44
Notre Dame Journal of Formal Logic
Volume XVII, Number 1, January 1976
NDJFAM

ALGEBRAIC SEMANTICS FOR $S2^0$ AND NECESSITATED EXTENSIONS

R. ROUTLEY and H. MONTGOMERY

Algebraic techniques are used to show that Feys' system $S2^0$ (*cf.* [1]) and certain necessitated extensions of $S2^0$, such as Lewis' systems S2 and S3, have the finite model property, and accordingly are decidable. Representation theorems are then used to establish set-theoretical semantics for the modal systems studied. Where the results obtained are not new they improve on earlier results (such as those of Lemmon in [3]) in two respects; first they provide direct algebraic treatments of the systems, and second they furnish better semantical results (see the discussion of theorem J for S2). The techniques used however follow those of McKinsey (in [4]) and Lemmon (in [2] and [3]). Since it is now known that these techniques do not work for all necessitated extensions of $S2^0$, a somewhat piecemeal approach is inevitable. Weak results are also obtained for Feys' system $S1^0$ and Lewis' system S1 (for details of these systems see [1]).

The sentential systems studied are of interest not so much as systems containing a viable necessity operator '$\Box$', but as intensional logics which axiomatise epistemic or other operators. For instance $S2^0$ can be interpreted as an epistemic logic such that '$\Box$' reads 'it is believed reasonably that', and S2 as an epistemic logic where '$\Box$' reads 'it is known that'. The set-theoretical semantics established are however independent of these epistemic interpretations.

The basic system examined, Feys' $S2^0$, has as postulates:

T1. $A \,\&\, B \prec A$
T2. $A \,\&\, B \prec B \,\&\, A$
T3. $(A \,\&\, B) \,\&\, C \prec .A \,\&\, (B \,\&\, C)$
T4. $A \prec A \,\&\, A$
T5. $A \prec B \,\&\, B \prec C \prec .A \prec C$
T6. $\Diamond(A \,\&\, B) \prec \Diamond A$

Strict Detachment (**SD**): $\vdash A, \vdash A \prec B \rightarrow \vdash B$
Adjunction (**A**): $\vdash A, \vdash B \rightarrow \vdash A \,\&\, B$
Substitutivity of Strict Equivalents (**SSE**): $\vdash A \leftrightharpoons B, \vdash C(A) \rightarrow \vdash C(B)$

Received August 11, 1969

The connectives '&', '$\sim$', and '$\Diamond$' are taken as primitive, and further connectives '$\vee$', '$\supset$', '$\equiv$', '$\Box$', '$\prec$', '$\prec\succ$', '$\Diamond^n$', and '$\Box^n$' are defined as usual. Also

$$\nabla A =_{Df} \Diamond A \;\&\; \Diamond{\sim}A;\ \triangle A =_{Df} \Box A \vee \Box{\sim}A;\ \mathsf{T} =_{Df} p \supset p.$$

Numerals preceded by '*F*' refer to items designated by the same numerals in [1]. The postulates of Feys' system S1^0 are obtained from those of S2^0 by deleting T6. A necessitated extension of S2^0 (S1^0) is an extension of S2^0 (S1^0) obtained by adding one or more axioms of the form $\Box C$.

Theorem 1 $\vdash_{\mathsf{L}} A$ *iff* $\vdash_{\mathsf{L}} \Box\mathsf{T} \prec A$, *where* L *is any system obtained from* S2^0 *by adding axioms of the form* $\Box C$ *for some* C.

Proof: Since $p \prec p$ is a theorem of all these systems, by F31.11, if $\vdash_{\mathsf{L}} p \prec p \prec A$ then $\vdash_{\mathsf{L}} A$. The proof of the converse is by induction over the length of the proof of A.

1. For every axiom A there is some B such that $A \prec\succ \Box B$. By F43.1, $\vdash_{\mathsf{L}} \Box B \prec . \mathsf{T} \prec B$. Hence since $\vdash_{\mathsf{L}} A$, $\vdash_{\mathsf{L}} \mathsf{T} \prec B$. Hence by Becker's rule F46.1, $\vdash_{\mathsf{L}} \Box\mathsf{T} \prec \Box B$, and by substitutivity $\vdash_{\mathsf{L}} \Box\mathsf{T} \prec A$.
2. For the rule of Adjunction F42.21 applies. For Strict Detachment apply F30.15 to $\vdash_{\mathsf{L}} \Box\mathsf{T} \prec A$ and $\vdash_{\mathsf{L}} A \prec B$ to give $\vdash_{\mathsf{L}} \Box\mathsf{T} \prec B$ whence $\vdash_{\mathsf{L}} B$. Finally if $\vdash_{\mathsf{L}} B$ follows from $\vdash_{\mathsf{L}} A$ by substitutivity of strict equivalents, the same substitution (after a change of variables where necessary) yields $\vdash_{\mathsf{L}} \Box\mathsf{T} \prec B$ from $\vdash_{\mathsf{L}} \Box\mathsf{T} \prec A$.

A similar result holds for extensions of S1^0.

Theorem 2 $\vdash_{\mathsf{L}} A$ *iff* $\vdash_{\mathsf{L}} \Box\mathsf{T} \prec A$, *where* L *is any system obtained from* S1^0 *by adding axioms of the form* $\Box C$ *for some* C.

Proof differs from that of Theorem 1 only at the following points:

1. When A is an axiom since $\vdash_{\mathsf{L}} A$, $\vdash_{\mathsf{L}} \mathsf{T} \prec B$ by F35.41. Also since $\vdash_{\mathsf{L}} \Box\mathsf{T}$, $\vdash_{\mathsf{L}} B \prec \mathsf{T}$ by F35.41. Hence by **SSE** $\vdash_{\mathsf{L}} \Box\mathsf{T} \prec \Box B$.
2. For the rule of Adjunction T-theorem F35.22 can be applied.

Definitions (*cf.* [2], [3], and [4]):

$$\begin{array}{rcl} \mathsf{N}x =_{Df} -\mathsf{P}\,-x & ; & \mathsf{C}x =_{Df} \mathsf{P}x \cap \mathsf{P}\,-x \\ x \supset y =_{Df} -x \cup y & ; & x \mathbin{)(} y =_{Df} x \supset y \,.\!\cap\!.\, y \supset x \\ x \prec y =_{Df} -\mathsf{P}(x \cap -y) & ; & x \prec\succ y =_{Df} x \prec y \,.\!\cap\!.\, y \prec x \\ 1x =_{Df} x \cup -x & ; & 0x =_{Df} -1x \end{array}$$

Since $1x \equiv 1y$ for any x and y, the subscripts will as usual be omitted.

$$x - y =_{Df} x \cap -y$$
$$x \leqslant y =_{Df} x \cap y \equiv x;\ y \geqslant x =_{Df} x \leqslant y$$

Strict identity, symbolised '$\equiv$', should be distinguished from extensional identity, symbolised '$=$'. These identity relations are explained in [6], [7], and [8]. The salient point here is that strict identities may be intersubstituted in modal sentence contexts, but extensional identities may only be intersubstituted in extensional sentence contexts and not in general when they are within the scope of a modal operator such as 'P'. The distinction between strict and extensional identity will be exploited in a subsequent paper, where semantics for systems obtained by adding merely contingent axioms to $S2^0$ are discussed. For examples of such systems see [5].

Definition: A structure $\mathfrak{M} = \langle M, \cap, \cup, -, \mathsf{P}\rangle$ is a *mac algebra* iff M is a set of elements, closed under operations $\cap, \cup, -, \mathsf{P}$, such that

(i) $\langle M, \cap, \cup, -\rangle$ is a non-degenerate Boolean algebra (with strong identity $\equiv$).
(ii) for all $x, y \in M$, $\mathsf{P}(x \cup y) \equiv \mathsf{P}x \cup \mathsf{P}y$, i.e., P is additive over $\cup$.
(iii) $\sim(\mathsf{N}1 \leqslant \mathsf{P}0)$.

Definition: A structure $\mathfrak{M} = \langle M, \cap, \cup, -, \mathsf{P}\rangle$ is a *joined mac algebra* iff it satisfies conditions (i), (ii), and

(iii^1) If $\mathsf{P}x \equiv \mathsf{P}0$ then $x \equiv 0$, for $x \in M$.

A mac algebra is a modal algebra with strong identity which satisfies the requirement that $\mathsf{N}1$ does not precede $\mathsf{P}0$, in other words that $\mathsf{P}0$ is not designated. A joined mac algebra is a modal algebra (in the sense of [2]) which satisfies McKinsey's requirement ([4], p. 120) that if $-\mathsf{P}x$ is designated then $x \equiv 0$. Since $1 \not\equiv 0$ is a non-degenerate Boolean algebra (iii^1) implies

(iii^{11}) $\mathsf{P}0 \not\equiv \mathsf{P}1$

a condition which would suffice in place of (iii) or (iii^1) in some of the main theorems (Theorems A-E) which follow. (iii) is chosen because it provides the weakest condition on Kripke models for $S2^0$.

Theorem 3 *In any modal algebra* $\mathfrak{M} = \langle M, \cap, \cup, -, \mathsf{P}\rangle$

(i) *for* $x, y \in M$, $\mathsf{N}(x \cap y) \equiv \mathsf{N}x \cap \mathsf{N}y$.
(ii) *for* $x, y \in M$, *if* $x \leqslant y$, *then* $\mathsf{N}x \leqslant \mathsf{N}y$ *and* $\mathsf{P}x \leqslant \mathsf{P}y$.
(iii) *for* $x \in M$, $\mathsf{N}1 \equiv \mathsf{N}x$ *iff* $\mathsf{N}1 \leqslant \mathsf{N}x$.
(iv) *for* $x \in M$, $\mathsf{P}x \equiv \mathsf{P}0$ *iff* $\mathsf{P}x \leqslant \mathsf{P}0$.

Definitions: (i) A structure $\mathfrak{M} = \langle M, D, \cap, \cup, -, \mathsf{P}\rangle$ is a *modal matrix* iff M is a set of elements closed under 2-place operations $\cap$ and $\cup$ and 1-place operations $-$ and P and D is a non-null subset of M. The matrix is *proper* iff $D \subset M$.
(ii) A function $\nu_{\mathcal{L}}$: $A \to M$, i.e., from wff of logic $\mathcal{L}$ to elements of M, provides a *valuation* (or assignment of values) *under matrix* $\mathfrak{M}$ and provided these conditions are satisfied:

$$\nu(A \,\&\, B) \equiv \nu(A) \cap \nu(B);\ \nu(\sim A) \equiv -\nu(A);\ \nu(\Diamond A) \equiv \mathsf{P}\nu(A)$$

i.e., provided ν is a homomorphism.
(iii) A matrix $\mathfrak{M}$ *satisfies* wff A iff for every valuation under M, $\nu(A) \in D$; otherwise A is *falsified* by $\mathfrak{M}$. A matrix satisfies a modal system $\mathcal{L}$, is an $\mathcal{L}$*-matrix*, iff it satisfies every theorem of $\mathcal{L}$; and it is *characteristic* for a

modal system $\mathcal{L}$, is an *$\mathcal{L}$-characteristic matrix*, iff it satisfies all and only the theorems of $\mathcal{L}$.

Definition: A modal matrix $\mathfrak{M} \equiv \langle M, D, \cap, \cup, -, \mathsf{P}\rangle$ is *usual* iff

(i) $\mathfrak{M}$ is proper, i.e., $D \subset M$;
(ii) D is a filter of $\mathfrak{M}$, i.e., for $x, y \in D$, $x \cap y \in D$ and for $x \in D$, $y \in M$, $x \cup y \in D$;
(iii) if $x \leftrightarrow y \in D$ then $x \equiv y$.

Lemma 1 *If* $\mathfrak{M}$ *is a modal matrix satisfying requirements* (ii) *and* (iii) *of the previous definition and satisfying* $p \dashv p$, *then:*

(i) $x \equiv y$ *iff* $x \leftrightarrow y \in D$.
(ii) $x \leqslant y$ *iff* $x \dashv y \in D$.

Proof: (i) Since $p \dashv p$ is a theorem $x \dashv x \in D$. Since D is a filter of $\mathfrak{M}$, $x \leftrightarrow x \in D$. Thus, if $x \equiv y$ then $x \leftrightarrow y \in D$. The other half of (i) is immediate from the definition above.

(ii) $x \leqslant y$ iff $x \cap y \equiv x$ iff $x \cap y \leftrightarrow x \in D$, by (i), iff $x \dashv y \in D$, since

$$x \cap y \leftrightarrow x \;.\!\equiv\!.\; x \cap y \dashv x \;.\!\cap\!.\; x \dashv x \cap y.$$
$$\equiv.\; -\mathsf{P}((x \cap y) \cap -x) \;.\!\cap\!.\; -\mathsf{P}(x \cap -(x \cap y)).$$
$$\equiv.\; -\mathsf{P}(x \cap y \cap -x \;.\!\cup\!.\; x \cap -x \;.\!\cup\!.\; x \cap -y).$$
$$\equiv.\; -\mathsf{P}(x \cap -y) \;.\!\equiv\!.\; x \dashv y.$$

The finite model property is first established in detail for S2^0, in Theorems A-E (for S2^0).

Theorem A $\mathfrak{M} = \langle M, D, \cap, \cup, -, \mathsf{P}\rangle$ *is a usual* S2^0*-matrix iff* $\langle M, \cap, \cup, -, \mathsf{P}\rangle$ *is a mac algebra* (*or a joined mac algebra*) *and* $D \equiv \{x\colon \mathsf{N}1 \leqslant x\}$.

Proof: 1. Let $\mathfrak{M} = \langle M, D, \cap, \cup, -, \mathsf{P}\rangle$ be a usual S2^0-matrix. Then

(i) $\langle M, \cap, \cup, -\rangle$ is a Boolean algebra. This is proved as in McKinsey [4] and Lemmon [2].
(ii) $D = \{x\colon \mathsf{N}1 \leqslant x\}$. If $\mathsf{N}1 \leqslant x$, then by Lemma 1, $\mathsf{N}1 \dashv x \in D$. Since $\vdash_{S2^0} \Box(p \supset p)$, $\mathsf{N}1 \in D$. Thus since $\mathfrak{M}$ is usual $x \in D$. Conversely if $x \in D$, apply the derived rule: if $\vdash_{S2^0} A$ then $\vdash_{S2^0} \Box \mathrm{T} \dashv A$. Thereby $\mathsf{N}1 \dashv x \in D$, so by Lemma 1, $\mathsf{N}1 \leqslant x$.
(iii) $\mathsf{P}(x \cup y) \leftrightarrow \mathsf{P}x \cup \mathsf{P}y \in D$ since $\vdash_{S2^0} \Diamond(A \vee B) \leftrightarrow \Diamond A \vee \Diamond B$. Hence by the identity requirement on usualness $\mathsf{P}(x \cup y) \equiv. \mathsf{P}x \cup \mathsf{P}y$.
(iv) To show $\sim(\mathsf{N}1 \leqslant \mathsf{P}0)$ suppose for a reduction, $\mathsf{N}1 \leqslant \mathsf{P}0$. Then $\mathsf{P}0 \in D$. But $-\mathsf{P}0 \equiv \mathsf{N}1 \in D$. Hence since D is a filter $0 \equiv \mathsf{P}0 \cap -\mathsf{P}0 \in D$. Thus for all $x \in M$, $x \in D$, contradicting usualness of the S2^0-matrix.
(iv′) If $\mathsf{P}x \equiv \mathsf{P}0$ then $\mathsf{N}-x \equiv \mathsf{N}1$, so by (ii) $\mathsf{N}-x \in D$. Also, using F43.1, $\mathsf{N}-x \dashv. 1 \dashv -x \in D$. Since $\mathfrak{M}$ is usual, $1 \dashv -x \in D$, and by Lemma 1, $1 \leqslant -x$. Since too $-x \leqslant 1$, $-x \equiv 1$, and thus $x \equiv 0$. Hence too $\mathsf{P}1 \not\equiv \mathsf{P}0$.

By (i)-(iv) it follows that $\langle M, \cap, \cup, -, \mathsf{P}\rangle$ is a mac algebra and that $D = \{x\colon \mathsf{N}1 \leqslant x\}$ and by (i)-(iv′) that the quintuple is a joined mac algebra.

2. Let $\langle M, \cap, \cup, -, \mathsf{P}\rangle$ be a mac algebra: to show that the postulates of $S2^0$ are satisfied by $\mathfrak{M} = \langle M, D, \cap, \cup, -, \mathsf{P}\rangle$, where $D = [x\colon \mathsf{N}1 \leqslant x]$, and that this modal matrix is usual.

(i) Consider the Axioms T1-T4. These are necessitated versions of postulates effectively guaranteed by the Boolean algebra $\langle M, \cap, \cup, -\rangle$. Consider T1. Its valuation $\nu(A \mathbin{\&} B \prec A) =. \nu(A) \cap \nu(B) \prec . \nu(A)$. Let $\nu(A)$ be x and $\nu(B) \equiv y$. Now for any $x, y \in M$, $1 \equiv. x \cap y \supset x$, so $\mathsf{N}1 \equiv. \mathsf{N}(x \cap y \supset x) \equiv. x \cap y \prec x$. Hence for all $x, y \in M$, $x \cap y \prec x \in D$, i.e., T1 is satisfied. Similarly for T2-T4.
(ii) By Boolean algebra $x \cup y \cap (-y \cup z) \leqslant x \cup y$; hence $\mathsf{N}(x \cup y) \cap \mathsf{N}(-y \cap z) \equiv \mathsf{N}((x \cup y) \cap (-y \cup z)) \leqslant \mathsf{N}(x \cup y)$. Thus:

$$1 \equiv \mathsf{N}(-x \cup y) \cap \mathsf{N}(-y \cap z) \supset \mathsf{N}(-x \cup y);$$

and

$$\mathsf{N}1 \equiv \mathsf{N}(x \prec y \cap y \prec z \supset. x \prec y); \text{ so } (x \prec y) \cap (y \prec z) \prec . x \prec z \in D.$$

Thus any valuation of T5 belongs to D, hence T5 is satisfied.
(iii) Since $x \cap y \leqslant x$, $\mathsf{P}(x \cap y) \leqslant \mathsf{P}x$. Thus as $\mathsf{P}(x \cap y) \prec \mathsf{P}x \in D$, T6 is satisfied.
(iv) The tasks of showing that $\mathfrak{M}$ is usual and that the rules of $S2^0$ preserve satisfaction almost coincide. $D = \{x\colon \mathsf{N}1 \leqslant x\}$ is a filter of $\mathfrak{M}$. Since $\mathsf{N}1 \in D$, D is not empty. If $x, y \in D$ then as $\mathsf{N}1 \leqslant x$ and $\mathsf{N}1 \leqslant y$, $\mathsf{N}1 \leqslant x \cap y$, so $x \cap y \in D$. Therefore **Adj** is also vindicated. If $x \in D$, $y \in M$ then $\mathsf{N}1 \leqslant x \leqslant x \cup y$, so $x \cup y \in D$. For strict detachment suppose $x \in D$ and $x \prec y \in D$. Then $x \leqslant y$ so $y \equiv x \cup y \in D$. For substitutivity, suppose $x \leftrightarrow y \in D$. Thereby $x \prec y \in D$ and $y \prec x \in D$, so $x \leqslant y$ and $y \leqslant x$, and $x \equiv y$.
(v) D is proper since $0 \notin D$. Suppose otherwise $0 \in D$. Then $\mathsf{N}1 \leqslant 0$. As $0 \leqslant \mathsf{N}1$ in a Boolean algebra, $0 \equiv \mathsf{N}1$. Also since $0 \leqslant 1$, $\mathsf{N}0 \leqslant \mathsf{N}1 \equiv 0$. Hence $\mathsf{N}0 \equiv \mathsf{N}1 \equiv 0$ and $\mathsf{P}0 \equiv \mathsf{P}1 \equiv 1$. Thus $\mathsf{N}1 \leqslant \mathsf{P}0$ contradicting $\sim(\mathsf{N}1 \leqslant \mathsf{P}0)$. Since $\mathsf{N}1 \in D$, D is not null.
(v′) In case the algebra is joined $\mathsf{N}0 \notin D$. Suppose otherwise $\mathsf{N}0 \in D$. Then $\mathsf{N}1 \leqslant \mathsf{N}0$. Since $0 \leqslant 1$, $\mathsf{N}0 \leqslant \mathsf{N}1$; so $\mathsf{N}0 \equiv \mathsf{N}1$. Therefore by (iii′) $0 \equiv 1$ contradicting the non-degeneracy of the algebra.

Definitions: A wff A is **S**-*satisfied* (*falsified*) by a mac algebra $\mathfrak{M} = \langle M, \cap, \cup, -, \mathsf{P}\rangle$ iff it is satisfied (falsified) by the corresponding $S2^0$-matrix $\langle M, D, \cap, \cup, -, \mathsf{P}\rangle$.

$$\mathbf{S} =_{Df} \{x\colon \mathsf{N}1 \leqslant x\}.$$

From these definitions **S**-satisfaction and **S**-falsification result.

Theorem E *Let* $\mathfrak{M} = \langle M, \cap, \cup, -, \mathsf{P}\rangle$ *be a mac algebra (or a connected mac algebra) and let* $a_1, \ldots, a_r$ *be a finite sequence of elements of* M. *Then there is a finite mac algebra (joined mac algebra)* $\mathfrak{M}_1 \equiv \langle M_1, \cap_1, \cup_1, -_1, \mathsf{P}_1\rangle$ *with at most* $2^{2^{(r+2)}}$ *elements such that:*

(i) *for* $1 \leqslant i \leqslant r$, $a_i \in M_1$;
(ii) *for* $x, y \in M_1$, $x \cap_1 y \equiv x \cap y$;

(iii) *for* $x, y \in M_1$, $x \cup_1 y \equiv x \cup y$;
(iv) *for* $x \in M_1$, $-_1 x \equiv -x$;
(v) *for* $x \in M_1$ *such that* $\mathrm{P}x \in M_1$, $\mathrm{P}_1 x \equiv \mathrm{P}x$.

Proof: Let $\mathfrak{M}_1$ be the Boolean subalgebra of $\mathfrak{M}$ generated by $a_1, \ldots, a_r$, $\mathrm{P}0$, $\mathrm{P}1$. By Boolean algebra results, there are not more than $2^{2^{(r+2)}}$ elements in M_1. Define $\cap_1, \cup_1, -_1$ as the restrictions of $\cap, \cup, -$ to M_1. For $x \in M_1$, x is *covered* by y iff $y \in M_1$ and $\mathrm{P}y \in M_1$ and $x \leqslant y$. That (i)-(v) are satisfied and that requirements (i) and (ii) on a mac algebra are met is proved as in Lemmon [2], p. 55 or McKinsey [4], pp. 124-125. Since $0 \in M_1$, $1 \in M_1$, $\mathrm{P}0 \in M_1$, $\mathrm{P}1 \in M_1$, $\mathrm{P}_1 0 \equiv \mathrm{P}0$. Also $\mathrm{N}_1 1 \equiv -_1 \mathrm{P}_1 0 \equiv -\mathrm{P}0 \equiv \mathrm{N}1$. So, since $\sim(\mathrm{N}1 \leqslant \mathrm{P}0)$, $\sim(\mathrm{N}_1 1 \leqslant \mathrm{P}_1 0)$. In case $\mathfrak{M}$ is joined, it suffices to show, because of Theorem 3 (iv), that if $\mathrm{P}_1 x \leqslant \mathrm{P}_1 0$ then $x \equiv 0$. But since $\mathrm{P}_1 0 \equiv \mathrm{P}0$ and $\mathrm{P}x \leqslant \mathrm{P}_1 x$, if $\mathrm{P}_1 x \leqslant \mathrm{P}_1 0$ then $\mathrm{P}x \leqslant \mathrm{P}0$. Hence $x \equiv 0$, when $\mathfrak{M}$ is joined.

Theorem C S2^0, *and each of its consistent extensions, has a characteristic usual modal matrix.*

Proof as in McKinsey [4], p. 122-123.

Theorem D $\vdash_{S2^0} A$ *iff* A *is* S*-satisfied by all mac algebras (or by all joined mac algebras).*

Proof: If $\vdash_{S2^0} A$, then A is satisfied by all S2^0 matrices, so it is satisfied by all usual S2^0 matrices $\langle M, \{x\colon \mathrm{N}1 \leqslant x\}, \cap, \cup, -, \mathrm{P}\rangle$. Thus A is S-satisfied by all mac algebras. If $\sim \vdash_{S2^0} A$ then there is a characteristic usual modal matrix which falsifies A; to this a mac algebra corresponds. Therefore A is not S-satisfied by all mac algebras.

Theorem E *Let* A *be a wff with* r *subformulas. Then* $\vdash_{S2^0} A$ *iff* A *is* S*-satisfied by all mac algebras (joined mac algebras) with not more than* $2^{2^{(r+2)}}$ *elements.*

Proof as in Lemmon [2], p. 56 (with $a_r \not\equiv 1$ replaced by $\sim(\mathrm{N}1 \leqslant a_r)$).

Corollaries 1. S2^0 *has the finite model property, and so is decidable.*
2. $\vdash_{S2^0} A$ *iff* A *is* S*-satisfied by all finite mac algebras.*

Some of the development for S2^0 is easily parallelled for S1^0.

Definition: A structure $\mathfrak{M} = \langle M, \cap, \cup, -, \mathrm{P}\rangle$ is a *tom algebra* iff M is a set of elements closed under operations $\cap, \cup, -, \mathrm{P}$ such that

(i) $\langle M, \cap, \cup, -\rangle$ is a non-degenerate Boolean algebra;
(ii) $\mathrm{P}(x \cap z) \leqslant \mathrm{P}(x \cap y) \cup \mathrm{P}(-y \cap z)$;
(iii) $\sim(\mathrm{N}1 \leqslant \mathrm{P}0)$.

Although the principle if $x \leqslant y$ then $\mathrm{P}x \leqslant \mathrm{P}y$ no longer holds generally, the principle that if $x \equiv y$ then $\mathrm{P}x \equiv \mathrm{P}y$ and $\mathrm{N}x \equiv \mathrm{N}y$ of course holds.

Theorem A (for S1^0) $\mathfrak{M} = \langle M, D, \cap, \cup, -, \mathrm{P}\rangle$ *is a usual* S1^0*-matrix iff* $\langle M, \cap, \cup, -, \mathrm{P}\rangle$ *is a tom algebra and* $D \equiv \mathsf{S}$.

Proof is similar to that of Theorem A for S2^0. Consider, e.g., 2 (v) of

Theorem A (for $S2^0$) *Suppose* $0 \in D$. *Then* $0 \equiv \mathrm{N}1$, *so* $1 \equiv \mathrm{P}0$. *Thus* $\mathrm{N}1 \leqslant \mathrm{P}0$ *contradicting* $\sim(\mathrm{N}1 \leqslant \mathrm{P}0)$.

Theorem C (for $S1^0$) $S1^0$ *has a characteristic usual modal matrix.*

Theorem D (for $S1^0$) $\vdash_{S1^0} A$ *iff* A *is* S*-satisfied by all tom algebras.*

Proofs are as for $S2^0$.

Theorems A-E are now developed for $S3^0$. The theorems are simplified if the standard $S3^0$ postulate

T7. $\Box(A \supset B) \prec . \Box(\Box A \supset \Box B)$

is replaced, first by its ($S2^0$) deductive equivalent

T7′. $\Box A \prec . \Box(\Box B \supset \Box A)$

(compare Lemmon [3], p. 195). The equivalence is proved thus:

$\vdash_{S2^0} A \prec B \prec . \Box A \supset \Box B$	by F33.311
$\vdash_{S2^0} (\Box A \supset . A \prec B) \prec . \Box A \supset . \Box A \supset \Box B$	by F42.12
$\vdash_{S2^0} (\Box A \supset . \Box A \supset \Box B) \prec . \Box A \supset \Box B$	by F34.1
$\vdash_{S2^0} \Box A \prec . A \prec B \prec . \Box A \prec \Box B$	by F46.1

Since, given T7′, $\Box(A \supset B) \prec . \Box A \prec \Box(A \supset B)$, T7 follows by F31.021. Conversely, since

$\vdash_{S2^0} \Box A \prec . B \prec A$

$\Box A \prec . \Box B \prec \Box A$, i.e., T7′, follows applying T7. Secondly, T7′ is ($S2^0$) deductively equivalent to

T7″. $\Box A \prec . \Box \mathrm{T} \prec \Box A$.

T7″ follows from T7′ by substitution. Conversely,

$\vdash_{S2^0} B \prec \mathrm{T}$	by F43.1
$\vdash_{S2^0} \Box B \prec \Box \mathrm{T}$	by F46.1
$\vdash_{S2^0}$ T7″ $\prec$ T7	by F45.30, F45.31

Lemma 2 $\vdash_{S3^0} \Diamond A \leftrightarrow . \Diamond(A \vee . \Diamond A \,\&\, \Box \mathrm{T})$.

Proof: One half follows from $p \prec . p \vee q$ and F41.41; the other half follows by contraposing T7″ and using F42.12.

Definition: A mac algebra (joined mac algebra) is *strictly directive* when

(iv) $\mathrm{P}x \equiv \mathrm{P}(\mathrm{P}x - \mathrm{P}0 . \cup . x)$ *for* $x \in M$.

Theorem A (for $S3^0$) $\mathfrak{M} = \langle M, D, \cap, \cup, -, \mathrm{P}\rangle$ *is a usual* $S3^0$*-matrix iff* $\langle M, \cap, \cup, -, \mathrm{P}\rangle$ *is a strictly directive mac algebra* (*or connected mac algebra*) *and* $D \equiv \{x\colon \mathrm{N}1 \leqslant x\}$.

Proof: This extends Theorem A for $S2^0$ in the relevant respects.

1. A usual $S3^0$-matrix guarantees (iv) as a consequence of Lemma 2.
2. In any mac algebra, since $0 \leqslant y$, $\mathrm{P}0 \leqslant \mathrm{P}y$. Hence in turn, $\mathrm{P}x - \mathrm{P}y \leqslant \mathrm{P}x - \mathrm{P}0$, $(\mathrm{P}x - \mathrm{P}y) \cup x \leqslant (\mathrm{P}x - \mathrm{P}0) \cup x$. Thus, by (iv),

(a) $P[(Px - Py) \cup x] \leqslant P[(Px - P0) \cup x] \equiv Px$.

Now:

(b) $Nx \supset N(Ny \supset Nx) \equiv -Nx \cup N(-Ny \cup Nx)$
$\equiv P - x \cup -P(-P - y \cap P - x)$

Also: $1 \equiv P(P - x \cap - P - y) \cup P - x \cup -P(P - x \cap -P - y)$
$\equiv P[(P - x \cap - P - y) \cup -x] \cup -P(-P - y \cap P - x)$
$\leqslant P - x \cup - P(-P - y \cap P - x)$ by (a)
$\leqslant Nx \supset N(Ny \supset Nx)$ by (b)

Hence $N1 \leqslant Nx \dashv N(Ny \supset Nx)$; so **T7′** is satisfied.

Theorem B (for S3^0) The enunciation of this theorem is exactly the same as that of Theorem B for S2^0, except that *'strictly directive mac algebra'* replaces *'mac algebra'*.

Proof: It needs to be shown

$$P_1x \equiv P_1[(P_1x - P_10) \cup x]$$

given $Px \equiv P[(Px - P0) \cup x]$. Since $P0 \in M_1$, $P_10 \equiv P0$. Let $y_1, y_2, \ldots, y_n$ cover x so that $P_1x \equiv Py_1 \cap Py_2, \ldots, Py_n$. Since $x \leqslant y_i$, $Px \leqslant Py_i$. Also $P_1x \leqslant Py_i$. Hence $P_1x - P_10 \leqslant Py_i - P0$; and so $(P_1x - P_10) \cup x \leqslant (Py_i - P0) \cup x \leqslant (Py_i - P0) \cup y_i$. But $P[(Py_i - P0) \cup y_i] \equiv Py_i \in M_1$, so that $(P_1x - P_10) \cup x$ is covered by $(Py_i - P0) \cup y_i$, for each i. Suppose the remainder of the cover of $(P_1x - P_10) \cup x$ is given by $z_1, \ldots, z_m$. Then

$$P_1[(P_1x - P0) \cup x] \equiv P[(Py_i - P0) \cup y_i] \cap \ldots P[(Py_n - P0) \cup y_n] \cap Pz_1, \ldots, Pz_m$$
$$\equiv Py_1 \cap \ldots Py_n \cap Pz_1, \ldots, Pz_m \equiv P_1x \cap Pz_1, \ldots, Pz_m \leqslant P_1x.$$

Conversely, as $0 \leqslant x$, $0 \leqslant P_10$ and (β) $P_1x \leqslant P_1x \cup P_10$. Since $P_10 \equiv P0$, $P0 \leqslant Px$, and $Px \leqslant P_1x$ generally, $P_10 \leqslant P_1x$. Hence $P_1x - P_10 \geqslant 0$; $P(P_1x - P_10) \geqslant P0 \equiv P_10$. So $P_1(P_1x - P_10) \geqslant P(P_1x - P_10) \geqslant P_10$. Now, by ($\beta$), $P_1x \leqslant P_1x \cup P_1(P_1x - P_10) \leqslant P_1((P_1x - P_10) \cup x)$.

Theorems C-E (for S3^0) *Enunciation and proofs are direct adaptations of those for* S2^0.

Hence S3^0 has the finite model property and is decidable.

Analogous results hold for the weak modal system C3$_0$, the system obtained from Lemmon's C2 by adding the postulate

A7′. $\Box(A \supset B) \supset \Box(\Box A \supset \Box B)$.

A7′ is deductively equivalent with respect to C2 to

A7. $\Box A \supset . \Box(\Box B \supset \Box A)$.

Lemma $\vdash_{C3_0} \Diamond A \equiv \Diamond(A \vee . \Diamond A \,\&\, \Box T)$.

Definition: A modal algebra (as defined in Lemmon [2]) is *directive* when

$$Px = P(Px - P0 \;.\!\cup\!.\; x) \text{ for } x \in M.$$

Theorem A (for $C3_0$) $\mathfrak{M} = \langle M, D, \cap, \cup, -, P\rangle$ *is a regular* $C3_0$*-matrix iff* $\langle M, \cap, \cup, -, P\rangle$ *is a directive modal algebra and* $D \equiv [x\colon x = 1]$.

Proof is similar to that for $S3^0$, replacing '$\equiv$' by '$=$' and '$\leqslant$' by '$\leq$'. Definitions of 'regular' and '$\leq$' are as in Lemmon [2].

Theorems B-E (for $C3_0$) Enunciations and proofs are adaptations of those for $S3^0$.

The system $D3_0$ is obtained by adding to $C3_0$ the deontic postulate

A5. $\Box A \supset \sim\Box\sim A$.

Definition: A modal algebra is *deontic* when $P1 = 1$.

Theorems A-E (for $D3_0$) Proofs combine those for $C3_0$ with Lemmon's results for D2 in [2].

Systems $S2^{sd}$ and $S3^{sd}$ are obtained from systems $S2^0$ and $S3^0$ respectively by adding the postulate,

$\Box$A5. $\Box A \dashv \sim\Box\sim A$.

Definition: A mac algebra is *strictly deontic* when $Nx \leqslant Px$.

Theorems A-E (for $S2^{sd}$ and $S3^{sd}$)

Proof: The relevant extras are these: Theorem A: 1. A usual $S2^{sd}$ (or $S3^{sd}$) matrix guarantees $Nx \dashv Px \in D$, and so $Nx \leqslant Px$, in virtue of $\Box$A5. 2. Since $Nx \leqslant Px$, $1 \equiv -Px \cup Px \leqslant -Nx \cup Px$. Thus $N1 \leqslant Nx \dashv Px$; so $\Box$A5 is satisfied.

Theorem B: Given $Nx \leqslant Px$, $N_1x \leqslant P_1x$ follows, since $Px \leqslant P_1x$ and $N_1x \equiv -P_1 - x \leqslant Nx$.

Definition: A mac algebra is *strictly epistemic* when

$$x \leqslant Px \text{ for } x \in M.$$

Strictly epistemic mac algebras correspond of course to S2-matrices, and strictly epistemic strictly directive mac algebras to S3-matrices. In case a strictly directive mac algebra is strictly epistemic the strictly directive requirement can be replaced by a strict transitivity requirement $Px \equiv P(Px - P0)$. Strictly epistemic mac algebras are strictly deontic.

Theorems A-E (for S2 and S3)

Proof: The relevant extras are these: Theorem A: 1. A usual S2 (or S3) matrix guarantees $x \leqslant Px$ in virtue of the S2 postulate $A \dashv \Diamond A$ (F36.0). 2. Since $x \leqslant Px$, $1 \equiv -x \cup x \leqslant -x \cup Px$, and $N1 \leqslant x \dashv Px$, so $A \dashv \Diamond A$ is satisfied.

Theorem B: Given $x \leqslant Px$, $x \leqslant P_1x$ follows, since $Px \leqslant P_1x$ quite generally.

A *model structure* (*m.s.*) is an ordered quadruple $\mathfrak{K} = \langle G, K, N, R\rangle$, where K is a set of items, $G \in K$, $N \subseteq K$, and R is a binary relation on K. An m.s. is a *Lewis model structure* (*l.m.s.*) iff $G \in N$.

Definition: $Q =_{Df} K - N$.

$\mathfrak{K}^+$, *the algebra on m.s.K*, is the ordered structure $\langle M, \cap, \cup, -, \mathsf{P}\rangle$ where

(i) $M = \mathcal{P}K$, i.e., the power set of K;
(ii) $\cap$, $\cup$, $-$ are the set-theoretic operations of meet, join, and complement restricted to M;
(iii) $\mathsf{P}A = \{H\colon (\mathcal{S}H')(H' \in A \ \&\ HRH') \vee H \in Q\}$, for $A \in M$.

The set theory assumed is familiar extensional set theory with single identity '=', single (improper) inclusion '$\subseteq$', and non-ontological quantifiers '$\mathcal{A}$' and '$\mathcal{S}$'.

Lemma $N \neq Q$.

Proof: $G \in N$ or $G \in Q$ but not both.

Lemma *In any algebra on any m.s.*

(i) $\mathsf{P}\Lambda = Q$, *where* Λ *is the null set.*
(ii) $-\mathsf{P}\Lambda = N$.
(iii) $Q \subseteq \mathsf{P}A$, for any $A \in M$.

Theorem F *If* $\mathfrak{K}$ *is a Lewis model structure then* $\mathfrak{K}^+$ *is a mac algebra.*

Proof: Conditions (i) and (ii) on a mac algebra are established as in Lemmon [2], Theorem 15. (The terminology is of course adjusted.) *ad* (iii) Since $\mathfrak{K}$ is a Lewis m.s., $G \in N$ and so $G \notin Q$. Therefore $(\mathcal{S}H)(H \in N \ \&\ H \notin Q)$. Now if $-\mathsf{P}\Lambda \subseteq \mathsf{P}\Lambda$, then $N \subseteq Q$; whence $\sim(\mathcal{S}H)(H \in N \ \&\ H \notin Q)$, and a contradiction. So $\sim(-\mathsf{P}\Lambda \subseteq \mathsf{P}\Lambda)$.

Theorem G *Any finite mac algebra is isomorphic to the algebra on some finite Lewis m.s.*

Proof: Let $\mathfrak{M} = \langle M, \cap, \cup, -, \mathsf{P}\rangle$ be a finite mac algebra. Then for some $\mathfrak{K}$, $\langle M, \cap, \cup, -\rangle$ is isomorphic to the algebra of subsets of $\mathfrak{K}$, by Stone's representation theorem, under isomorphism ϕ say. It suffices to add to Lemmon's proof in [2] of Theorem 17, the following detail showing that when $\mathfrak{M}$ is a mac algebra, $\mathfrak{K}$ is a Lewis m.s. Since in a mac algebra $\sim(-\mathsf{P}0 \leqslant \mathsf{P}0)$, by the isomorphism $\sim(\phi - \mathsf{P}0 \subseteq \phi\mathsf{P}0)$, where $\subseteq$ corresponds to $\leqslant$. Thus, where P^* is the possibility operation in $\mathfrak{K}^+$, $\sim(-\mathsf{P}^*\phi 0 \subseteq \mathsf{P}^*\phi 0)$, i.e., $\sim(-\mathsf{P}^*\Lambda \subseteq \mathsf{P}^*\Lambda)$. Hence, by lemmata, $\sim(N \subseteq Q)$. Thus $(\mathcal{S}H)(H \in N)$. Call such an H, G, i.e., $G = \in H\colon H \in N$. Since $G \in N$, $\mathfrak{K}$ is a Lewis m.s.

Theorem H $\vdash_{S2^0} A$ *iff*

(i) *A is* S-*satisfied by* $\mathfrak{K}^+$ *for all l.m.s.* $\mathfrak{K}$.
(ii) *A is* S-*satisfied by* $\mathfrak{K}^+$ *for all finite l.m.s.* $\mathfrak{K}$.

Proof: If $\vdash_{S2^0} A$ then A is S-satisfied by all mac algebras, by Theorem D; and so by $\mathfrak{K}^+$, for all (finite) l.m.s. $\mathfrak{K}$, since these are mac algebras, by Theorem F. Conversely if $\sim\vdash_{S2^0} A$ then some finite mac algebra S-falsifies A, by Theorems D and E. Hence, by Theorem G, A is S-falsified by $\mathfrak{K}^+$ for some (finite) m.s. $\mathfrak{K}$.

A *valuation model* for a wff A *on* an m.s. $\mathfrak{K}$ is a binary function $\mathsf{v}(p, H)$, where p ranges over sentential variables of A and H over items of K, whose values lie in $\{\mathsf{T}, \mathsf{F}\}$. A *value* $\mathsf{v}(B, H)$ for any subformulae B of A *for* a given valuation model for A on an m.s.$\mathfrak{K}$ is defined recursively as follows: if B is atomic $\mathsf{v}(B,H)$ is as in the valuation model; $\mathsf{v}(\sim B, H) = \mathsf{T}$ iff $\mathsf{v}(B, H) = \mathsf{F}$; $\mathsf{v}(B \supset C, H) = \mathsf{T}$ iff $\mathsf{v}(B, H) = \mathsf{F}$ or $\mathsf{v}(C, H) = \mathsf{T}$; $\mathsf{v}(\Box B, H) = \mathsf{T}$ iff $\mathsf{v}(B, H') = \mathsf{T}$ for all $H' \in K$ such that HRH', and $H \in N$.

Wff A is *true in* valuation model $\mathsf{v}(p, H)$ for A on m.s. $\mathfrak{K} = \langle G, K, N, R\rangle$ *at* $H' \in K$ iff $\mathsf{v}(A, H') = \mathsf{T}$. A is $S2^0$-*true on* model $\mathsf{v}(p, H)$ on Lewis m.s. $\mathfrak{K} = \langle G, K, N, R\rangle$ iff $\mathsf{v}(A, G) = \mathsf{T}$. A is $S2^0$-*valid in* $\mathfrak{K} = \langle G, K, N, R\rangle$ iff A is true in all models on m.s.$\mathfrak{K}$ at all $H' \in N$, i.e., at $H' \in K$ where $H' \notin Q$ for all $H' \in K$. A is $S2^0$-*valid on* Lewis m.s. $\mathfrak{K} = \langle G, K, N, R\rangle$ iff A is $S2^0$-true in all models $\mathsf{v}(p, H)$ on l.m.s. $\mathfrak{K}$. A is $S2^0$-*valid* iff A is $S2^0$-valid in all Lewis m.s.

Lemma *A is $S2^0$-valid iff A is $S2^0$-valid on all l.m.s.*

A is $S2^0$-*valid over* m.s.$\mathfrak{K} = \langle G, K, N, R\rangle$ iff, for every $H' \in K$, $H' \in N$ materially implies A is true in all models in m.s.$\mathfrak{K}$ at all $H' \in K$.

Where $\mathsf{v}(p, H)$ is a valuation model for a wff A, which contains variable p_i, on an m.s.$\mathfrak{K}$,

$$\mathsf{V}(p_i) =_{Df} \{H\colon H \in K \;\&\; \mathsf{v}(p_i, H) = \mathsf{T}\},$$

and an assignment $\mathfrak{A}$ to the variables $p_1, \ldots, p_n$ of A from $\mathfrak{K}^+$ is defined:

$$\mathfrak{A} = \langle \mathsf{V}(p_1), \ldots, \mathsf{V}(p_n)\rangle.$$

For any assignment $\mathfrak{A}$ from $\mathfrak{K}^+$ to the variables of A, $\mathsf{V}_{\mathfrak{A}}(B)$ is the value assigned to subformula B of A in $\mathfrak{K}^+$ for the assignment $\mathfrak{A}$. Where $\mathfrak{A}$ is an assignment $\mathfrak{A} = \langle A_1, \ldots, A_n\rangle$, with $A_i \subseteq K$, from $\mathfrak{K}^+$ to the variables of A, a valuation model $\mathsf{v}_{\mathfrak{A}}(p, H)$ for A on $\mathfrak{K}$ is defined thus: $\mathsf{v}_{\mathfrak{A}}(p_i, H) = \mathsf{T}$ iff $H \in A_i$.

Lemma (i) *Where $\mathsf{v}(p, H)$ is a valuation model for wff A on m.s. $\mathfrak{K} = \langle G, K, N, R\rangle$, for all $H \in K$, $\mathsf{v}(A, H) = \mathsf{T}$ iff $H \in \mathsf{V}_{\mathfrak{A}}(A)$.*
(ii) *Where $\mathfrak{A}$ is an assignment to the variables of wff A from $\mathfrak{K}^+$ for some m.s. $\mathfrak{K} = \langle G, K, N, R\rangle$, for all $H \in K$, $\mathsf{v}_{\mathfrak{A}}(A, H) = \mathsf{T}$ iff $H \in \mathsf{V}_{\mathfrak{A}}(A)$.*

Proofs as in Lemmon [2], p. 61.

Theorem I (i) *Where $\mathfrak{K} = \langle G, K, N, R\rangle$ is any Lewis m.s., A is* S-*satisfied by $\mathfrak{K}^+$ iff A is $S2^0$-valid in, or on, $\mathfrak{K}$.*
(ii) *Where $\mathfrak{K}$ is any m.s., A is satisfied by $\mathfrak{K}^+$ iff A is $S2^0$-valid over $\mathfrak{K}$ (provided paradoxical implications are admitted).*

Proof: (i) (a) Let A be S-satisfied by $\mathfrak{K}^+$, and consider a valuation model $\mathsf{v}(p, H)$ for A on $\mathfrak{K}$. Then $\mathsf{V}_{\mathfrak{A}}(A) \supseteq N^*K$, where N^* is the necessity operator in $\mathfrak{K}^+$ and $\mathfrak{A}$ is any assignment from $\mathfrak{K}^+$ to A's variables. Now $N^*K = -\mathsf{P}*\Lambda = -Q = N$. Thus $\mathsf{V}_{\mathfrak{A}}(A) \supseteq N$. Since then, for all $H \in N$, $H \in \mathsf{V}_{\mathfrak{A}}(A)$, it follows by Lemma (i) that for all $H \in N$, $\mathsf{v}(A, H) = \mathsf{T}$. Hence A is $S2^0$-valid in $\mathfrak{K}$. Since too $G \in N$, $\mathsf{v}(A, G) = \mathsf{T}$. Hence A is $S2^0$-valid on $\mathfrak{K}$.

(b) Let A be S2^0-valid in $\mathfrak{K}$, and consider an assignment $\mathfrak{A}$ to the variables of A. Since A is S2^0-valid in $\mathfrak{K}$, for all $H \in N$, $\mathsf{v}_{\mathfrak{A}}(A, H) = \mathsf{T}$. Hence by Lemma (ii) for all $H \in N$, $H \in \mathsf{V}_{\mathfrak{A}}(A)$. As then $\mathsf{V}_{\mathfrak{A}}(A) \supseteq N$, $\mathsf{V}_{\mathfrak{A}}(A) \supseteq N{*}K$. Thus A is S-satisfied by $\mathfrak{K}^+$. For A is S-satisfied by $\mathfrak{K}^+$ iff the value of A for assignments from $\mathfrak{K}^+$ to its variables is designated, i.e., includes $N{*}K$. Next let A be S2^0-valid on l.m.s. $\mathfrak{K}$ and consider any assignment $\mathfrak{A}$ to the variables of A from $\mathfrak{K}^+$. Since A is S-valid on $\mathfrak{K}$, $\mathsf{v}_{\mathfrak{A}}(A, G) = \mathsf{T}$, whence by Lemma (ii), $G \in \mathsf{V}_{\mathfrak{A}}(A)$. Since however, G may be any element of $\mathfrak{K}$, since that is $G = \epsilon H$: $H \in K$, $N \subseteq \mathsf{V}_{\mathfrak{A}}(A)$; and that A is S-satisfied follows as before.
(ii) Where $\mathfrak{K}$ is an m.s. which is not a Lewis m.s. N is null. Then, for every H if $H \in N$ then $H \in \mathsf{V}_{\mathfrak{A}}(A)$ is true vacuously provided the if-then is paradoxical. Likewise, for any H, if $H \in N$ then $\mathsf{v}(A, H) = \mathsf{T}$, holds vacuously.

Theorem J (i) $\vdash_{S2^0} A$ *iff* A *is* S2^0-*valid.*
(ii) $\vdash_{S2^0} A$ *iff* A *is* S2^0-*valid over all m.s.* (*provided paradoxical implications are exploited*).

Proof: (i) $\vdash_{S2^0} A$ iff A is S-satisfied by $\mathfrak{K}^+$ for all l.m.s. $\mathfrak{K}$, by Theorem H (i), iff A is S2^0-valid in all l.m.s. $\mathfrak{K}$, by Theorem I (i), iff A is S2^0-valid.
(ii) Similar to (i) but using Theorem I (ii).

Definition: An m.s.$\mathfrak{K}$ is *strictly epistemic* ($\mathcal{S}$ epistemic) iff $(\mathcal{A}H)(H \in Q \vee HRH)$.

Theorems F-J (for S2)

Theorem F (for S2) *If* $\mathfrak{K}$ *is a strictly epistemic l.m.s., then* $\mathfrak{K}^+$ *is a strictly epistemic mac algebra.*

Proof: Because of Theorem F (for S2^0) it suffices to show that when $(\mathcal{A}H)(H \in Q \vee HRH)$, $A \subseteq \mathsf{P}A$ for $A \subseteq K$. Now given the premiss, $H \in A$ materially implies $H \in A$ & $(HRH \vee H \in Q)$, which in turn implies $(H \in A$ & $HRH) \vee H \in Q$, and so implies $(\mathcal{S}H')(H' \in A$ & $HRH') \vee H \in Q$, i.e., $H \in \mathsf{P}A$, as required for a classical inclusion.

Theorem G (for S2) *Any finite strictly epistemic mac algebra is isomorphic to the algebra on some finite strictly epistemic l.m.s.*

Proof is similar to Lemmon [2], Theorem 19, p. 59.

For the remaining systems considered it is enough to establish Theorems F and G. For Theorems H-J they follow as before. The main connections may be summed up in a table like this:

System	Corresponding algebra	Corresponding m.s.
S2^0	mac	l.m.s.
S2	strictly epistemic mac	strictly epistemic l.m.s.

(compare Lemmon [3], p. 207-208). We condense, e.g., the S2 line of this table to:

System S2 $\sim$ strictly epistemic mac algebra $\sim$ strictly epistemic l.m.s.

Definitions of truth and validity are of course appropriately modified to reflect these connections. For instance, wff A is S2-*true* on model $\mathbf{v}(p, H)$ on strictly epistemic l.m.s. $\mathfrak{K} = \langle G, K, N, R\rangle$ iff $\mathbf{v}(A, G) = \mathbf{T}$. Definitions of S2-*valid* (*in*) and S2-*valid on* are similarly modified by replacing '(Lewis) m.s.' by 'strictly epistemic (Lewis) m.s.'

Theorems H-J (for S2) are similar in statement (and proof) to those for S2^0, except that 'strictly epistemic (Lewis) m.s.' systematically replaces '(Lewis) m.s.' and 'S2-valid' replaces 'S2^0-valid'. Consider to illustrate:

Theorem J (for S2) (i) $\vdash_{S2} A$ *iff A is* S2-*valid*;
(ii) $\vdash_{S2} A$ *iff A is* S2-*valid over all strictly epistemic m.s.* (*provided paradoxical implications are exploited*).

That is, given the proviso, iff for every H', $H' \epsilon N$ *materially* implies A is true in all models on all strictly epistemic m.s. at all $H' \epsilon K$.

Theorem J (i) strikes us as a better result than Lemmon's Theorem 26 in [3], p. 202, which corresponds rather to Theorem J (ii). Unfortunately Lemmon offers no sufficient definition of his key notion 'weak validity'; for truth at $l \epsilon N$, in terms of which weak validity is to be defined, is nowhere defined in Lemmon's papers. The obvious way of defining truth at $l \epsilon N$—by adding '& $l \notin Q$' to the definition of truth at $l \epsilon K$ given in [2], p. 60—renders Lemmon's Theorem 26 in [3] incorrect. A way to repair Lemmon's result is to use a connective which effectively drops off the cases where truth is evaluated at some $l \epsilon Q$; and this can be done by exploiting paradoxical features of '$\supset$', by requiring (in Lemmon's terminology): A is true for model $\Phi(v, K)$ at $l \epsilon (K - Q)$ in an m.s. $\langle K, Q, U\rangle$ iff $l \not\subset Q \supset \Phi'(A, l) = \mathbf{T}$. Similarly a better result* for S2-provability is given by the corollary to Theorem E for S2, in terms of satisfaction in all finite strictly epistemic l.m.s., than is given by Lemmon's Theorem 23 ([3], p. 201), in terms of weak satisfaction in all finite **e**-algebras. For the matrices corresponding to Lemmon's finite **e**-algebras may be improper. But since improper matrices satisfy everything, in virtue of features of paradoxical implication, they can be thrown in without upset.

Definition: An m.s. $\mathfrak{K}$ is *strictly directive* (s *directive*) iff R is transitive from pairs of elements in N, i.e., for H_1, $H_2 \epsilon N$ and $H_3 \epsilon K$, H_1RH_2 and H_2RH_3 imply H_1RH_3.

Theorems F-J (for S3^0)

Theorem F (for S3^0) *If* $\mathfrak{K}$ *is a strictly directive l.m.s., then* $\mathfrak{K}^+$ *is a strictly directive mac algebra.*

*Just as satisfaction in normal (transitive) **e**-algebras gives, as Lemmon claims in [3], p. 201, better results for T and S4 than, what is equivalent, weak satisfaction in closed (transitive) **e**-algebras.

Proof: It suffices to show, where $\mathfrak{R}$ is strictly directive, $\mathrm{P}A = \mathrm{P}(\mathrm{P}A - Q .\cup. A)$ for $A \subseteq K$. A one way inclusion is immediate. For the other suppose $H \in \mathrm{P}(\mathrm{P}A - Q .\cup. A)$. If $H \in Q$ then $H \in \mathrm{P}A$ by a lemma. If then $H \in N$, by definition of 'P', for some H', $H' \in \mathrm{P}A - Q$ or $H' \in A$ and HRH'. If $H' \in A$ and HRH' then $H \in \mathrm{P}A$ as required, so it remains to consider the case where for some $H', H' \in \mathrm{P}A - Q$ and HRH'. Then $H' \in N$ and for some $H'', H'' \in A$ & $H'RH''$. Since $H, H' \in N$ and HRH' and $H'RH''$ by strict directiveness HRH''. Thus since $H'' \in A$, by predicate logic, $H \in \mathrm{P}A$.

Theorem G (for S3^0) *Any finite strictly directive mac algebra is isomorphic to the algebra on some finite strictly directive l.m.s.*

Proof: By isomorphism ϕ, $\mathrm{P}(\mathrm{P}A - Q \cup. \mathrm{P}A) = \mathrm{P}A$, for $A \subseteq K$. Suppose H_1, $H_2 \in N$, H_1RH_2, and H_2RH_3. Then $H_1 \in \mathrm{P}\{H_2\}$ and $H_2 \in \mathrm{P}\{H_3\} - Q$ (see Lemmon [2], p. 56). Thus $\{H_2\} \subseteq \mathrm{P}\{H_3\} - Q \subseteq. \mathrm{P}\{H_3\} - Q \cup \{H_3\}$; hence $\mathrm{P}\{H_2\} \subseteq \mathrm{P}(\mathrm{P}\{H_3\} - Q \cup. \{H_3\})$ [$= \mathrm{P}\{H_3\}$]. Since $H_1 \in \mathrm{P}\{H_2\}$, $H_1 \in \mathrm{P}\{H_3\}$, that is H_1RH_3 as required.

The remaining results for S3^0 simply follow out the connections:

System S3^0 $\sim$ strictly directive mac algebra $\sim$ strictly directive l.m.s.

Theorems F-J (for S3)

System S3 $\sim$ s epistemic s directive mac algebra
$\sim$ s epistemic s directive l.m.s.

Proofs of Theorems F-G combine those for S2 and for S3^0.

Theorems F-J (for C3$_0$)

System C3$_0$ $\sim$ s directive modal algebra $\sim$ s directive m.s. Proofs of theorems extend those in Lemmon [2] for C2 in much the way that those for S3^0 extend those for S2^0.

Definition: An m.s. $\mathfrak{R}$ is *strictly deontic* (s *deontic*) iff $(\mathcal{A}H)(\mathcal{S}H')(HRH' \vee H \in Q)$.

Theorems F-J (for D3$_0$)

System D3$_0$ $\sim$ s directive s deontic modal algebra
$\sim$ s directive s deontic m.s.

Theorems F-J (for S2^{sd} and S3^{sd})

System S2^{sd} $\sim$ s deontic mac algebra $\sim$ s deontic l.m.s.
System S3^{sd} $\sim$ s directive s deontic mac algebra
$\sim$ s directive s deontic l.m.s.

The methods of the paper also suffice to treat many other necessitated extensions of S2^0 which have not so far been discussed in detail in the literature, for example, the systems S4^0 and S8 (both explained in [1]).

REFERENCES

[1] Feys, R., *Modal Logics*, E. Nauwelaerts, Louvain (1965).

[2] Lemmon, E. J., "Algebraic semantics for modal logics I," *The Journal of Symbolic Logic*, vol. 31 (1966), pp. 46-65.

[3] Lemmon, E. J., "Algebraic semantics for modal logics II," *The Journal of Symbolic Logic*, vol. 31 (1966), pp. 191-218.

[4] McKinsey, J. J. C., "A solution of the decision problem for the Lewis systems S2 and S4, with an application to topology," *The Journal of Symbolic Logic*, vol. 6 (1941), pp. 117-134.

[5] Montgomery, H., and R. Routley, "Conventionalist and contingency-oriented modal logics," *Notre Dame Journal of Formal Logic*, vol. XII (1971), pp. 131-152. *Cf.* also this Journal, vol. XIV (1973), p. 583.

[6] Routley, R., and V. Macrae, "On the identity of sensations and physiological occurrences," *American Philosophical Quarterly*, vol. 3 (1966), pp. 87-110.

[7] Routley, R., "Existence and identity in quantified modal logics," *Notre Dame Journal of Formal Logic*, vol. X (1969), pp. 113-149.

[8] Routley, R., "Non-existence does not exist," *Notre Dame Journal of Formal Logic*, vol. XI (1970), pp. 289-320.

Monash University
Clayton, Victoria, Australia

and

University of Canterbury
Canterbury, New Zealand