Notre Dame Journal of Formal Logic
Volume XX, Number 3, July 1979
NDJFAM

GÖDEL'S PROOF AND THE LIAR PARADOX

JILL HUMPHRIES

Through the years there has been considerable discussion and not a little confusion about the relation of Gödel's incompleteness proofs to the paradoxes of set theory. His arguments, particularly the first incompleteness proof[1] have been thought to be closely related to the liar and Richard's paradoxes, and do appear to have some resemblance to them in their pattern. Gödel himself commented on this relationship[2] and Mostowski[3] considers Gödel's argument to be simply a formalized version of Richard's paradox in which the defects in Richard's informal argument that generate the contradiction have been avoided by Gödel's formalization. At the same time, however, Gödel emphasized that his arguments are constructive arguments in which the undecidable sentence truly says of itself that it is undecidable and not a paradox or a sentence that is neither true nor false. It is important then, to provide a clarification of Gödel's arguments and to determine whether they are indeed related to these paradoxes. In this paper I want to argue (1) that both the preliminary proof and the detailed formal proof in Gödel's 1931 paper involve diagonal procedures and (2) that neither proof is related to the liar paradox although the preliminary proof appears to have a superficial resemblance to the liar paradox.

To do this it is necessary first to review briefly the distinction made by Tucker[4] between heterological procedures used in the liar paradox and the diagonal procedure involved in Richard's paradox. A heterological procedure generates a paradox in which there is an oscillating pattern of the form 'If P then not P and if not-P then P'. Each of the two statements is an informal contradiction in the sense that P and not-P cannot be true at the same time, and together they yield a truth-functional contradiction of the form 'P and not-P'. According to Tucker's analysis, heterological procedures are faulty non-constructive procedures. The error in the procedure is a confusion between first and second order predicates where a second order predicate that describes some property of some first order predicates is treated as a first order predicate so that there is no first order predicate to which the second order predicate can be applied.

Received October 5, 1974

For example, 'heterological' is a second order predicate which describes a property of some first order predicates, the property of not possessing the property they express. But in Grelling's paradox, where there is an attempt to decide whether 'heterological' is heterological, 'heterological' is treated as a first order predicate; there is no first order predicate such as 'long' to which 'heterological' can be applied and the result is the statements 'If 'heterological' is heterological then it is not heterological' and 'If 'heterological' is not heterological then it is heterological' a pair of statements with the oscillating pattern characteristic of heterological procedures. Similarly, in the liar paradox there is no first order statement to which the second order predicate 'is false' can be applied; 'is false' is treated as a first order predicate and generates the contradictions 'If 'This sentence is false' is false then it is true' and 'If 'This sentence is false' is true then it is false'.

The diagonal procedure on the other hand is a constructive procedure that does not of itself generate any contradictions. It is a procedure that specifies that for an enumeration E of a class Σ an element D can be constructed that is a member of Σ but does not belong to the enumeration E; D differs systematically from each of the elements in E and as such is indenumerable relative to E. But while the diagonal procedure does not of itself generate any contradictions it has been involved in some of the paradoxes of set theory, in particular Richard's paradox. It can be argued, however, that the contradiction in Richard's paradox is not the result of an improper use of the diagonal procedure but is the result of an appeal to an actual infinity of numbers.

In Richard's paradox, an enumeration E of number-theoretical definitions is constructed and then by the diagonal procedure an additional number N is defined that differs systematically from each definition in the enumeration. At this point there is no paradox in the argument. By applying the diagonal procedure a definition has been constructed that is indenumerable relative to the enumeration E in a straightforward constructive manner. The paradox is found in the further argument that (1) all definitions of numbers that can be defined in a finite number of words are contained in the list E, and (2) N is also defined in a finite number of words. This argument, however, would seem to rest on an appeal to an actual infinity of numbers and their definitions (that the enumeration E contains *all* definitions of finitely definable numbers). The difficulty here is that the notion of an actual infinity is a dubious if not inconsistent notion and if it is rejected there is no reason to assume that E does contain all finite definitions of numbers. Rather, by using the diagonal procedure in the first part of the argument, Richard can be considered to have given an informal proof of the indenumerability of numbers and their definitions. Thus it is not the diagonal procedure itself, but a further argument that generates the paradox in Richard's paradox.[5]

Given this distinction between heterological and diagonal procedures, it can be shown that Gödel's arguments are not related to the liar paradox.

Gödel's argument demonstrates that there are well-formed formulae in *Principia Mathematica* (**P.M.**) that are not provable and their negations are not provable either. At the foundation of the argument is the arithmetization of the formulae of **P.M.** in which each formula is assigned a unique number; it is then demonstrated that within this numerical system there are numbers which cannot be computed by the computation methods of the system. The corresponding formulae of **P.M.** are, then, undecidable formulae and can be determined from the numerical systems. The uniqueness of the numbers corresponding to the formulae in **P.M.** is guaranteed by the systematic assignment of numbers to the primitive terms of **P.M.** and the formulae constructed from them. The primitive terms are assigned the integers:

1	3	5	7	9	11	13
0	f	-	v	(x)	(	)

A numerical variable is assigned a prime number p greater than 13, a sentential variable a number p^2 and a predicate variable a number p^3. The number associated with each formula in **P.M.** is the product of the finite sequence of positive integers raised to the power of the numbers of the corresponding primitive terms. More generally, if a is a primitive term then it is correlated with the number $\varphi(a)$ and if $R(a_1,a_2, \ldots, a_n)$ is a class of formulae it is assigned a class of numbers $R'(x_1,x_2, \ldots, x_n)$ such that $a_1,a_2, \ldots, a_n$ is correlated with $x_1,x_2, \ldots, x_n$ if and only if $x_i = \varphi(a_i)$. In the preliminary version,[6] formulae of **P.M.** with one free variable (class-expressions) are ordered according to their length and alphabetically when they are of the same length, the nth class-expression of the series being denoted by $R(n)$. If a is an arbitrary class-expression, $[a;n]$ denotes the formula that results from substituting the symbol for n for the free variable in a. Thus $[R(n);n]$ denotes the formula that is the result of substituting the symbol for n in the nth class-expression $R(n)$. The class of formulae $[R(n);n]$ that is not provable is denoted by $\overline{\mathrm{Prov}}[R(n);n]$ and a class K of numbers is defined as:

$$n \epsilon K \equiv [\overline{\mathrm{Prov}}\ R(n);n]$$

There is also a corresponding class-expression S such that $[S;n]$ is interpreted as 'n belongs to K' and since S is a class-expression it must be in the enumeration $R(n)$, say at the qth place so that $S = R(q)$.

With this notation it is possible to produce a formula that is undecidable in **P.M.** This is the formula $[R(q);q]$, the formula that is the result of replacing the symbol for q in $R(q)$, i.e. $[S;q]$. If $[S;q]$ is provable then it would be true that q belongs to K and so $\overline{\mathrm{Prov}}[S;q]$ would also hold: if $[S;q]$ is provable then it is not provable. Similarly if the negation of $[S;q]$ is provable then $\overline{n \epsilon K}$ would hold and $\mathrm{Prov}[S;q]$ would be true; but then both $[S;q]$ and $[\overline{S;q}]$ would be provable and **P.M.** would be inconsistent. Hence if **P.M.** is consistent neither $[S;q]$ nor its negation are provable and so $[S;q]$ is undecidable.

Gödel claimed that his argument was similar to Richard's paradox and also that it was closely related to the liar paradox in that the undecidable sentence is a self-descriptive sentence that asserts its own unprovability. This however, clearly cannot be correct. The liar paradox does not involve a case of genuine self-description; hence, if $[S;q]$ is a self-descriptive sentence that asserts its own unprovability, it cannot be related to the liar paradox in this way. Similarly, the procedures involved in the liar paradox and Richard's paradox are entirely different; hence Gödel's argument could not be related closely to both of them, and since, as Gödel emphasized, his argument is constructive, it should involve a diagonal procedure rather than a heterological procedure. There does however, seem to be a problem with the nature of the argument used by Gödel in the preliminary proof in that it does seem to have the oscillating pattern characteristic of a heterological argument: if $[S;q]$ is provable then it is not provable, and if $[\overline{S;q}]$ is provable then it is not provable. An explanation of this similarity of his argument to the pattern found in the liar paradox is required then, if the confusion between the two is to be removed.

First, the undecidable sentence $[S;q]$ differs significantly from the liar paradox in that it is a genuinely self-descriptive sentence. In the liar paradox there is no first order sentence P to which the predicate 'false' can be applied; the sentence 'The sentence '. . . .' is false' is neither true or false because there is no sentence '. . . .' that *can* be either true or false, and so, because it lacks content, it cannot be a self-descriptive sentence. In contrast with this, the undecidable sentence $[S;q]$ does have content and is true. $[S;q]$ says that q belongs to K; but q is the class-expression 'n belongs to K' and so $[S;q]$ says that the class expression that expresses 'q belongs to K' itself belongs to K. $[S;q]$ is, however, simply the mathematical description of the undecidable formula in **P.M.** But because S is $R(q)$ and is defined within the system, it is possible to determine the formula S and the number q and so determine the formula described by $[S;q]$. For this reason $[S;q]$ has the content that is lacking in the liar paradox: it truly says that a formula in **P.M.** is undecidable. It is a rather complex but constructive case of self-description and does not bear any relation to the sentence 'The sentence '. . . .' is false'.

The second problem is the apparent heterological pattern of Gödel's argument. Tucker[7] has argued that Gödel's argument is quasi-heterological, one that has the same oscillating pattern as a heterological argument but is constructive since the recursive foundation underlying the argument provides content for $[S;q]$. He argues that the arithmetization of **P.M.** provides more than a one-one correlation between numbers and formulae in **P.M.**, that it also provides a way of embedding metamathematical statements within the arithmetical system, and that because the recursive formulae are constructed from the defining equations of the system there is an effective procedure for defining their meaning. This is all quite correct, but simply arguing that because the recursive structure provides content to the undecidable sentence Gödel's argument is only quasi-heterological, constructive but with the pattern 'if P, then not-P and if not-P, then P' does not solve the problem of *why* it has this form, or of what procedures are used in the argument. Whatever procedure is involved, if the argument is to be constructive, the recursive foundation is needed in that it guarantees that in constructing $[S;q]$ Gödel is not pulling rabbits out of a hat. Consequently an explanation of the pattern of the argument is still required. The procedure that is used has to be identified and it has to be shown that it is a different procedure from the one used in the liar paradox.

The explanation is that in this case appearances are misleading. The argument involves a straightforward diagonal procedure, the apparent heterological pattern of the argument being the result of what has to be demonstrated in an incompleteness proof, and also to some extent the brevity of the argument. First, the undecidable formula differs systematically from each formula correlated with the number $R(n)$. It is the *result* of substituting the symbol for q for the free variable in S, the formula that is in the qth place in $R(n)$. Each formula in $R(n)$, however, is such that n is either a member of K, the class of unprovable formulae, or is a member of $\overline{K}$ the class of provable formulae. But $[S;q]$ differs from each formula in $R(n)$ in that neither it nor its negation can be correlated with an n that is a member of either K or $\overline{K}$. Let r be a number to which $[S;q]$ is correlated and let s be a number to which $\overline{[S;q]}$ is correlated. Neither r nor s can be numbers in $R(n)$: if $[S;q]$ is not provable then $\overline{[S;q]}$ should be provable so that $r \epsilon K$ and $s \epsilon \overline{K}$, and if $\overline{[S;q]}$ is not provable then $[S;q]$ should be provable so that $r \epsilon \overline{K}$ and $s \epsilon \overline{K}$. But neither $[S;q]$ nor $\overline{[S;q]}$ is provable so either r and s are both members of both K and $\overline{K}$, or they are members of neither K nor $\overline{K}$. If the former case held **P.M.** would be inconsistent, so if **P.M.** is consistent neither $[S;q]$ nor $\overline{[S;q]}$ can be correlated with any n in $R(n)$.

Since $[S;q]$ is a meaningful formula that has been constructed from a class-expression that is in the enumeration $R(n)$ but differs systematically from each expression in $R(n)$ it is clear that it has been constructed by a diagonal procedure, though this is not explicitly indicated in Gödel's argument. The reason for this, and also for the air of heterologicality surrounding the argument is that Gödel did not emphasize that the undecidable formula is the formula that is the *result* of substituting the symbol for q in the qth class expression S, the formula $[S;q]$ and that the *argument* he used to demonstrate undecidability is an argument to the effect that the provability of either $[S;q]$ or its negation cannot be assumed without generating a contradiction, rather than an argument demonstrating that they cannot be in the enumeration $R(n)$.[8] But when this argument is examined closely it can be seen that it does not have any relation to a heterological pattern of argument. To demonstrate that $[S,q]$ is undecidable Gödel has to demonstrate two things: that $[S;q]$ is not provable and that $\overline{[S;q]}$ is not provable, and to do this he uses two distinct *reductio ad absurdum* arguments that together do *not* generate a contradiction:

(1.) *If* Prov $[S;q]$, *then* Prov $[\overline{S;q}]$	$P \supset -P$
$/\therefore$ Prov $[\overline{S;q}]$	$/\therefore -P$
(2.) *If* Prov $[\overline{S;q}]$, *then* Prov $[S;q]$ *and* $\overline{\text{Prov}}\,[S;q]$	$Q \supset (P \& -P)$
$/\therefore$ $\overline{\text{Prov}}\,[S;q]$	$/\therefore -Q$
(3.) $\overline{\text{Prov}}\,[S;q]$ *and* $\overline{\text{Prov}}\,[\overline{S;q}]$	$-P \& -Q$

The difference between this and the argument in a heterological procedure, for example the liar paradox can easily be seen:

Let S be 'The sentence '. . . .' is false'

(1.) *If S is true, then S is false*	$P \supset -P$
$/\therefore$ *S is false*	$/\therefore -P$
(2.) *If S is false, then S is true*	$-P \supset P$
$/\therefore$ *S is true*	$/\therefore P$
(3.) *S is false and S is true*	$P \& -P$

In Gödel's argument there are two distinct formulae P and Q and for each of them it is demonstrated that the assumption of their provability entails their unprovability, two arguments of the form $(P \supset -P) \supset -P$, though the second argument is more complex than the first. But in the Liar paradox, or any other argument that results from a heterological procedure, there are two arguments of the forms $(P \supset -P) \supset -P$ and $(-P \supset P) \supset P$ so that a contradiction $P \& -P$ is generated. Nor does this difference have anything to do with self-reference or the lack of it. $[S;q]$ happens to be self-descriptive, though as Gödel has pointed out[9] it need not be; but the pattern of inference in Gödel's argument is still very different from heterological arguments with no element of self-reference. In a heterological argument where no self-reference is involved there are two sentences P and Q which if true are false and conversely. But in this case each sentence refers to the other so that the end result is two contradictions $P \& -P$ and $Q \& -Q$ whereas in Gödel's argument neither $[S;q]$ nor $[\overline{S;q}]$ refers to the other, and no contradictions are generated.

The preliminary version of Gödel's proof does not, then, have any relation to a heterological argument and is not even a quasi-heterological argument. It involves a straightforward diagonal procedure, although this aspect of the argument is not emphasized. What appears to be a heterological pattern of argument is simply a demonstration that the necessary and sufficient conditions for undecidability hold for a class-expression of **P.M.** when arithmetic is embedded in it.

The detailed version of Gödel's proof of undecidability differs to some extent from the preliminary version, but the differences make this version even less similar to the liar paradox and heterological patterns of argument in general. As in the preliminary version, a diagonal procedure is used, although this aspect is, again, not emphasized in the argument, and undecidability is demonstrated for ω-consistent classes rather than simply consistent classes. With the relevant definitions, Gödel's argument can be given briefly and the difference in the form of the argument becomes quite evident. Only an informal interpretation of the definitions will be given here since the purpose of considering the detailed version is to elucidate the form of the argument rather than a detailed examination of the proof.

Let η be an arbitrary recursive ω-consistent class of formulae.

(i) $\mathsf{Sb}(x_y^\nu)$—the formula that results from x when the free numerical variable ν is replaced by y in x.

Thus:

(ii) $\mathsf{Sb}(y_z(_{(y)}^{19}))$—the formula that is the result of substituting in y the numeral for y for the free numerical variable 19.
(iii) $x \mathsf{Gen}\, y$—the generalization of y by x.
(iv) $\mathsf{Neg}\, x$—the negation of x
(v) $x \mathsf{B}_\eta y$—x is the proof of y

With these preliminary definitions the undecidable formula can be constructed, and the concepts required to demonstrate its undecidability can be defined.

$$\mathsf{Bew}_\eta \equiv (\mathrm{E}y)(y \mathsf{B}_\eta x) \tag{1}$$
$$\mathsf{Q}(x,y) \equiv \overline{x \mathsf{B}_\eta [\mathsf{Sb}(y_{z(y)}^{19})]} \tag{2}$$

$x \mathsf{B}_\eta y$ and $\mathsf{Sb}(y_{z(y)}^{19})$ are both recursive and so $\mathsf{Q}(x,y)$ is also recursive. There is, then, a predicate q with free variables 17 and 19 such that:

$$\overline{x \mathsf{B}_\eta [\mathsf{Sb}(y_{z(y)}^{19})]} \rightarrow \mathsf{Bew}_\eta [\mathsf{Sb}(q_{z(x)\,z(y)}^{17\ \ 19})] \tag{3}$$
$$x \mathsf{B}_\eta [\mathsf{Sb}(y_{z(y)}^{19})] \rightarrow \mathsf{Bew}_\eta\, \mathsf{Neg} [\mathsf{Sb}(q_{z(x)\,z(y)}^{17\ \ 19})] \tag{4}$$

p is a class-expression with the free variable 19:

$$p = 17 \mathsf{gen}\, q \tag{5}$$

r is a recursive class-expression with free variable 17:

$$r = \mathsf{Sb}(q_{z(p)}^{19}) \tag{6}$$
$$\mathsf{Sb}(p_{z(x)}^{19}) = 17 \mathsf{Gen} r \tag{7}$$
$$\mathsf{Sb}(q_{z(x)\,z(p)}^{17\ \ 19}) = \mathsf{Sb}(r_{z(x)}^{17}) \tag{8}$$

The formula $17\mathsf{Gen}r$, the formula that results from substituting in p the numeral for p for the free variable 19 can be interpreted as 'p is not provable' or 'the class expression that expresses 'not provable in x' is unprovable'. It is, then, the analogue of $[S;q]$ and can be shown to be undecidable. By substituting p for y in (3) and (4) and by (8):

$$\overline{x \mathsf{B}_\eta (17\mathsf{Gen}r)} \rightarrow \mathsf{Bew}_\eta [\mathsf{Sb}(r_{z(x)}^{17})] \tag{9}$$
$$x \mathsf{B}_\eta (\mathsf{Gen}r) \rightarrow \mathsf{Bew}_\eta [\mathsf{Neg}\ \mathsf{Sb}(r_{z(x)}^{17})] \tag{10}$$

(If x is not the proof of $17\mathsf{Gen}r$ then, that this is the case is provable, and if x is the proof of $17\mathsf{Gen}r$ then that this is the case is provable.) Also, by (1):

$$\mathsf{Bew}_\eta(17\mathsf{Gen}r) \rightarrow (\mathrm{E}n)[n \mathsf{B}_\eta (17\mathsf{Gen}r)] \tag{11}$$

(I) 17Genr *is not provable.*

If 17Genr is provable then by (11) $\mathrm{Bew}_\eta[\mathrm{Neg}\ \mathrm{Sb}(r_{z(n)}^{17})]$ would hold. But it would also be the case then that $\mathrm{Sb}(q_{z(p)}^{19})$ is true, i.e., that p is not provable in η, and so

$$\mathrm{Sb}(q_{z(n)z(p)}^{17\ \ 19}) \text{ or } \mathrm{Sb}(r_{z(n)}^{17})$$

would also hold. Thus, if 17Genr is provable, η would be simply inconsistent (and also ω-inconsistent); hence, since η is ω-consistent 17Genr is not provable.

(II) Neg(17Genr) *is not provable.*

By (I) 17Genr is not provable and:

$$\overline{\mathrm{Bew}_\eta}(17\mathrm{Genr}) \rightarrow (n)[n\mathrm{B}(17\mathrm{Genr})] \tag{12}$$

and,

$$\overline{\mathrm{Bew}_\eta}(17\mathrm{Genr}) \rightarrow (n)\mathrm{Bew}_\eta[\mathrm{Sb}(r_{z(n)}^{17})] \text{ (from (9))} \tag{13}$$

But ω-consistency is defined as a class in which there is no a such that:

$$(n)\mathrm{Bew}_\eta[\mathrm{Sb}(a_{z(n)}^{\nu})] \ \&\ \mathrm{Bew}_\eta[\mathrm{Neg}(\nu\mathrm{Genr})] \tag{14}$$

where a is a class-expression and ν is a free variable of a. Hence, if Neg(17Genr) is provable then both

$$(n)\mathrm{Bew}_\eta[\mathrm{Sb}(r_{z(n)}^{17})] \text{ and } \mathrm{Bew}_\eta[\mathrm{Neg}(17\mathrm{Genr})]$$

would hold, and so since η is ω-consistent, Neg(17Genr) is not provable.

(III) *Neither* 17Genr *nor* Neg(17Genr) *is provable, and so* 17Genr *is undecidable.*

That 17Genr is constructed by a diagonal procedure is not, at first glance obvious, but it can be shown to be a diagonal derivative by an argument similar to that given above for $[S;q]$. The class of formulae $\mathrm{Sb}(y_{z(y)}^{19})$ is the analogue of the enumeration $\mathrm{R}(n)$ and is made up of mutually exclusive subclasses $\overline{x\mathrm{B}_\eta(\mathrm{Sb}_{z(y)}^{19})}$ and $x\mathrm{B}_\eta(\mathrm{Sb}_{z(y)}^{19})$ and while 17Genr is a member of the class η it cannot be a member of the class $\mathrm{Sb}(y_{z(y)}^{19})$. By (1) both $\mathrm{Bew}_\eta x \equiv (\mathrm{E}y)(y\mathrm{B}_\eta x)$ and $\overline{\mathrm{Bew}_\eta x} \equiv (y)\overline{(y\mathrm{B}_\eta x)}$ and so:

$$\overline{\mathrm{Bew}_\eta}(17\mathrm{Genr}) \rightarrow (n)[n\mathrm{B}(17\mathrm{Genr})] \ \&\ (\mathrm{E}n)[n\mathrm{B}_\eta[\mathrm{Neg}(17\mathrm{Genr})]] \tag{15}$$

and

$$\overline{\mathrm{Bew}_\eta}[\mathrm{Neg}(17\mathrm{Genr})] \rightarrow (n)[n\mathrm{B}_\eta[\mathrm{Neg}(17\mathrm{Genr})]] \ \&\ (\mathrm{E}n)[n\mathrm{B}_\eta(17\mathrm{Genr})] \tag{16}$$

But both $\overline{\mathrm{Bew}_\eta}(17\mathrm{Genr})$ and $\overline{\mathrm{Bew}_\eta}[\mathrm{Neg}(17\mathrm{Genr})]$ hold and so both

$$(n)[\overline{n\mathrm{B}_\eta(17\mathrm{Genr})}] \ \&\ (\mathrm{E}n)[n\mathrm{B}_\eta[\mathrm{Neg}(17\mathrm{Genr})]]$$

and

$$\overline{(n)[n\mathrm{B}_\eta\mathrm{Neg}(17\mathrm{Genr})]} \ \&\ (\mathrm{E}n)n\mathrm{B}_\eta(17\mathrm{Genr})$$

also hold. Thus either both 17Genr and Neg(17Genr) belong to both

$$\overline{x\mathrm{B}_\eta[\mathrm{Sb}(y_{z(y)}^{19})]} \text{ and } x\mathrm{B}_\eta[\mathrm{Sb}(y_{z(y)}^{19})]$$

or they belong to neither of these sub-classes. In a former case η would be inconsistent, and so if η is consistent, 17Genr cannot belong to the class $\mathrm{Sb}(y_{z(y)}^{19})$. Since it is the result of substituting the numeral for p for the free variable 19 in p the formula 17Genr is undoubtedly a member of η, but since it differs systematically from each formula in $\mathrm{Sb}(y_{z(y)}^{19})$ it must be a diagonal derivative.

It is also evident that the argument given to demonstrate the undecidability of 17Genr differs from that given for the undecidability of $[S;q]$ and that it bears even less resemblance to the liar paradox than the argument in the preliminary version does. This can be seen quite clearly from the logical *form* of the steps in the argument:

(I) $\mathrm{Bew}_\eta(17\mathrm{Genr}) \rightarrow \mathrm{Bew}_\eta[\mathrm{Neg}\ \mathrm{Sb}(r_{z(n)}^{17})]\ \&\ \mathrm{Bew}_\eta[\mathrm{Sb}(r_{z(n)}^{17})]$ $\quad P \supset (-Q\ \&\ Q)$
$/ \therefore\ \overline{\mathrm{Bew}}_\eta(17\mathrm{Genr})$ $\quad / \therefore\ -P$

(II) $\mathrm{Bew}_\eta(17\mathrm{Genr})$ $\quad -P$
$\mathrm{Bew}_\eta(17\mathrm{Genr}) \rightarrow (n)[\mathrm{Bew}_\eta \mathrm{Sb}(r_{z(n)}^{17})]$ $\quad -P \supset R$
$(n)[\mathrm{Bew}_\eta \mathrm{Sb}(r_{z(n)}^{17})]$ $\quad R$
η is ω-consistent $\equiv -[(n)[\mathrm{Bew}_\eta \mathrm{Sb}(r_{z(n)}^{17})]\ \&\ \mathrm{Bew}_\eta[\mathrm{Neg}(17\mathrm{Genr})]]$
$\quad C \equiv (R\ \&\ S)$
$\overline{\mathrm{Bew}}_\eta[\mathrm{Neg}(17\mathrm{Genr})] \rightarrow (n)[\mathrm{Bew}_\eta \mathrm{Sb}(r_{z(n)}^{17})]\ \&\ \mathrm{Bew}_\eta[\mathrm{Neg}(17\mathrm{Genr})]$
$\quad S \supset (R\ \&\ S)$
η is ω-consistent $\equiv \overline{\mathrm{Bew}}\,[\mathrm{Neg}(17\mathrm{Genr})]$ $\quad C \equiv -S$
$/ \therefore\ \overline{\mathrm{Bew}}_\eta[\mathrm{Neg}(17\mathrm{Genr})]$ $\quad / \therefore\ -S$

(III) $/ \therefore\ \overline{\mathrm{Bew}}_\eta(17\mathrm{Genr})\ \&\ \overline{\mathrm{Bew}}_\eta[\mathrm{Neg}(17\mathrm{Genr})]$ $\quad / \therefore\ -P\ \&\ -S$

The argument demonstrating $\overline{\mathrm{Bew}}(17\mathrm{Genr})$ is a *reductio ad absurdum* argument similar to that demonstrating Prov $[S;q]$ but the argument demonstrating $\overline{\mathrm{Bew}}_\eta\mathrm{Neg}(17\mathrm{Genr})$ is very different though it is a perfectly valid argument. But, as in the preliminary version, the conclusion is of the form $P\ \&\ -S$, or $P\ \&\ -Q$, not a contradiction $P\ \&\ -P$ and there is no argument of the form $[(P \supset -P)\ \&\ (-P \supset P)] \supset (P\ \&\ -P)$; the arguments (I) and (II) demonstrate the necessary and sufficient conditions for the undecidability of 17Genr and while the nature of these arguments tends to obscure the fact that a diagonal procedure has been used it nevertheless is the case the 17Genr is a diagonal derivative that differs systematically from each formula in $\mathrm{Sb}(y_{z(y)}^{19})$. The argument cannot, then be related in any way to the liar paradox or heterological procedures in general.

NOTES

1. Gödel [1].

2. Gödel [1], p. 9. See also Tucker [4] and Tucker [5] for a discussion of other attempts to assimilate Gödel's argument to the liar paradox.

3. Mostowski [3].

4. Tucker [4] and Tucker [5].

5. See Humphries [2] for a detailed analysis of Richard's paradox.

6. Gödel [1], pp. 7-9.

7. Tucker [4] and Tucker [6].

8. Gödel could have argued instead that since in the formula $[\mathsf{R}(n);n]$ the free variable in S *is* replaced by the symbol for the number with which *it* is correlated the resulting formula differs from each expression in $\mathsf{R}(n)$ and hence since these expressions form two mutually exclusive sub-classes K and $\overline{K}$ neither $[S;q]$ nor its negation can belong to either class and so is undecidable. This is the converse of the argument above but this does not render the whole discussion circular. It would be circular to argue that Gödel's argument for undecidability entails that $[S;q]$ differs systematically from each expression in $\mathsf{R}(n)$ and *therefore* he could argue that $[S;q]$ is the diagonal derivative and so undecidable. But the argument above is to demonstrate that Gödel has *in fact* used a diagonal procedure in the construction of $[S;q]$, not a demonstration of undecidability or an argument about what alternative procedures might have been adopted in such a demonstration. But since K and $\overline{K}$ are mutually exclusive $[S;q]$ (or any other formula) is undecidable if and only if it is not in $\mathsf{R}(n)$, and so an argument that demonstrated that $[S;q]$ is a diagonal derivative could also be used to demonstrate undecidability in the same way that Richard's argument can be used to demonstrate incompleteness without the additional move of deriving a contradiction.

9. Gödel [1], p. 9.

REFERENCES

[1] Gödel, K., "On formally undecidable propositions of *Principia Mathematica* and related systems," in *The Undecidable*, M. Davis (ed.), Raven Press, Hewlett, New York (1965), pp. 5-38.

[2] Humphries, J., *Automata Theory as a Model of Biological Replication, Adaptation and Evolution*, Ph.D. Thesis, University of Waterloo (1973).

[3] Mostowski, A., *Sentences Undecidable in Formalized Arithmetic*, North-Holland Publishing Co., Amsterdam (1952).

[4] Tucker, J., "Gödel and Epimenides," *Proceedings of The Aristotelian Society*, vol. 58 (1958), pp. 25-48.

[5] Tucker, J., "Constructivity and grammar," *Proceedings of The Aristotelian Society*, vol. 62 (1962), pp. 45-66.

[6] Tucker, J., *Lectures on the Foundations of Mathematics*, Unpublished monograph, University of Waterloo (1967).

University of Pittsburgh at Johnstown
Johnstown, Pennsylvania