recognizable algebra R, the map $r : \mathfrak{A} \to \mathrm{R}(\mathfrak{A})$ is a functor from and to a category of models (the maps being $\mathcal{L}$-embeddings).

3. Further Results. First we consider an $\mathcal{L}$-chain of structures, i.e. a chain $\langle \mathfrak{A}_\zeta : \zeta < \eta \rangle$ for which $\mathfrak{A}_\rho \mathcal{L} \text{-} \prec \mathfrak{A}_\sigma$ for $\rho < \sigma < \eta$. We write $\mathfrak{A} = \bigcup_{\zeta<\eta} \mathfrak{A}_\zeta$. When $\mathcal{L}$ is an infinitary language, $\mathcal{L} = \mathcal{L}_{\alpha\beta}$, we need an analog of the Union of chains theorem (see [1] pages 79-80). This is stated as the next lemma, and its proof is an extension of the proof of the Union of chains theorem.

Lemma 2. *Let* $\langle \mathfrak{A}_\zeta : \zeta < \eta \rangle$ *be an* $\mathcal{L}$*-chain where* $\mathcal{L} = \mathcal{L}_{\alpha\beta}$. *If* $\mathrm{cf}(\eta) \geq \beta$, *then for every* $\zeta < \eta$, $\mathfrak{A}_\zeta \mathcal{L} \text{-} \prec \mathfrak{A}$.

Proposition 2. *Let* $\langle \mathfrak{A}_\zeta : \zeta < \eta \rangle$ *be an* $\mathcal{L}$*-chain and let* $\mathrm{cf}(\eta) \geq \alpha$. *Then for every recognizable algebra* R, $\mathsf{R}(\mathfrak{A}) = \varinjlim(\mathsf{R}(\mathfrak{A}_\zeta))$. (For the definition of $\varinjlim$ see [2], pages 128-130.)

Proof: We define homomorphisms $h_{\rho\sigma}$ of $\mathsf{R}(\mathfrak{A}_\rho)$ into $\mathsf{R}(\mathfrak{A}_\sigma)$ for all $\rho \leq \sigma < \eta$ as follows, $h_{\rho\sigma}: |\phi|_{\mathfrak{A}_\rho} \to |\phi|_{\mathfrak{A}_\sigma}$ for every $|\phi| \in \mathsf{R}(\mathfrak{A}_\rho)$. To show that $\mathsf{R}(\mathfrak{A})$ is the $\varinjlim$ we apply Lemma 2. Thus we need the hypothesis that $\mathrm{cf}(\eta) \geq \beta$. We must also make sure that every formula of $\mathcal{L}(A)$ is also a formula of $\mathcal{L}(A_\zeta)$ for some $\zeta < \eta$. The hypothesis that $\mathrm{cf}(\eta) \geq \alpha$ takes care of this problem.

Next we consider recognizable algebras of relations. Since $\mathcal{L}^2$ contains variables of degree γ, $\mathcal{L}^2(A)$ contains a symbol for each γ-ary relation of A.

Definition 3. A recognizable algebra of relations is defined as a recognizable algebra of formulas with $\mathcal{L}^2$ substituted for $\mathcal{L}$ in Definition 1.

We use $\mathfrak{P}$ to stand for a recognizable algebra of relations. Just as in Definition 2 we may define corresponding recognizable algebras of relations.

Theorem 3. $\mathfrak{A} \mathcal{L}^2 \text{-} \equiv \mathfrak{B}$ *iff for every pair of corresponding recognizable algebras of relations*, $\mathsf{P}(\mathfrak{A})$ *and* $\mathsf{P}(\mathfrak{B})$, $\mathsf{P}(\mathfrak{A}) \mathcal{L} \text{-} \equiv \mathsf{P}(\mathfrak{B})$.

Proof: We use a translation which is similar to the one used in the proof of Theorem 1. Note that the proof in this case works for $\mathcal{L} \in \mathcal{N}$.

The next result holds if $\mathcal{L} = \mathcal{L}_{\omega\omega}$ and $\mathsf{T}(V^\gamma) \neq \mathsf{E}$.

Proposition 3. $\mathfrak{A}$ *is finite iff every recognizable algebra* $\mathsf{R}(\mathfrak{A})$ *is finite.*

Acknowledgments. This paper was written while the author held a postdoctoral fellowship at the University of Florida. He wishes to thank both Professor A. R. Bednarek for his kind encouragement and Professors G. E. Reyes, P. S. Schnare, and M. Yasuhara for helpful comments concerning the topic of this paper.

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