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DEDUCTION THEOREMS IN SIGNIFICANCE LOGICS

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Introduction The deduction theorem for implication in sentential logic is a very useful aid in proving theorems, so as significance logics are generally fairly simple extensions of sentential logic, with perhaps some restrictions on modus ponens and substitution, it is of interest to examine the kinds of deduction theorems that can be proved for them. In this paper we will consider the system C_0, S_0, C_1, S_1, IS_1, L_3S_1, AS_1, HS_1, C_2, S_2, C_3, S_3, C_4, S_4, C_5, S_5, C_6, and S_6 as given in Goddard and Routley ([1]). In S_4 Goddard and Routley prove and use a deduction theorem, however, in S_1 such a theorem is also used but their argument for the correctness of the theorem in this case is faulty and in fact no such theorem can be proved. This also applies to IS_1, L_3S_1, AS_1, and HS_1, but in all other systems some form of the deduction theorem can be proved.

The system C_0 The system C_0 is equivalent to classical sentential logic (even though variables and well formed formulas (wffs) can take three truth values: **t** (true), **f** (false), and **n** (nonsignificant)). It therefore follows that the following standard deduction theorem holds:

DTC_0 *If A and B are wffs and* $A \vdash B$, *then* $\vdash A \supset B$.

The systems S_4 *and* S_6 These systems have uniform substitution and modus ponens for $\rightharpoondown$ as their only rules, so DTC_0 holds with $\rightharpoondown$ instead of $\supset$.

The system C_6 (*or* D_6) This system has uniform substitution and modus ponens for $\rightarrow$ as the only rules so DTC_0 holds with $\rightarrow$ instead of $\supset$.

The system S_0 This system has two sorts of variables, the variables of C_0, now called S-unrestricted variables (p, q, p', q', . . .) and also S-restricted variables (r, s, r', s', . . .). The S-restricted variables can take only the "significant" truth values **t** and **f**. Significance-restricted formulas (srfs) are then formed using the primitive connectives and S-restricted variables just as wffs are formed using the primitive connectives and the variables of sentential logic. Wffs in S_0 are formed using both types of variables and the connectives.

The axioms of S_0 are written in terms of S-restricted variables only

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and the rules (material detachment or modus ponens and substitution) do not allow us to deduce any theorem that is not an srf. The axioms are again those of sentential logic so we have:

DTS_0 *If A and B are* srf*s and $A \vdash B$ then $\vdash A \supset B$.*

The system C_1 This system is as C_0 but has in addition the significance operator **S** with the formation rule: *If Δ is a wff so is* $\mathbf{S}\Delta$. It has the axioms of C_0, some axioms for **S**, the substitution rule of C_0, and modified modus ponens:

R2′ *If $\vdash A$ and $\vdash A \supset B$, then $\vdash B$, provided no variable is uncovered in A and covered in B.*

A variable P is covered in a wff A iff P occurs in A and every occurrence of P in A is within the scope of some occurrence of **S** when A is written in primitive notation. A variable P is uncovered in A iff P occurs in A and not every occurrence of P in A is within the scope of an occurrence of **S**.

Because of this modified version of modus ponens the deduction theorem of C_0 must be modified and will require a detailed proof:

DTC_1 *If A and B are wffs, no variable covered in A or B is uncovered in B or A and $A \vdash B$, then $\vdash A \supset B$.*

Proof: We prove that each step $A \vdash A_i$ in the proof of $A \vdash B$ can be replaced by $\vdash A \supset A_i$. Any such step will have A_i an axiom, $A = A_i$, A_i derived from a previous step by substitution for variables not free in A or A_i derived from two previous steps by modified modus ponens. If A_i is an axiom we obtain by substitution in axiom 1.1′ of C_1:

$$\vdash A_i \supset (A \supset A_i).$$

Then by modified modus ponens and $\vdash A_i$, if no variable uncovered in A_i is covered in A:

$$\vdash A \supset A_i.$$

If $A = A_i$ we have by substitution in axiom 1.2′ of C_1:

$$\vdash A \supset. (A \supset A) \supset A :\supset: A \supset (A \supset A) .\supset. A \supset A.$$

By substitution in Axiom 1.1′ we have:

$$\vdash A \supset. (A \supset A) \supset A$$

and

$$\vdash A \supset (A \supset A),$$

as the restriction in modified modus ponens is satisfied we have

$$\vdash A \supset A$$

or

$$\vdash A \supset A_i.$$

If in the original proof $A \vdash A_i$ is obtained from $A \vdash A_j$ by substitution for variables free in A_j, but not in A then the same substitution will lead from $\vdash A \supset A_j$ to

$$\vdash A \supset A_i.$$

We now consider the case where in the original proof $A \vdash A_i$ was derived from $A \vdash A_j \supset A_i$ and $A \vdash A_j$ by modified modus ponens. This means that we can have no variable uncovered in A_j and covered in A_i.

We also assume that we have already proved

$$\vdash A \supset A_j$$

and

$$\vdash A \supset (A_j \supset A_i)$$

where no variable covered in A is uncovered in A_i or A_j and no variable covered in A_i or A_j is uncovered in A. By substitution and Axiom 1.2′ we have

$$\vdash A \supset (A_j \supset A_i) \mathrel{.{\supset}{:}} A \supset A_j \mathrel{.{\supset}.} A \supset A_i$$

and as obviously no variable uncovered in $A \supset (A_j \supset A_i)$ is covered in $A \supset A_j \mathrel{.{\supset}.} A \supset A_i$, we have:

$$\vdash A \supset A_j \mathrel{.{\supset}.} A \supset A_i.$$

Now no variable uncovered in A_j can be covered in A_i or A and no variable uncovered in A can be covered in A_i, so by $\vdash A \supset A_j$ and modified modus ponens we have:

$$\vdash A \supset A_i.$$

Thus DTC_1 holds.

The system C_2 This system is as C_1 with uniform substitution and modified modus ponens, but it has one extra rule involving a new primitive operator **T**:

RC3 *If* $\vdash A_1 \,\&\, A_2 \,\&\, \ldots \,\&\, A_k \mathrel{.{\supset}.} B_1 \vee B_2 \vee \ldots \vee B_m$
then $\vdash \mathbf{T}A_1 \,\&\, \mathbf{T}A_2 \ldots \,\&\, \mathbf{T}A_k \mathrel{.{\supset}.} \mathbf{T}B_1 \vee \mathbf{T}B_2 \vee \ldots \vee \mathbf{T}B_m$,

where $A_1, \ldots, A_k, B_1, \ldots, B_m$ contain only classical connectives and all variables of $B_1, \ldots, B_m$ are among those of $A_1, \ldots, A_k$.

DTC_1, with the extra condition that A contains only classical connectives, holds in this system. To the proof we need to add the case that deals with RC3.

If in the original proof $A \vdash \mathbf{T}A_1 \,\&\, \ldots \,\&\, \mathbf{T}A_k \mathrel{.{\supset}.} \mathbf{T}B_1 \vee \ldots \vee \mathbf{T}B_m$ is obtained by Rule RC3 from $A \vdash A_1 \,\&\, \ldots \,\&\, A_k \mathrel{.{\supset}.} B_1 \vee \ldots \vee B_m$, we can assume

$$A \mathrel{{\supset}{:}} A_1 \,\&\, \ldots \,\&\, A_k \mathrel{.{\supset}.} B_1 \vee \ldots \vee B_m$$

where no variable covered in A is uncovered in $A_1 \ldots A_k$, $B_1 \ldots B_m$ and no variable covered in $A_1, \ldots A_k, B_1, \ldots$ or B_m is uncovered in A. Hence $\vdash A \ \& \ A_1 \ldots A_k \supset. B_1 \vee \ldots \vee B_m$ can be proved and provided A contains only classical connectives we have by RC3:

$$\vdash \mathbf{T}A \ \& \ \mathbf{T}A_1 \ldots \mathbf{T}A_k \supset. \mathbf{T}B_1 \vee \ldots \vee \mathbf{T}B_m$$

Hence

$$\vdash \mathbf{T}A \supset: \mathbf{T}A_1 \ \& \ldots \& \ \mathbf{T}A_k \supset. \mathbf{T}B_1 \vee \ldots \vee \mathbf{T}B_m$$

and using $\vdash A \supset \mathbf{T}A$ we obtain

$$\vdash A \supset: \mathbf{T}A_1 \ \& \ldots \& \ \mathbf{T}A_k \supset. \mathbf{T}B_1 \vee \ldots \vee \mathbf{T}B_m$$

The systems C_3, C_4, *and* C_5 These systems have wffs that are not wffs of C_2 and have the modified modus ponens of C_2 replaced by two rules:

RC2 *If* $\vdash A$ *and* $\vdash A \supset B$, *then* $\vdash B$, *provided*

(i) *no variable is uncovered in* A *and covered in* B,
(ii) $A \supset B$ *is a wff of* C_2.

RC4 *If* $\vdash \mathbf{T}A$ *and* $\vdash A \supset B$, *then* $\vdash B$.

To incorporate RC4 into the proof of our deduction theorem we would require a derived rule

$$A \supset \mathbf{T}B, A \supset. B \supset C \vdash A \supset C$$

which we do not have.

The deduction theorem of C_2(DTC$_1$) still holds in C_3, C_4, and C_5 provided all steps in the proof involve wffs of C_2.

The system S_1 Three equivalent formulations of this are given in [1], we will consider ${}_3S_1$, which has the axioms of S_0, some axioms for $\mathbf{S}$, (unrestricted) modus ponens, a substitution rule for unrestricted variables and the following substitution rule for restricted variables:

R1.2 *If* $\vdash A$ *and* $\vdash \mathbf{S}B$, *then* $\vdash \mathbf{S}_B^R A|$, *where* R *is an* $\mathbf{S}$-*restricted variable and* A *and* B *are wffs.*[1]

Because of this rule no deduction theorem can be proved in S_1. We have for example by R1.2 and the first axiom of S_1:

$$\mathbf{S}B \vdash B \supset (B \supset B),$$

but we have no way of proving

$$\vdash \mathbf{S}B \supset. B \supset (B \supset B) \tag{1}$$

even though $\mathbf{S}B$ is significant ($\vdash \mathbf{SS}B$ holds for all B in S_1).

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1. Wffs in S_1 are defined as in S_0. $\mathbf{S}_B^R A|$ stands for the uniform substitution of B for all free occurrences of R in A.

Goddard and Routley in their section on S_1 in [1] assume that a deduction theorem (that of S_0) holds here simply because S_1 contains all the theorems of their RSL (sentential logic). The deduction theorem, however, is a metatheorem and hence need not hold. If DTS_0 were to hold the undesirable result that all wffs are significant could be proved in S_1 as follows: By (1) and the rule

$$A \vdash \mathbf{S}A$$

which holds in S_1,

$$\vdash \mathbf{S}(\mathbf{S}B \supset . B \supset (B \supset B)).$$
$$\vdash \mathbf{SS}B$$

and

$$\vdash \mathbf{S}(p \supset q) \supset (\mathbf{S}p \supset \mathbf{S}q) \tag{2}$$

hold in S_1 so by substitution and modus ponens:

$$\vdash \mathbf{S}(B \supset (B \supset B))$$

and by the S_1 axiom:

$$\vdash \mathbf{S}(p \supset q) \supset \mathbf{S}p \tag{3}$$

and substitution we have for the arbitrary wff B:

$$\vdash \mathbf{S}B.$$

The systems L_3S_1, AS_1, HS_1, S_2, S_3, *and* S_5 These systems have the rules of S_1 and so no deduction theorem is provable.

The system IS_1 This system contains all the theorems of S_1 and in addition axioms for material implication $\rightarrow$. No deduction theorem is provable for $\supset$ as in S_1 and none is provable for $\rightarrow$ as $\vdash A \rightarrow B$ is not provable for B an axiom and A an arbitrary (or even **S**-restricted) wff.

It may be of interest to consider what minor changes could be made to S_1 and other systems based on it to allow the proof of a deduction theorem. We could change to one set of **S** unrestricted variables and change axioms such as

$$\vdash r \supset (r' \supset r)$$

with **S** restricted variables to

$$\mathbf{S}p, \mathbf{S}p' \vdash p \supset (p' \supset p), \tag{4}$$

we then no longer need a substitution rule for **S**-restricted variables. However, we have now replaced several axioms by rules which in turn have to be considered in the proof of a deduction theorem and the handling of such rules would require unacceptable new axioms. If we change (4) to:

$$\vdash \mathbf{S}p \supset . \mathbf{S}p' \supset (p \supset . p' \supset p)$$

we need no new rules, but we already have an unacceptable axiom as (2), (3) and the rule $A \vdash \mathbf{S}A$ will give us $\vdash \mathbf{S}p$. It seems, therefore, that no deduction theorem is possible in a system similar to S_1.

REFERENCES

[1] Goddard, L. and R. Routley, *The Logic of Significance and Context*, Scottish Academic Press, Edinburgh (1973).

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