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A GENERALIZATION OF COMBINATORIAL OPERATORS

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Let $\varepsilon = (0, 1, 2, \ldots)$. We mean by *number*, an element of ε; by *set*, a subset of ε. Let V be the class of all sets, Q the class of all finite sets. Combinatorial operators, introduced by John Myhill [4], are certain maps from V into V such that sets in Q are mapped into Q. We are concerned with what happens if the operator is allowed to map finite sets to infinite sets. If we require certain uniformity conditions, many of the properties of combinatorial operators still hold. Some of these operators, called uniform semicombinatorial operators, are inherited in a natural way from recursive combinatorial operators of several variables. The main result of this paper* is the existence of a family of uniform semicombinatorial operators none of which can be obtained in this way.

We will use the following notations: For $n \geq 1$, ε^n, V^n, Q^n, and so on, denote the ordinary Cartesian products. If $\alpha \in V^n$, $1 \leq t \leq n$, α_t denotes the t'th component of α, and similarly for ε^n. When dealing with ordered k-tuples of sets, $\subseteq$ and $\sim$ are understood coordinate-wise; however, $\gamma \subset \alpha$ means $\gamma \subseteq \alpha$ and $\gamma \neq \alpha$. If f is a function, $f(x)$ and f_x are used interchangeably, and δf denotes the domain of f. We will use the following Gödel numbering for Q: $\rho_0 = \emptyset$, and for $n \geq 1$,

$$\rho_n = (i_1, \ldots, i_k), \text{ where } n = 2^{i(1)} + \ldots + 2^{i(k)},$$

and the i_j are distinct. Denote $\operatorname{card} \rho_n$ by r_n. We assume knowledge of the definitions and elementary properties of recursive equivalence types, as given in [3], Chapter II or [2]. We denote the collection of isolated sets by $\mathcal{I}$.

1 *Combinatorial and semicombinatorial operators*

An *operator* of n variables ($n \geq 1$) is a mapping from a subclass of V^n containing Q^n into V. For any operator Φ, we write

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$$\bigcup(\text{Range } \Phi) = \Phi^{\varepsilon}.$$

A *combinatorial operator* (CO) Φ of n variables is an operator with domain V_n satisfying:

(i) $\alpha \in Q^n \Longrightarrow \Phi(\alpha) \in Q$,
(ii) $(\alpha, \beta \in Q^n$ and $\alpha \sim \beta) \Longrightarrow \Phi(\alpha) \sim \Phi(\beta)$,
(iii) there is a map, denoted by Φ^{-1}, from Φ^{ε} into Q^n such that for $x \in \Phi^{\varepsilon}$, $\alpha \in V^n$

$$x \in \Phi(\alpha) \Longleftrightarrow \Phi^{-1}(x) \subseteq \alpha.$$

The map Φ^{-1} given by (iii) is unique and is called the *quasi-inverse* for Φ. Any operator Φ defined on V^n which has a quasi-inverse satisfies

$$\Phi(\alpha) = \bigcup\{\Phi(\gamma)\colon \gamma \in Q^n \text{ and } \gamma \subseteq \alpha\}, \text{ for } \alpha \in V^n.$$

It is useful to define the *associated operator* Φ_0 from Q^n into V, by

$$\Phi_0(\alpha) = \Phi(\alpha) - \bigcup_{\gamma \subset \alpha} \Phi(\gamma), \text{ for } \alpha \in Q^n.$$

A CO Φ of n variables is *recursive* if the function $g\colon \varepsilon^n \to \varepsilon$ given by

$$\rho(g(i)) = \Phi[\rho(i_1), \ldots, \rho(i_n)],$$

is recursive. If Φ is any operator of n variables satisfying (i) and (ii), the *induced function of* Φ, denoted by f_Φ, is the function from ε^n into ε given by

$$f_\Phi(i) = \text{card } \Phi[\nu(i_1), \ldots, \nu(i_n)],$$

where $\nu_0 = \emptyset$, and for $j \geqslant 1$, $\nu_j = (0, 1, \ldots, j-1)$. A function $f\colon \varepsilon^n \to \varepsilon$ is called a *combinatorial function* if $f = f_\Phi$, for some CO Φ of n variables.

Definition: A *semicombinatorial operator* (SCO) Ψ is an operator of one variable with domain V satisfying:

(i′) $(\alpha, \beta \in Q$ and $\alpha \sim \beta) \Longrightarrow \Psi(\alpha) \simeq \Psi(\beta)$.
(ii′) Ψ has a quasi-inverse.

As in the case of a CO it can be proved that every SCO has a unique quasi-inverse. An SCO is *isolic* if it maps Q into $\mathcal{J}$. We define the *induced function* F_Ψ of an SCO Ψ, analogously as for a CO, as follows:

$$F_\Psi(i) = \text{Req } \Psi(\nu_i), \text{ for } i \in \varepsilon.$$

Clearly, the family of (isolic) SCOs which map Q into Q coincides with the family of COs of one variable.

Definition: An SCO Ψ is *uniform* if

(iii′) the function $g\colon \Psi^{\varepsilon} \to \varepsilon$, given by $\rho_{g(x)} = \Psi^{-1}(x)$, has a partial recursive extension,
(iv′) there is a partial recursive function of three variables $f(a, b, x)$, such that if $r_a = r_b$, then f_{ab} is a one-one function of x, with

$$\Psi(\rho_a) \subseteq \delta f_{ab} \text{ and } f_{ab}[\Psi(\rho_a)] = \Psi(\rho_b).$$

Each such function $f(a, b, x)$ is called a *u-function* for Ψ. A function from ε into Ω is called *semicombinatorial* (SC) if it is induced by some uniform SCO. An SC function is *isolic* if it maps ε into Λ, i.e., if it is induced by a uniform isolic SCO. Analogues of most of the propositions of [3], Chapter I, are valid for uniform SCOs and SC functions. These analogues will be denoted by an asterisk. The following four examples of such propositions are among the most important.

P8* *Let Ψ be a uniform* SCO, *and Ψ_0 its associated operator. Then*

$$(\alpha,\ \beta \in Q \text{ and } \alpha \neq \beta) \Longrightarrow \Psi_0(\alpha) \mid \Psi_0(\beta).$$

P11* *Let Ψ be a uniform isolic* SCO. *Then Ψ_0 maps Q into $\mathcal{J}$, and*

$$(\alpha,\ \beta \in Q \text{ and } \alpha \sim \beta) \Longrightarrow \Psi_0(\alpha) \simeq \Psi_0(\beta).$$

P19* *For each function F from ε into Λ, there is a unique sequence $\{C_i\}$ of isolic integers such that for all n*

$$F(n) = \sum_{i=0}^{n} C_i \binom{n}{i}.$$

P20* *Let F: $\varepsilon \to \Lambda$, and let $\{C_i\}$ be the unique sequence given by* P19*. *Then F is* SC *iff $C_i \in \Lambda$ for all i.*

By P20*, a function from ε to ε is combinatorial iff it is SC. In addition, it can be shown that the family of isolic SC functions includes all constant functions F: $\varepsilon \to \Lambda$ and is closed under addition and multiplication. Finally, the following relations from [3], p. 51, are valid for a uniform isolic SCO Ψ, for $\alpha,\ \beta \in V$:

(1) $\alpha \in \mathcal{J} \Longrightarrow \Psi(\alpha) \in \mathcal{J}$,
(2) $\alpha \simeq \beta \Longrightarrow \Psi(\alpha) \simeq \Psi(\beta)$.

Thus a uniform isolic SCO induces a function from Ω to Ω which maps Λ into Λ. Relation (1) follows from (iii′); (2) may be proved using the following lemma, which is verified in [5], pp. 47-53:

Lemma *If Ψ is a uniform isolic* SCO, *then the associated operator Ψ_0 has a u-function.*

2 *Inherited semicombinatorial operators* We will need the following properties of Φ_0, where Φ is a CO of n variables; they are the respective n-variable analogues of P8, P10, and P11 of [3].

(3) $(\alpha,\ \beta \in Q^n \text{ and } \alpha \neq \beta) \Longrightarrow \Phi_0(\alpha) \cap \Phi_0(\beta) = \emptyset$,
(4) $\Phi(\alpha) = \bigcup\{\Phi_0(\gamma)\colon \gamma \in Q^n \text{ and } \gamma \subseteq \alpha\}$, *for* $\alpha \in V^n$,
(5) $(\alpha,\ \beta \in Q^n \text{ and } \alpha \sim \beta) \Longrightarrow \Phi_0(\alpha) \sim \Phi_0(\beta)$.

Proposition *Let $k \geqslant 1$, and let Φ be a recursive* CO *of $k+1$ variables. Then for any $\mu \in V^k$, the operator Ψ given by*

(6) $\Psi(\alpha) = \Phi(\mu, \alpha)$, *for* $\alpha \in V$,

is a uniform SCO. *Furthermore, if $\mu \in \mathcal{J}^k$, then Ψ is isolic.*

Proof: We first verify conditions (i′)-(iv′) for Ψ. Condition (i′) holds by (ii) of [3], p. 52. Conditions (ii′) and (iii′) follow from the fact that the projection of Φ^{-1} onto the $(k+1)$st coordinate is a quasi-inverse for Ψ. Concerning (iv′): For all a, b such that $r_a = r_b$, let p_{ab} be the natural bijection from ρ_a onto ρ_b. From the recursiveness of Φ and (3), (4), (5), it follows that there is a partial recursive function of three variables $f(a, b, x)$, such that if $r_a = r_b$, then f_{ab} is a one-one function from $\Phi(\varepsilon^k, \rho_a)$ onto $\Phi(\varepsilon^k, \rho_b)$, and for $\tau \in Q^k$, $\alpha \subseteq \rho_a$,

$$f_{ab}[\Phi_0(\tau, \alpha)] = \Phi_0(\tau, p_{ab}(\alpha)).$$

Therefore by (4),

$$f_{ab}[\Phi(\mu, \rho_a)] = \Phi(\mu, \rho_b),$$

i.e.,

$$f_{ab}[\Psi(\rho_a)] = \Psi(\rho_b).$$

The above shows that Ψ is a uniform SCO. Finally, if $\mu \in \mathcal{I}^k$ and $\alpha \in Q$, then $\Phi(\mu, \alpha) \in \mathcal{I}$ by (iii) of [3], p. 52, and hence Ψ is isolic.

Definition: A uniform SCO Ψ is *inherited* if Ψ is a recursive CO or if, for some $k \geqslant 1$, there are a recursive CO Φ of $k+1$ variables and a k-tuple of sets μ such that (6) holds. An SC function is *inherited* if it is induced by some inherited SCO.

The family of isolic, inherited SC functions includes all constant functions from ε into Λ, and is closed under addition and multiplication. The non-recursive combinatorial functions are isolic SC functions (by P20*) but they are not inherited. The theorem below asserts the existence of a family of infinite-valued isolic SC functions which are not inherited.

We will need the following definitions and notation. A set is *indecomposable* if it cannot be expressed as the union of two infinite separable sets. An RET X is *indecomposable* if it cannot be expressed as the sum of two infinite RETs, i.e., if every set in X is indecomposable. Clearly, all indecomposable RETs are isols. By [2], Theorem 43(b), there are $2^{\aleph_0}$ infinite indecomposable isols. Denote the collection of infinite indecomposable sets by $\mathcal{I}_0$, and the collection of infinite indecomposable isols by Λ_0. The following conditional is obvious but useful:

$$X \in \Lambda_0 \Longrightarrow (\forall n \in \varepsilon)(X - n \in \Lambda_0).$$

For $\alpha \in V$, $n \in \varepsilon$, let

$$[\alpha, n] = \{x\colon \rho_x \subseteq \alpha \text{ and } r_x = n\}.$$

Clearly,

$$\alpha \simeq \beta \Longrightarrow [\alpha, n] \simeq [\beta, n],$$

and we define for $A \in \Omega$,

$$[A, n] = \mathrm{Req}[\alpha, n], \text{ where } \alpha \text{ is any set in } A.$$

Theorem *Let* $X \in \Lambda_0$. *Define* $C_i = X - i$, *for* $i \in \varepsilon$, *and*

$$F(n) = \sum_{i=0}^{n} C_i \binom{n}{i}, \text{ for } n \in \varepsilon.$$

Then F *is an isolic* SC *function which is not inherited.*

Proof: The function $F(n)$ is SC by P20*. Suppose F is inherited. Let Ψ be an inherited SCO which induces F. Since F attains infinite values, Ψ is not a recursive CO. Therefore for some $k \geq 1$, there are a recursive CO Φ of $k + 1$ variables and a k-tuple of sets μ, such that (6) holds for all α. We may assume without loss of generality that all the sets $\mu_m (1 \leq m \leq k)$ are infinite. For suppose not, say μ_k is finite. Then the operator Φ^* of k variables given by

$$\Phi^*(\alpha_1, \ldots, \alpha_k) = \Phi(\alpha_1, \ldots, \alpha_{k-1}, \mu_k, \alpha_k),$$

is a recursive CO such that for all $\alpha \in V$,

$$\Psi(\alpha) = \Phi^*(\mu_1, \ldots, \mu_{k-1}, \alpha).$$

Denote for $i \in \varepsilon^k$, $n \in \varepsilon$,

$$\text{(7)} \qquad c(i, n) = f_{\Phi_0}(i_1, \ldots, i_k, n) \in \varepsilon.$$

(This is well-defined by (5).) Denote the ordered k-tuple of zeros by 0_k. We will need the following two lemmas.

Lemma L1 *For* $\alpha \in Q$,

$$\Psi_0(\alpha) = \bigcup \{\Phi_0(\tau, \alpha) \colon \tau \in Q^k \text{ and } \tau \subseteq \mu\}.$$

Lemma L2 *Suppose* $\Psi_0(\alpha) \in \mathcal{I}_0$ *for some* $\alpha \in Q$. *Then there is a number* t, $1 \leq t \leq k$, *such that* $\mu_t \in \mathcal{I}$ *and*

$$\text{Req } \mu_t = \text{Req } \Psi_0(\alpha) - c(0_k, n),$$

where $n = \text{card } \alpha$.

Lemma L1 is a direct consequence of the definition of Ψ_0 and (4); L2 will be verified later. The proof of the theorem can be completed using L2 as follows: For any $n \in \varepsilon$, it can be shown that

$$\text{(8)} \qquad \text{Req } \Psi_0(\nu_n) = C_n = X - n,$$

using a proof similar to that in [3], p. 20. Let $s(0) = 0$, and for $j \geq 0$, let

$$s(j + 1) = s(j) + c(0_k, s(j)) + 1.$$

By (8) and L2, there is for each $j \in \varepsilon$, a set $\mu_{t(j)} \in (\mu_1, \ldots, \mu_k)$ such that $\mu_{t(j)} \in \mathcal{I}$ and

$$\text{Req } \mu_{t(j)} = \text{Req } \Psi_0(\nu_{s(j)}) - c(0_k, s(j)).$$

It follows by (8) and the definition of the sequence $s(j)$ that for each $j \in \varepsilon$, $\text{Req } \mu_{t(j)} > \text{Req } \mu_{t(j+1)}$. Therefore, the sets $\mu_{t(j)}$ are all distinct, which is a contradiction. This proves that F cannot be inherited.

We now prove L2. Assume $\alpha \in Q$ is such that $\Psi_0(\alpha) \in \mathcal{J}_0$, and let $\eta = \Psi_0(\alpha)$. For $\tau \in V^k$ and $i \in \varepsilon^k$, we will say τ is of *type* i, if $\tau \in Q^k$, $\tau \subseteq \mu$ and for $1 \leq m \leq k$, $\operatorname{card} \tau_m = i_m$. Define for $i \in \varepsilon^k$,

$$\eta(i) = \bigcup \{\Phi_0(\tau, \alpha) \colon \tau \text{ is of type } i\}. \tag{9}$$

By L1,

$$\eta = \bigcup_{i \in \varepsilon^k} \eta(i). \tag{10}$$

Note that the unions in (9) and (10) are disjoint by (3). Denote for $1 \leq m \leq k$, $U_m = \operatorname{Req} \mu_m$. We claim:

(a) For $i \in \varepsilon^k$, $\operatorname{Req} \eta(i) = [U_1, i_1] \cdot \ldots \cdot [U_k, i_k] \cdot c(i, n)$;
(b) For $i \in \varepsilon^k - (0_k)$, the set $\eta(i)$ is infinite or empty;
(c) There is a unique $i \in \varepsilon^k - (0_k)$ such that $\eta(i)$ is infinite;
(d) $\operatorname{Req} \eta = c(0_k, n) + \operatorname{Req} \eta(i)$, where i is the unique k-tuple given by (c);
(e) If i is as in (d), then for some t, $1 \leq t \leq k$, $\operatorname{Req} \eta(i) = U_t$.

Statements (d) and (e) together imply the desired result. Statement (a) is a direct consequence of the definitions and relations (7) and (9); (b) follows from (a) and the fact that the RETs U_m are infinite. Concerning (c): Recall that η is an infinite indecomposable set. We have $\eta(i)$ is infinite for at least one $i \neq 0_k$, since otherwise (b), (9), and (10) imply

$$\eta = \eta(0_k) = \Phi_0(\phi_k, \alpha)$$

(where ϕ_k denotes the k-tuple of empty sets), and hence $\eta \in Q$. Also, for any $i \in \varepsilon^k$,

$$\eta(i) \,\Big|\, \bigcup_{j \neq i} \eta(j),$$

since Φ is recursive. Therefore, by (10), at most one of the sets $\eta(i)$ is infinite. This proves (c). Statement (d) follows from (b), (c), (7), and (10). Finally, (e) can be proved from (a) using the fact that for A infinite, $n \geq 2$, $[A, n]$ is decomposable. This proves L2 and hence the Theorem.

Remark: According to a suggestion by Erik Ellentuck, the result of the Theorem also holds if X is any infinite isol which is multiple-free, i.e., such that

$$2Y \leq X \Rightarrow Y \in \varepsilon.$$

Since there are infinite multiple-free regressive isols ([1]), this proves the existence of a non-inherited isolic SC function with range a subset of the class of infinite regressive isols.

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