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Complexity of Model-Theoretic Notions

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The main result of this paper is that if T is a countable theory and Γ a type, then the predicate "T has a model omitting Γ" is sharply Σ_1^1 in the sense of recursion theory, and particularly that every Σ_1^1 predicate is recursive in this predicate. We will also show that the predicate "T is ω_0-categorical" is arithmetical.

1 Definitions and preliminaries Let L be the language of the full countable predicate calculus. A *theory* is a subset of the formulas of L. If T is a theory then $L(T)$ denotes the sublanguage of L which involves the symbols found in members of T. An *n-type* in T is a subset of the formulas of $L(T)$ whose variables are among $v_1, \ldots, v_n$ and which is consistent with T. A *type* is an n-type for some n. If T is a theory, Γ is an n-type in T, and $B(v_1, \ldots, v_n)$ is a formula of $L(T)$, we say that $B(v_1, \ldots, v_n)$ is a *generator* of Γ in T if for each formula $A(v_1, \ldots, v_n) \in \Gamma$, $T \vdash B(v_1, \ldots, v_n) \rightarrow A(v_1, \ldots, v_n)$.

We assume that some standard scheme for Gödel numbering languages is given having all the usual properties, and so we will often speak of theories, types, etc., as though they were sets of natural numbers. For example the predicate "T has a model omitting Γ" is a predicate whose arguments T and Γ are actually sets of natural numbers. If x is the Gödel number of a formula, let A^x denote the formula; and if A is a formula, let $\overline{A}$ denote the Gödel number of A.

Let $\mathfrak{N} = \langle N, +^{\mathfrak{N}}, \cdot^{\mathfrak{N}}, <^{\mathfrak{N}}, 0, 1, \ldots \rangle$ be the standard model of arithmetic and let $L(\mathfrak{N})$ be the language of $\mathfrak{N}$ with nonlogical symbols $+, \cdot, <, \underline{0}, \underline{1}, \ldots$ ($L(\mathfrak{N})$ is of course a subset of L). Let F_w be the set of all quantifier-free sentences of $L(\mathfrak{N})$ which are true in $\mathfrak{N}$. By an ω-model of a theory $T \subset L(\mathfrak{N})$ we mean a model of T whose universe consists exactly of the interpretations of the constant symbols $\underline{0}, \underline{1}, \underline{2}, \ldots$. Clearly the only ω-model of F_w is $\mathfrak{N}$ itself. There are many sentences of $L(\mathfrak{N})$ which are consistent with F_w but false in $\mathfrak{N}$. If A

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is any such sentence, there can be no ω-model of $F_w \cup \{A\}$, and so the type $\Gamma_\infty = \{v_1 \neq \underline{0}, v_1 \neq \underline{1}, \ldots\}$ is not omitted in any model of $F_w \cup \{A\}$.

In what follows:

$P_1, P_2, \ldots$ are new unary predicate symbols not found is $L(\mathfrak{N})$
$X_1, X_2, \ldots, X, Y, Z$ are variables ranging over subsets of N
$x_1, x_2, \ldots, x, y, z, t$ are variables ranging over N
$n_1, n_2, \ldots$ are members of N.

We denote by $L(\mathfrak{N}, P_1, \ldots, P_j)$ the language whose symbols are those of $L(\mathfrak{N})$ plus $P_1, \ldots, P_j$. When we write a formula $A(P_1, \ldots, P_j; v_1, \ldots, v_k)$ of $L(\mathfrak{N}, P_1, \ldots, P_j)$ only $v_1, \ldots, v_k$ are really variables; $P_1, \ldots, P_j$ simply lists which new predicate symbols appear in the formula.

Let $R(X_1, \ldots, X_j, x_1, \ldots, x_k)$ be a predicate whose first j arguments range over subsets of N and whose remaining arguments range over N. We say that R is *arithmetical* if there is some formula $A(P_1, \ldots, P_j; v_1, \ldots, v_k)$ of $L(\mathfrak{N}, P_1, \ldots, P_j)$ such that for each sequence of subsets $X_1, \ldots, X_j$ and each sequence $n_1, \ldots, n_k$ of natural numbers, $R(X_1, \ldots, X_j, n_1, \ldots, n_k)$ holds iff $\langle \mathfrak{N}, X_1, \ldots, X_j \rangle \models A(P_1, \ldots, P_j; \underline{n}_1, \ldots, \underline{n}_k)$. We say that R is Σ_1^1 if for some arithmetical predicate S, $R(X_1, \ldots, X_j, n_1, \ldots, n_k)$ holds iff $\exists X\, S(X, X_1, \ldots, X_j, n_1, \ldots, n_k)$. R is Π_1^1 iff the above holds with $\exists$ replaced by $\forall$. Finally we say that R is *hyperarithmetical* if R is both Σ_1^1 and Π_1^1. We will assume a little familiarity with recursion theory, such as is found in Shoenfield [2].

2 ***The predicate*** *OM(X, Y)*

Consider the predicate $Om(X, Y)$ where X is a theory, Y is a type in X, and Y is omitted in some model of X.

Proposition 2.1 *The predicate $Om(X, Y)$ is sharply Σ_1^1. Moreover every Σ_1^1 predicate is recursive in Om, so Om is a complete Σ_1^1 predicate.*

Proof: First, if T is a theory and Γ is an n-type, then Γ is omitted in some model of T iff there is a consistent extension T' of T such that Γ has no generator in T' ([1], Cor. 2.2.10, p. 80). Now consider the following arithmetical predicates:

$Con(Z, y)$:	$Z \cup \{y\}$ is a consistent theory
$Imp(Z, Y, y)$:	y is a generator of Y in Z.

By our note above, we have

$$Om(X, Y) \longleftrightarrow X \text{ is a theory} \wedge Y \text{ is a type in } X \wedge \exists Z(X \subset Z \wedge Con(Z, \overline{v_1 = v_1}) \wedge \neg \exists y(Con(Z, y) \wedge Imp(Z, Y, y))).$$

The predicates inside the parentheses are arithmetical, and the $\exists Z$ makes $Om(X, Y)$ a Σ_1^1 predicate. Moreover let $R(x)$ be a Σ_1^1 predicate (to save notation we will just consider a predicate of a single number argument). Then for some formula $A(P_1; v_1)$ of $L(\mathfrak{N}, P_1)$ we have for each natural number n

$$R(n) \longleftrightarrow \exists X(\langle \mathfrak{N}, X \rangle \models A(P; \underline{n})).$$

The right-hand side of this equivalence is true iff the theory $T_A(n) = F_w \cup \{A(P_1; \underline{n})\}$ in the language $L(\mathfrak{N}, P_1)$ has an ω-model. So $R(n) \longleftrightarrow Om(T_A(n), \Gamma_\infty)$, where Γ_∞ is as defined in Section 1. Since $T_A(n)$ is recursive in n, we have that $R(x)$ is recursive in $Om(X, Y)$, which proves our result.

In general, $Om(X, Y)$ is Σ_1^1, but one can easily find restrictions on X, Y which make $Om(X, Y)$ of lower complexity. For example, if every complete extension of X is a finite extension, then $Om(X, Y)$ is arithmetical.

*3 **The predicate "T is ω_0-categorical"*** The Omitting Types Theorem is used to prove the classical Ryll-Nardjewski Theorem about countable ω_0-categorical theories ([2], p. 91):

Theorem 3.1 *Let T be a countable complete theory having only infinite models. Then the following are equivalent:*

(1) *T is ω_0-categorical.*
(2) *For each n, there are only finitely many complete n-types consistent with T.*
(3) *Every complete type consistent with T has a generator in T.*

From this it can be seen that the predicate "T is ω_0-categorical" (as a predicate of the set variable T) is Π_1^1, for (3) is expressible as $\forall\Gamma(\Gamma$ is a complete type in $T \rightarrow \exists y Imp(T, \Gamma, y))$. The predicate within parentheses is arithmetical and the set quantifier $\forall$ makes the whole formula Π_1^1. But (2) provides an easy proof of the following:

Theorem 3.2 *The predicate "X is a complete ω_0-categorical theory" is arithmetical.*

Proof: Let $K(X)$ be the conjunction of the following predicates:

$A(X)$: X is a complete theory
$B(X)$: X includes the sentences saying the universe is infinite
$C(X)$: For each n, there is a finite sequence α of formulas with free variables $v_1, \ldots, v_n$, such that

(1) each α_i generates a complete type over X
(2) the sentence $\forall v_1 \ldots \forall v_n \bigvee\!\!\bigvee_{i \in dom\,\alpha} \alpha_i(v_1, \ldots, v_n)$ is in X.

It is clear that each of A, B, C is arithmetical, and by (2) of Theorem 3.1, $K(X)$ holds iff X is ω_0-categorical.

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[2] Shoenfield, J. R., *Mathematical Logic,* Addison-Wesley, Reading, Massachusetts, 1967.

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