Notre Dame Journal of Formal Logic
Volume 26, Number 1, January 1985

Validity in Intensional Languages: A New Approach

WILLIAM H. HANSON and JAMES HAWTHORNE

Although the use of possible worlds in semantics has been very fruitful and is now widely accepted, there is a puzzle about the standard definition of validity in possible-worlds semantics that has received little notice and virtually no comment. A sentence of an intensional language is typically said to be valid just in case it is true at every world under every model on every model structure of the language. Each model structure contains a set of possible worlds, and models are defined relative to model structures, assigning truth-values to sentences at each world countenanced by the model structure. The puzzle is why more than one model structure is used in the definition of validity. There is presumably just one class of all possible worlds and just one model structure defined on this class that does correctly the things that model structures are supposed to do. (These include, but need not be limited to, specifying the set of individuals in each world as well as various accessibility relations between worlds.) Why not define validity simply as truth at every world under every model on this one model structure? What is the point of bringing in more model structures than just this one?

We investigate these questions in some detail and conclude that for many intensional languages the puzzle points to a genuine difficulty: the standard definition of validity is insufficiently motivated. We begin (Section 1) by showing that a plausible and natural account of validity for intensional languages can be based on a single model structure, and that validity so defined is analogous in important respects to the standard account of validity for extensional languages. We call this notion of validity "validity$_1$", and in Section 2 we contrast it with the standard notion, which we call "validity$_2$". Several attempts are made to discover a rationale for the almost universal acceptance of validity$_2$, but in most of these attempts we come up empty-handed. So in Section 3 we investigate validity$_1$ for some intensional languages. Our investigation includes providing axiomatizations for several propositional and predicate logics, most of which are provably complete. The completeness proofs are given in the Appendix, which also contains a sketch of a compactness proof for one of the predicate logics.

Received October 31, 1983; revised March 6, 1984

1 Truth and validity in extensional and intensional languages By an *extensional language* we mean one containing (at most) the usual quantifiers and truth-functional connectives, identity, individual constants and variables, sentence letters, and n-ary predicate letters ($n \geq 1$). The notion of an *interpretation* (a nonempty set plus an assignment of truth values to sentence letters, objects from the set to individual constants, and n-ary relations on the set to n-ary predicate letters) is taken as basic and understood, as are the usual definitions of satisfaction and truth under an interpretation. Interpretations of extensional languages will sometimes be called *E-interpretations* in order to distinguish them from interpretations of intensional languages to be defined later. The assignments made by an E-interpretation to the nonlogical symbols of an extensional language will be called the *extensions* of these symbols under the E-interpretation in question. An extensional language L will often have an *intended interpretation* which provides the *intended extensions* of its nonlogical symbols. Given all of these notions, we can say that a sentence Φ of an extensional language L is *true* iff Φ is true under the intended interpretation of L, and that Φ is *valid* or *logically true* iff it is true under all interpretations of L. A valid sentence is thus one that comes out true no matter what nonempty set is taken as the range of its variables and no matter what extensions based on this set are assigned to its nonlogical symbols.

The aim of this definition of validity is clearly to capture the intuitive notion of truth under all possible circumstances (or in all possible worlds) for every extensional meaning assignment to nonlogical terms. The definition as stated may actually go somewhat beyond this goal, however, for it allows any nonempty set as the domain of an E-interpretation. In so doing, it bypasses such questions as "Are there possible worlds containing only a finite number of individuals?" If mathematics is necessary (i.e., true in all possible worlds) and if mathematical objects are treated realistically, the answer to this question will be "No". But the definition of validity just given is designed to be as metaphysically accommodating as possible. It allows as domains of interpretations the most austere and bizarre worlds that anyone might think possible. Thus, if a sentence is valid, it is surely true under all meaning assignments in all possible worlds, but, depending on one's theory of possible worlds, there might be sentences true under all meaning assignments in all possible worlds that are not valid. There are two reasons why this apparent defect in the definition of validity is not as serious as it may seem. First, by countenancing the domain of each interpretation as the formal counterpart of a possible world it avoids intractable disputes about the nature and number of possible worlds and thereby yields a notion of validity that is precise and capable of being investigated by rigorous methods. Second, it could be argued that our intuitive notion of validity encompasses not just truth under all meaning assignments in all possible worlds, but truth when interpreted on or applied to any problem or subject matter as well. If this is correct, then the fact that the definition of validity refers to all interpretations is a virtue. For any interpretation can be thought of as the domain of a problem or subject matter along with a meaning assignment connecting the objects involved to the language. Actually we don't consider the apparent defect a defect at all, for we are willing to construe possible worlds broadly enough to include the domain of any subject matter. On this broad construal, each nonempty set of objects constitutes a possible world.

The foregoing is presented as a standard against which to measure definitions of truth and validity for intensional languages. By an *intensional language* we mean one that contains, in addition to the resources of extensional languages, sentential connectives for necessity, obligation, counterfactual conditionals, or other such notions. Pressed to be more precise about what we mean by an intensional language, we would say that it is one that contains intensional contexts, where an intensional context is one within which all and only expressions having the same intension can be exchanged *salva veritate*. Formal accounts of truth and validity for intensional languages often define the *intension* of an expression as a function that assigns to each possible world w the extension of that expression at w. On this account a *proposition* is a function from worlds to truth values, an n-ary *relation-in-intension* is a function from worlds to sets of n-tuples of individuals, an *individual concept* is a function from worlds to individuals. These notions can be made precise using familiar model-theoretic methods due to Kripke and others. Thus if L is a language containing n non-truth-functional connectives, a *model structure* for L is a sequence $(G, K, R_1, \ldots, R_n, \psi)$, where K is a nonempty class (called an index class), $G \in K$, the R_i are relations on K satisfying certain formal constraints (e.g., reflexivity, symmetry, transitivity), and ψ is a function such that for each index $w \in K$, $\psi(w)$ is a nonempty set. A *model* is then defined as an ordered pair $(v, \mathcal{M})$, where $\mathcal{M}$ is a model structure and v is a valuation function that makes an assignment to each sentence and predicate letter of L at each index (i.e., each member of the index class) of $\mathcal{M}$. These assignments then lead, via familiar recursion rules, to a truth value for each sentence of L at each index of $\mathcal{M}$.

This theory of models provides us with a precise way of defining intensions as functions from possible worlds to extensions, but it also allows much more. For although it will be convenient to think of the index class K of any model structure as a class of possible worlds, not all such classes really contain possible worlds, even on our broad construal of that notion. And obviously only one such class is the class of all possible worlds. Thus to give a formal account of intensions as functions on possible worlds it would seem we should restrict our attention to just one model structure, call it $\mathcal{S}$, in which K is the class of all possible worlds, G is the actual world, $\psi(w)$ is the set of all the things that exist in w, for each $w \in K$, and each R_i is a relation between possible worlds (a world accessibility relation) that forms the basis of a recursion rule for one of the non-truth-functional connectives of L. Intensions can now be defined as functions from the indices of $\mathcal{S}$ to extensions definable using $\mathcal{S}$. Each valuation v of a model $(v, \mathcal{S})$ thus assigns an intension to each sentence of L. Many different models can of course be defined on $\mathcal{S}$, and there may be one model on $\mathcal{S}$, call it $(m, \mathcal{S})$, that provides what we would ordinarily call *the* intension (or perhaps the intended intension) of each nonlogical symbol of L. For example if L contains the predicate 'is red', then m will assign to this predicate at each world w the set of things that are red in w. Thus the intension, or meaning, of 'is red' is on this analysis a function whose value for a world w is the set of things red in w. We call $\mathcal{S}$ the *intended model structure* and m the *intended meaning assignment* for L.

Words from natural languages, such as 'red', are not usually included in formal languages, so there is usually no *standard* intended intension for the nonlogical symbols of such languages. The interest in an intended meaning assignment is thus usually transient; a user exploits a specific meaning assignment only for some immediate purpose. In contrast, the intended model structure $\mathcal{S}$ is eminent among model structures for intensional languages, for it is central to the *theory of meaning* of such languages.[1] This centrality is reflected in three ways. First $\mathcal{S}$ provides the basis for valuation functions that are genuine meaning assignments (not just functions defined on some arbitrarily selected index set) and which therefore determine an extension for each nonlogical symbol at each possible world. Second, since the actual world is an index of $\mathcal{S}$, each meaning assignment defined on $\mathcal{S}$ determines what we might call the extension *simpliciter*—the extension at the actual world—of each nonlogical symbol. Finally, the accessibility relations of $\mathcal{S}$ fix the meanings of the intensional connectives and prevent them from varying from model to model.[2]

If L is an intensional language, we define an *interpretation* of L (also called an *I-interpretation*) as an ordered pair (v, w) consisting of a valuation v defined on $\mathcal{S}$ and a world w from the index class K of $\mathcal{S}$. An I-interpretation is similar to an E-interpretation in that it provides a set that functions as a domain of quantification and an assignment to each nonlogical symbol of the language. For the I-interpretation consisting of the valuation v and the world w the set in question is $\psi(w)$, and the value assigned to a nonlogical symbol is the intension of that symbol under v. An I-interpretation also provides the basis for an assignment of truth values to sentences of L. For (v, w) these will be the truth values assigned to sentences at w under v. We shall call (m, G) the *intended I-interpretation* of L, and say that a sentence of L is *true* iff it is true under (m, G). This definition of truth is intuitively correct, since m gives us the intended intensions of the expressions of L, and truth is just truth at the actual world. It is also exactly analogous to the definition of truth for extensional languages given above: truth under the intended interpretation. This account of truth, or something very much like it, is the basis of several contemporary attempts to explain certain non-truth-functional contexts.[3]

It is useful to frame definitions of semantical terms for intensional languages in terms of I-interpretations because this facilitates comparison with definitions of the corresponding terms for extensional languages. Thus we have already seen that truth turns out to be truth under the intended interpretation for both extensional and intensional languages. Another term definable using I-interpretations (although no parallel term is definable for extensional languages using E-interpretations) is *analyticity*. Analytic sentences have been thought of as those whose truth depends solely on the meanings of the words they contain. If we identify the meanings of nonlogical words with intensions, then we can plausibly define the analytic sentences of L as those that are true under every I-interpretation containing the valuation m. Thus an analytic sentence is one whose truth depends on the intensions of the expressions it contains (hence the restriction to m), but not on any nonlinguistic fact (hence the consideration of all possible worlds).

We can also use I-interpretations to give plausible definitions of logical truth and validity. A sentence ϕ of L is *logically true* iff ϕ is true under every

I-interpretation. Logical truth is thus equivalent to truth at every world under every valuation definable on the model structure $\mathcal{S}$. Since consideration of all valuations definable on $\mathcal{S}$ exhausts the ways in which intensions can be assigned to nonlogical symbols of L, logical truth amounts to truth in all possible worlds under all assignments of intensions. Our account of validity is now simply to identify it with logical truth: a sentence ϕ of L is *valid* iff ϕ is logically true. This account of validity seems plausible in its own right, and thinking of it in terms of I-interpretations shows that it is analogous to the definition of validity for extensional languages. The latter point is worth emphasizing. In determining validity for both extensional and intensional languages we consider the result of taking different sets of individuals as the range of the quantifiers (in the intensional case these are the different sets of individuals existing in the various possible worlds) and then making different assignments to the nonlogical symbols relative to this choice of a domain of quantification (in the intensional case these are the different assignments of intensions). When a sentence comes out true under all such assignments relative to all such domains, we consider it valid.

We believe that the semantical notions we have defined for intensional languages are natural, plausible, and interesting. There is, however, a potentially unsatisfactory feature of these definitions analogous to that mentioned earlier in our discussion of extensional languages. This is the fact that they all refer to the model structure $\mathcal{S}$ whose index set K is the set of all possible worlds. Since the nature and number of possible worlds are matters of dispute and puzzlement, formal work in the semantics of intensional languages may seem all but impossible if $\mathcal{S}$ and K are understood in this way. We think the proper response to this worry is the one we mentioned earlier in connection with extensional languages: some idealization of the notion of a possible world is needed if semantics is to yield any interesting results, and that notion should be construed broadly rather than narrowly. Using this general approach we should seek to determine what formal contraints $\mathcal{S}$ and K must satisfy in order for interesting theorems to hold, and then ask ourselves if these contraints count for or against a particular metaphysical view of possible worlds. There are many such views, and we now offer sketches of three of them. What is most noteworthy about these views, from our point of view, is that as long as they satisfy certain minimal constraints, they are all compatible with the formal results presented in Section 3.

One way of understanding possible worlds is what we previously called the broad construal. On this view any nonempty set of objects constitutes a possible world, and so we could take K to be the class of all nonempty sets and $\psi(w)$ to be w, for each $w \in K$. This would be a nominalistic view of possible worlds. A world is fully individuated only by the totality of objects which exist in it. Objects are primitive on this view. They cannot be individuated in terms of anything more primitive.

Alternatively, we may want to assign the same set to many different possible worlds so as to reflect the intuitive idea that exactly the same individuals may exist in more than one world. On this approach K and ψ can't be specified as succinctly as on the former. K would have to be some suitably large index class and ψ a not entirely trivial function defined on K which has the class of all nonempty sets as its range. This alternative accommodates a pluralistic realist metaphysics. One version of this view is that a world is fully individuated by the totality of its objects and the properties and relations they exhibit there. Another "more metaphysical" version holds that two different worlds may contain precisely the same objects exhibiting precisely the same properties and relations. In both versions objects, properties, and relations are primitives; they cannot be individuated in terms of anything more primitive. In the second version, possible worlds are primitive as well.

A third alternative for K and ψ arises from a monistic realist metaphysics. This is the view that objects are only fully individuated by the totality of properties and relations they exhibit, and a world is fully individuated by the totality of its objects. Only the properties and relations are primitive on this view. If two worlds have an object in common, then they must have all objects in common, since the object they share must exhibit precisely the same properties and relations to the other objects. Cross-world identification of objects is purely conceptual on this view. An object in one world may have one or more counterparts in another world, but no object inhabits more than one world.

Although the metaphysical views motivating these three approaches differ significantly from one another, the difference between the approaches themselves is insignificant for some interesting results in logic, assuming that on each approach K is at least denumerable, and that for each nonzero countable cardinal τ, there are at least denumerably many $w \in K$ such that the cardinality of $\psi(w)$ is τ. (These assumptions are used in the completeness proofs given in the Appendix. Other assumptions needed only for the system *PAN* are given in Section 3.) Cardinality considerations such as these have been sufficient for important results in semantical investigations of extensional languages. Section 3 and the Appendix show that the present assumptions yield interesting results for intensional languages.

The notion of validity defined above, when made precise by specifying K and ψ in the way just discussed, we call *validity*$_1$. It is the main object of study in Section 3. While we find validity$_1$ a natural, interesting, and, as Section 3 will show, fruitful notion, it does have one curious feature: it is quite different from the definition of validity for intensional languages commonly found in the literature. That definition is that a sentence ϕ of an intensional language L is valid if and only if ϕ is true at every index under every model *on every model structure* for L. We call this notion *validity*$_2$. In the next section we discuss some relations between validity$_1$ and validity$_2$, and inquire about the appropriateness of validity$_2$.

2 Validity$_1$ and validity$_2$ If K and ψ of the single model structure $\mathcal{S}$ involved in validity$_1$ can be specified in a way that makes K plausibly represent the set of all possible worlds, then validity$_1$ is far more intuitively satisfying than validity$_2$. We think our comments on truth, analyticity, logical truth, and validity$_1$ in the previous section make this clear. What then is the reason for the widespread interest in validity$_2$? We seriously doubt that it is due to others having failed in their attempts to give precise and plausible specifications of validity$_1$, since we know of virtually no one other than Carnap and Kanger who has even tried.[4] In Section 3 we report on our efforts in this direction and show that some interesting formal results about validity$_1$, including recursive axiomatizability, are provable for certain intensional languages. In this section, however, we consider several reasons that might be given for studying validity$_2$ and find most of them seriously deficient.[5]

The first thing to notice is that validity$_2$ is in an obvious way a generalization of validity$_1$, for validity$_2$ involves not only the model structure $\mathcal{S}$ but every model structure that has some of the same formal properties as $\mathcal{S}$. (The formal properties in question are of course the conditions on the accessibility relations R_i that must be met by each model structure of a language L.) But the mere fact that validity$_2$ generalizes in this way on a plausible and interesting notion of validity does not make it plausible or interesting. How might it be thought to? Are we to think of the model structures other than $\mathcal{S}$ that are involved in the definition of validity$_2$ as representing ways that possible worlds might have been? Surely this is unsatisfactory. The whole point of invoking possible worlds in the first place was to provide a foundation for our notion of possibility. Any attempt to analyze this notion in terms of possible sets of possible worlds would be gratuitous.

A better way of understanding validity$_2$ may be to think of the model structures other than $\mathcal{S}$ not as shadowy representatives of the ways possible worlds might have been, but rather as providing a variety of different kinds of interpretations of the language in question. This approach to validity$_2$, which we call the structural approach, may be what Thomason has in mind when he says

> I should mention at this point that I am conceiving of the notion of a model structure (and hence of a possible world), as used in the semantics of modal logic, as an abstract or structural concept having a number of kinds of realizations.... For instance, possible worlds may be interpreted temporally, metaphysically, linguistically (as "state descriptions") or probabilistically (as sample points in a probability space).[6]

On the structural approach, the mere fact that two model structures share certain formal features (e.g., that a certain relation between possible worlds and another relation between moments of time are both reflexive and transitive) is seen as sufficient for using both in determining whether a sentence is valid$_2$. A valid$_2$ sentence is thus one which expresses a truth of logic when its intensional connectives are read in a variety of different ways. The trouble with validity$_2$, when so understood, is that it can't be expected to give us a complete account of the logic of any of these readings. For instance, if we are interested in alethic modality, we cannot take the fact that a sentence is invalid$_2$ to indicate that it fails to be a truth of logic when its intensional connectives are read as alethic modalities. It may only indicate that it fails of logical truth when its intensional connectives are given temporal readings. Thus on the structural approach, validity$_2$ can be thought of as giving a complete account only of what is common to the logic of a great many intensional connectives, most of which are never even specified.

Once it becomes clear that the structural approach leads to this curious interpretation of intensional connectives, it loses much of its interest. For surely we want a model-theoretic semantics to give us, for each operator in the language, a specific reading in which we are interested. We want it to give us complete accounts of both truth and validity for the language, where each operator is understood in a specific, fixed way. The structural approach does not do this. If someone nevertheless insists that the structural approach gives an adequate justification of validity$_2$, we can only point out that he or she considers it adequate to treat some operators of a language differently than others. For it is only the intensional connectives, not the truth-functional connectives or quantifiers, that must on this approach be thought of as embodying the common structural features of several different connectives.[7]

Having mentioned temporal locutions in our discussion of the structural approach to validity$_2$, it seems appropriate to say something about tense logics at this point. Expressions like "It is and always will be the case that" are similar to intensional contexts, and the standard semantics for tense logic is similar to that for intensional logic, but based on an index set containing moments of time rather than possible worlds. This is quite appropriate if the only nonextensional connectives in the language are temporal ones. What we are interested in is the treatment of validity usually found in this kind of semantics, a treatment quite similar to validity$_2$. We think this treatment may be justified if a certain view about the nature of time is correct, even though we see no general justification of validity$_2$ for intensional languages. Consider two alternatives. If we believe that the logic of temporal locutions is such that the moments of time must be ordered as, say, the real numbers taken in order of magnitude, then it will be proper to employ in the semantics of tensed languages only model structures that have the real numbers as their index sets. Indeed, if it were not for the fact that we want to allow that different individuals might exist at a given moment of time, we could restrict attention to a single model structure. If on the other hand we believe that logic dictates only that the moments of time are, say, simply ordered but that the structure of time is otherwise contingent, then the semantics must contain model structures with index sets having many different structures.[8] If we adopt the former approach to tense logic we are claiming that there is only one possible structure of time, but if we adopt the latter approach we are allowing that time may have any one of a large class of structures. There are rough but obvious similarities between validity$_1$ and the former approach, and between validity$_2$ and the latter. And the latter approach has some intuitive plausibility in tense logic, for it makes sense to say that although time in fact has a certain structure, it might have had any of a number of other structures. Yet the analogous statement about possible worlds—that the structure of possible worlds might have been different than it is—does not make sense. The difficulties in finding any way of making sense of validity$_2$ are persistent, and the widespread interest in validity$_2$ remains puzzling.

Perhaps the most promising way of justifying that interest (although it may not explain it historically) is to be found in recent work in pragmatics. The basic idea is that the proposition expressed by a sentence depends on the context in which the sentence is uttered. Thus Stalnaker [24], for example, takes what he calls an interpreted sentence to be a function from contexts to propositions, where a proposition is, as usual, a function from possible worlds to truth values. And he believes that not only sentences containing demonstratives and indexicals, but also those containing modal expressions (which are, of course, intensional sentences) should be dealt with in this way.

> When you say "We shall overcome", I need to know who you are, and for whom you are speaking. If you say "that is a great painting", I need to know what you are looking at, or pointing to, or perhaps what you referred to in your previous utterance. Modal terms also are notoriously dependent on context for their interpretation. For a sentence using *can*, *may*, *might*, *must* or *ought*, to determine a proposition unambiguously, a domain of 'all possible worlds' must be specified or intended. It need not be *all* conceivable worlds in any absolute sense, if there is such a sense. Sentences involving modals are usually to be construed relative to all possible worlds consistent with the speaker's knowledge, or with some set of presuppositions, or with what is morally right, or legally right, or normal, or what is within someone's power. Unless the relevant domain of possible worlds is clear in the context, the proposition expressed is undetermined. [24], pp. 384–385

If Stalnaker is right in thinking that the possible worlds that figure in the semantical analysis of at least some intensional languages are best thought of as relative to contexts, then validity$_2$ becomes a much more plausible notion. Each of the different model structures used in the definition of validity$_2$ can now be thought of as representing the worlds possible relative to some context. The formal semantical analysis expressed in the notion of validity$_2$ takes the form it does because of the informal pragmatic base on which it rests. Of course there is a certain idealization involved here. Since any nonempty class K can serve as the index class of a model structure, acceptance of validity$_2$ tacitly endorses the idea that for any (finite or infinite) cardinal A, there is a context relative to which there are just A possible worlds. But this is not a serious objection. The notion of a context is too vague to allow us to choose the index classes of model structures in any other way. This idealization is of a piece with the one we noted in Section 1 that allowed any nonempty set to be the domain of an E-interpretation. Without such idealizations, formal work in semantics would be all but impossible.

We thus seem to have found a way of justifying the interest in validity$_2$ for intensional languages after all, at least if the contention that intensional expressions are context-relative is correct. The main difficulty with this justification is that when logicians attempt to formalize the pragmatic notions on which it is based, what they come up with does not always fit the justification. Kaplan's "On the logic of demonstratives" [11], for example, incorporates contexts as well as possible worlds into model structures, and it treats propositions as the values of functions whose arguments are contexts. But the set of possible worlds dealt with in a structure is fixed for the structure; it does not vary with contexts. Furthermore, different structures may contain different sets of contexts, and validity is defined as truth at every context in every structure. This latter move makes it look as though contexts themselves are relative to something, and that this is why many different sets of them (each represented by a different structure) are used in the definition of validity. But to what are contexts relative, we may ask, and why? These questions are analogous to the ones about possible worlds and validity$_2$ with which we began.

In spite of these difficulties with formal work in pragmatics, we admit that the approach advocated in the quotation from Stalnaker does provide a coherent and not implausible way of understanding validity$_2$. We seriously doubt, however, that much of the formal work done on validity$_2$ in the last twenty-five years has been motivated by this way of understanding things, and we suspect that many of the logicians who have done this work would not even accept it as a rationale for their work. What is needed is greater awareness on the part of intensional logicians of philosophical questions that arise about the most basic concepts with which they deal. Much more philosophical discussion would be required before the Stalnaker approach could be confidently adopted as the rationale for validity$_2$.

Before presenting our own alternative to validity$_2$ in Section 3, we want to consider one more attempt to invest this notion with interest. Perhaps we have made a mistake in concentrating so much of our attention on validity$_2$. For if validity$_2$ could be shown to coincide with validity$_1$, then it could be seen as a mere technical device for studying an intrinsically interesting notion of validity. Indeed, this may be possible for some intensional languages—those that can be shown to have universal model structures. A universal model structure for a language is one such that a sentence is valid under this model structure (i.e., true at all indices under all models on this structure) iff it is valid$_2$. Universal model structures exist for the most common systems of propositional modal logic as well as for some quantified modal systems (e.g., the systems $T+BF$, $S4+BF$, $LPC+S5$ of Hughes and Cresswell [5], Chapters 8–10, all of which contain the Barcan formula, as well as $LPC+T$ and $LPC+S4$, which do not). If the universal model structure of a system can be plausibly identified with the model structure $\mathcal{S}$ used in the definition of validity$_1$, then, for the language in question, validity$_1$ and validity$_2$ coincide. Unfortunately there is an important class of intensional languages that have no universal model structures. Consider any intensional language L containing a connective for $S5$ logical necessity (i.e., a connective L such that $L\phi$ is true at any index under a model iff ϕ is true at every index under this model). It is easy to show that the sentences

(a) $\sim L(\exists x)(y)y=x$

and

(b) $\sim L(\exists x)(\exists y)(x\neq y\,\&\,(z)(z=x\vee z=y))$

are not valid$_2$ in L. For it is always possible to specify a model structure of L in which each domain associated with an index contains the same finite number of individuals. If that number is one, this model structure will invalidate (a); if it is two, it will invalidate (b). And it is clear that the only way a model structure of L can invalidate sentences like (a) and (b) is by associating domains of the same cardinality with each index. But then no single model structure can invalidate both (a) and (b). Hence there is no universal model structure for validity$_2$ in L. So no matter how we specify $\mathcal{S}$ for L, validity$_1$ will not coincide with validity$_2$.

Of course it may be possible to show that some intensional languages do have universal model structures that can plausibly be identified with $\mathbb{S}$, but the burden of proof is on the advocate of valildity$_2$. In any event, we think we have said enough to cast serious doubt on the wisdom of treating validity$_2$ as the standard notion of validity for intensional languages. In the next section we provide an alternative approach.

3 Formal treatment of validity$_1$ We have criticized the way semantics for intentional languages is usually developed and interpreted, and we have suggested a more plausible line. In particular, we have suggested that the interesting notion of validity is validity$_1$ because it is defined to coincide with logical truth on the intended model structure (i.e., truth at all possible worlds under all meaning assignments to terms). Now we will discuss, in rapid order, several semantical systems for intensional languages which exemplify our preferred approach. Among these will be propositional and quantified systems for a single modality interpreted, alternatively, as logical necessity, analytic necessity, and nomic (e.g., physical) necessity. The treatment of "contingent necessity" in the system for nomic necessity suggests a reason for continued interest in validity$_2$.

3.1 PL and QL The language of *PL* consists of a standard language for propositional logic and a modal operator 'L'. Sentences of *PL* are defined according to the usual recursive formation rules. Metalinguistic variables 'α', 'β', 'γ', etc. stand for sentences of *PL*, and '$(\alpha \,\&\, \beta)$', '$\sim\alpha$', and '$L\alpha$' stand for the conjunction, negation, and logical necessitation of sentences of *PL*. The semantics of *PL* is defined on the intended model structure (G, K, ψ) discussed in Section 1. Meaning assignments to sentence letters proceed as follows: for each $w \in K$ and v a valuation function for *PL*, and for each proposition letter α of *PL*, $v(\alpha, w) \in \{t, f\}$. Truth of a sentence under a meaning assignment at a world is defined using the usual recursion clauses. For each $w \in K$ and valuation function v:

(1) for each proposition letter α, $T(\alpha, v, w)$ iff $v(\alpha, w) = t$
(2) for each sentence α of *PL*, $T(\sim\alpha, v, w)$ iff not $T(\alpha, v, w)$
(3) for each sentence α and β of *PL*, $T((\alpha \,\&\, \beta), v, w)$ iff $T(\alpha, v, w)$ and $T(\beta, v, w)$
(4) for each sentence α of *PL*, $T(L\alpha, v, w)$ iff $\forall v' \forall w' T(\alpha, v', w')$.

Analytic truth of a sentence under a meaning assignment v is defined as truth under that assignment at all worlds: $AT(\alpha, v)$ iff $\forall w \in K\ T(\alpha, v, w)$. Logical truth is defined as analytic truth under all meaning assignments: $LT(\alpha)$ iff $\forall v AT(\alpha, v)$. The valid sentences are just the logically true ones: $Val_1(\alpha)$ iff $LT(\alpha)$.

In *PL* the notion of logical truth is expressed in the object language by the modal operator L, since for any given v, w, and α the following holds: $T(L\alpha, v, w)$ iff $LT(\alpha)$. All theorems of propositional *S5* are valid sentences of *PL*. The following axiom schemata and rules of inference, which characterize *S5*, are sound for *PL*, where $(\alpha \supset \beta)$ and $M\alpha$ abbreviate $\sim(\alpha \,\&\, \sim\beta)$ and $\sim L \sim\alpha$, respectively:

A1 *If* α *is an instance of a tautology (of propositional logic), then* $\vdash\alpha$.
A2 *If* $\vdash\alpha$ *and* $\vdash\alpha\supset\beta$, *then* $\vdash\beta$.
A3 *If* $\vdash\alpha$, *then* $\vdash L\alpha$.
A4 $\vdash L\alpha\supset\alpha$.
A5 $\vdash L(\alpha\supset\beta)\supset(L\alpha\supset L\beta)$.
A6 $\vdash M\alpha\supset LM\alpha$.

But these axioms are not complete. For example, any sentence of form '$M\alpha$' where α is a proposition letter is valid in *PL* (i.e., for any such α there is some v and w for which $v(\alpha, w)=t$, so $\forall v'\forall w' T(\sim L\sim\alpha, v', w')$). Still, membership in the set of valid sentences of *PL* is decidable. For any sentence α of *PL* there is an effective procedure for producing a logically equivalent sentence in modal conjunctive normal form, i.e., a sentence which is a conjunction of sentences of the form $(L\alpha_1 \vee \ldots \vee L\alpha_n \vee M\beta \vee \gamma)$, where the α_i, β, and γ are nonmodal sentences of propositional logic. This modal conjunctive normal form theorem holds because the *S*5 axioms are sound. So α is (decidably) valid just in case each conjunct is (decidably) valid. And a conjunct is (decidably) valid just in case some sentence of form $(\alpha_i \vee \beta)$ is a tautology, or $(\gamma\vee\beta)$ is a tautology.[9]

QL is just *PL* extended to first-order quantifier logic with identity. Here 'α', 'β', 'γ', etc. represent (possibly open) well-formed formulas; 'x', 'y', 'x_1', 'x_2', etc. represent free variables; '$=$' represents identity; and '(x)', '(y)', etc. represent universal quantifiers. The semantics is again defined on the intended model structure (G, K, ψ). The meaning assignments are the valuations v such that: for each proposition letter α and $w\in K$, $v(\alpha, w)\in\{t, f\}$; for each n-ary predicate letter α and $w\in K$, $v(\alpha, w)\subseteq\psi(w)^n$. Note that for *QL* a meaning assignment to a predicate letter α always specifies the extension of α at a world as a collection of things *in that world*. Assignments of things to free variables are functions s such that for every x and $w\in K$, $s(x, w)\in\psi(w)$. These functions are like meaning assignments in that what they assign to a symbol at a world is always something from that world.

Satisfaction is a quarternary relation such that for each v, s, and $w\in K$:

(1) $sat\,(x=y, v, s, w)$ iff $s(x, w)=s(y, w)$
(2) for α a proposition letter, $sat\,(\alpha, v, s, w)$ iff $v(\alpha, w)=t$
(3) for α an n-ary predicate letter, $sat\,(\alpha x_1\ldots x_n, v, s, w)$ iff $\langle s(x_1, w), \ldots, s(x_n, w)\rangle\in v(\alpha, w)$
(4) $sat\,(\sim\alpha, v, s, w)$ iff not $sat\,(\alpha, v, s, w)$
(5) $sat\,((\alpha\,\&\,\beta), v, s, w)$ iff $sat\,(\alpha, v, s, w)$ and $sat\,(\beta, v, s, w)$
(6) $sat\,((x)\alpha, v, s, w)$ iff $\forall s'(\forall y(s'(y, w)=s(y, w)$ or y is $x)$ only if $sat\,(\alpha, v, s', w))$
(7) $sat\,(L\alpha, v, s, w)$ iff $\forall v'\forall s'\forall w'\ sat\,(\alpha, v', s', w')$.

Metalinguistic predicates for truth at a world under a meaning assignment, analytic truth under a meaning assignment, and logical truth are defined as follows:

$T(\alpha, v, w)$ iff $\forall s\ sat\,(\alpha, v, s, w)$
$AT(\alpha, v)$ iff $\forall w T(\alpha, v, w)$
$LT(\alpha)$ iff $\forall v AT(\alpha, v)$.

Validity is again defined to coincide with logical truth: $Val_1(\alpha)$ iff $LT(\alpha)$. Again, the notion of logical truth is expressed in the object language by 'L', since for any given v, w, α, and s, we have $sat\,(L\alpha, v, s, w)$ iff $LT(\alpha)$.

The axiom schemata and inference rules of propositional *S*5, together with the following axiom schemata and rules for predicate logic with identity, an anti-essentialist axiom, and an infinite list of logical possibility axioms, are sound in *QL*. (Here $\supset$, $\equiv$, $\vee$, $\exists$, and M are the obvious abbreviations).

A7 *If* $\vdash \alpha$, *then* $\vdash (x)\alpha$.

A8 $\vdash (x)\alpha x \supset \alpha y$, *where* αy *is the result of the substitution of* y *for just those occurrences of* x *in* αx *which are free and outside the scope of modal operators, and* y *is free in* αy *wherever* x *is free in* αx.

A9 $\vdash (x)(\alpha \supset \beta) \supset (\alpha \supset (x)\beta)$, *provided every occurrence of* x *in* α *is bound or within the scope of a modal operator.*

A10 $\vdash x = x$.

A11 $\vdash (x = y \supset (\alpha x \equiv \alpha y))$, *where* αy *is the result of the substitution of* y *for just those occurrences of* x *in* αx *which are free and outside the scope of modal operators, and* y *is free in* αy *wherever* x *is free in* αx.

A12 $\vdash (L\alpha \equiv L(x)\alpha)$.

LP$_1$ $\vdash MC_1$, *where* C_1 *is* $(\exists x_1)(x_2)x_2 = x_1$.

LP$_2$ $\vdash MC_2$, *where* C_2 *is* $(\exists x_1)(\exists x_2)(\sim x_1 = x_2 \,\&\, (x_3)(x_3 = x_1 \vee x_3 = x_2))$.

⋮

LP$_n$ $\vdash MC_n$, *where* C_n *says, in the obvious way, that there are exactly* n *things.*

These axioms are not complete for *QL*. But the theorems of this system form a subset of the valid sentences of *QL* which is closed under uniform substitution of open formulas for predicate letters (where the substituted formula contains the same variables free and outside the scope of modal operators as are free in the predicate letter substituted for).[10] A similar relation holds between the theorems of *S*5 and the valid sentences of *PL*. As with *PL*, sentences of the form $M\alpha$, for an atomic sentence α, are also valid.

Not only is this axiomatic system not complete for *QL*, *QL* is not recursively axiomatizable. If it were, then first-order predicate logic would be decidable. For, from a recursive axiomatization of *QL* one could generate an effective enumeration of its valid sentences. And in such an enumeration either α or $M \sim \alpha$ would eventually occur, for any sentence α of first-order logic. This would yield a decision procedure for first-order logic.

3.2 PA and QA *PA* and *QA* are propositional and first-order systems for analyticity. Their syntax is that of *PL* and of *QL* with L replaced by A and M replaced by U. The semantics of *PA* is the same as that of *PL* but with semantical rule (4) replaced by the following:

(4′) For each sentence α of *PA*, $T(A\alpha, v, w)$ iff $\forall w' T(\alpha, v, w')$.

The semantics of *QA* copies that of *QL* except for rule (7):

(7′) $Sat\,(A\alpha, v, s, w)$ iff $\forall w' \forall s'\, sat\,(\alpha, v, s', w')$.

In the semantics of $PA(QA)$ truth at a world under a meaning assignment, analytic truth under a meaning assignment, and logical truth are all defined as for $PL(QL)$. A is an object language operator for analyticity in both systems: for any α, $T(A\alpha, v, w)$ iff $AT(\alpha, v)$.

In the L-systems sentences of form $M\alpha$, for atomic sentence α, are valid because:

$$\begin{aligned} LT(M\alpha) &\text{ iff } \forall v'\forall w' T(M\alpha, v', w') \\ &\text{ iff } \forall v'\forall w'\exists v\exists w T(\alpha, v, w) \\ &\text{ iff } \exists v\exists w T(\alpha, v, w). \end{aligned}$$

But, for α atomic, sentences of form $U\alpha$ are never valid in the A-systems. There is some v which assigns α the meaning of a contradiction (i.e., $AT(\alpha \equiv (B \,\&\, {\sim}B), v))$. For this v, not $\exists w T(\alpha, v, w)$. So not $LT(U\alpha)$, since

$$\begin{aligned} LT(U\alpha) &\text{ iff } \forall v\forall w' T(U\alpha, v, w') \\ &\text{ iff } \forall v\forall w'\exists w T(\alpha, v, w) \\ &\text{ iff } \forall v\exists w T(\alpha, v, w). \end{aligned}$$

The $S5$ axioms and inference rules for PL with L and M replaced by A and U, respectively, are sound and complete (see the Appendix) for PA. And since theoremhood for $S5$ is decidable, validity for PA is decidable. The axioms and inference rules for QL with the substitution of A and U for L and M are sound and complete (see the Appendix) for QA. QA is a quantified $S5$ system, but Axiom 12 and the AP-axioms (which copy the LP axioms) are nonstandard for such systems. The reference to free variables in the scope of modal operators in Axioms 8, 9, and 11 is also nonstandard. Axiom 11 makes QA a contingent identity system, and Axiom 12 is an anti-essentialist axiom. The AP axioms assert the analytic possibility that there are precisely n objects, for each natural number n greater than zero. They are valid for QA because there is a finite nonempty domain of each cardinality associated with some world in the intended model structure (G, K, ψ). This feature of QA also ensures the validity of all sentences of form $U((x)\alpha \vee (x){\sim}\alpha)$.[11] These sentences can be derived from (Axiom AP_1) together with the theorem $((\exists x_1)(x_2)x_2 = x_1 \supset ((x)\alpha \vee (x){\sim}\alpha))$. Indeed any sentence with a logical form that guarantees its truth at a world solely because of the size of the world's domain will be analytically possible.

Axiom 12 indicates the reason Axioms 8, 9, and 11 treat free variables in the scope of a modal operator as though they are bound. The analyticity operators effectively bind all free variables within their scope. ($U\alpha \equiv U(\exists x)\alpha$ is also derivable.) Semantically this seems plausible. The soundness of Axiom 12 flows from semantical rule (7′):

$$sat\,(A\alpha, v, s, w) \text{ iff } \forall w'\forall s'\ sat\,(\alpha, v, s', w)\ .$$

This rule is a plausible adaptation to the theory of analyticity of rule 7 for QL:

$$sat\,(L\alpha, v, s, w) \text{ iff } \forall v'\forall w'\forall s'\ sat\,(\alpha, v', s', w')\ .$$

Rule (7) is right. The logical truth of an n-ary open sentence applied to an n-tuple of objects should not depend on the meanings of predicates, or the nature of the world, or the natures of the particular objects involved. Logical truth concerns only logical form. Analytic truth concerns only logical form and the meanings of predicates. Thus, an open sentence is analytic of an object if and only if it is analytic of every object in every possible world.

In Section 1 we described several views on the individuation of worlds and their objects. On the monistic realist view there is no distinguished cross-world identification of objects. Objects are only individuated by the totality of properties and relations in which they participate in a world. Any linking of objects across worlds is merely conceptual. Cross world individuals are to be represented by individual concepts, and this can be done in terms of meaning assignments to predicate terms. An individual concept may be defined in the object language as follows: for monadic predicate term F, $A(x)(Fx \supset (y)(Fy \supset y = x))$. Semantically, $T(A(x)(Fx \supset (y)(Fy \supset y = x)), v, w)$ iff $\forall w'(v(F, w') = \varnothing$ or $(\exists a)(a \in \psi(w')$ and $v(F, w') = \{a\}))$. On this view individual concepts *define* cross-world individuals. The only legitimate sense in which a cross-world individual exists is as an individual concept. In the object language one can assert that the individual exists $((\exists x)Fx)$, that it satisfies a predicate, α, analytically $(A(x)(Fx \supset \alpha x))$, and that it is identical to or analytically identical to another individual $((x)(y)((Fx \,\&\, Gy) \supset x = y)$ and $A(x)(y)((Fx \,\&\, Gy) \supset x = y)$, respectively). QA suffices for the object language expression of all cogent propositions involving analyticity and individuals, i.e., all propositions cogent on the individual concept view of the cross-world identity of individuals.

All other metaphysical views described in Section 1 maintain that the individuation of objects is world independent. On these views the notion of an individual concept described above is still legitimate, but some individual concepts will pick out "real individuals" (i.e., they will pick out the same individual at each world in which it exits, and go uninstantiated at all other worlds). Those views do not invalidate the notion of analyticity defined in QA. But they do imply the cogency of another "more essentialist" notion of analyticity:

(7*) *sat* $(A^*\alpha, v, s, w)$ iff $\forall w' \forall s'$ $([\forall x$ free in α, $s'(x, w') = s(x, w)]$ only if *sat* $(\alpha, v, s', w'))$.

The antecedent on the right side of (7*) is needed because the assignments of individuals to variables, as defined at the beginning of this section, need not pick out the same object at different worlds for the same variable. (Redefining the assignments to do so would require patching in an account of how the variable is treated when the object doesn't exist at a world. Rule (7*) accomplishes the same thing.)

Let QA^* be the system obtained by replacing (7) by (7*) in QA. Sentences of the form $(A^*\alpha \equiv A^*(x)\alpha)$ are not generally valid in QA^*. In particular, for a monadic predicate letter 'F', there is a meaning assignment v and an individual $s(x, w)$ in $\psi(w)$ such that *sat* (A^*Fx, v, s, w) but not *sat* $(A^*(x)Fx, v, s, w)$. This can only happen if the individual in question (call it 'a') exhibits F wherever it appears, but not every individual does this (i.e., $\forall w'$ if $a \in \psi(w')$, then $a \in v(F, w')$, but $\exists w' \exists b \in \psi(w')$ such that $b \notin v(F, w')$). Only in cases where individuals like a are not picked out by an individual concept under a meaning assignment does the expressive power of QA^* differ from that of QA. If a is picked out by $v(G, \ )$ (i.e., if

$v(G, w') = \{a\}$ for $a \in \psi(w')$, and $v(G, w') = \varnothing$ otherwise) then we can get the previous effect in QA. For let a satisfy the conditions given in the two most recent parentheses and let $s(x, w) = a$. Then $sat((Gx \,\&\, A(x)(Gx \supset Fx)), v, s, w)$, but $sat(\sim A(x)Fx, v, s, w)$. So with the aid of individual concepts we can say in QA that a has $v(F, \ )$ analytically, and we can also say that not everything does. QA^* supercedes QA only in the ability of its object language to analytically attribute properties to bare, conceptually unrepresented, cross-world individuated particulars. The logic of QA^* may be worthy of a more thorough investigation, but not here. We don't know whether it is recursively axiomatizable.

QA^* has one other notable feature not shared by QA: $(x = y \supset A^*x = y)$ is valid in QA^*, whereas QA is a contingent identity system. Indeed, $(\sim Ax = y)$ is valid in QA. But if these features of QA seem implausible there is QA^* or an alternative system which lies between QA and QA^*. We call it $QA{=}$:

(7=) $sat(A = \alpha, v, s, w)$ iff $\forall w' \forall s'(\forall x \forall y$ free in $\alpha[s'(x, w') = s'(y, w')$ iff $s(x, w) = s(y, w)]$ only if $sat(\alpha, v, s', w'))$.

Rule (7=) requires its notion of analyticity to respect only the identifications of values of free variables, not their cross-world individuality.

$QA^=$ is not a contingent identity system; $(x = y \supset A^= x = y)$ is valid. But $QA^=$ is reducible to QA in the sense that for every sentence of $QA^=$ there is an equivalent sentence of QA. To see this first note that every open sentence $A^=\alpha$ with free variables $x_1 \ldots x_n$ of $QA^=$ is logically equivalent to a sentence of form $V_i(\pi_i \,\&\, A^=(x_1) \ldots (x_n)(\pi_i \supset \alpha))$, where each π_i is a distinct way of identifying and distinguishing values of all free variables of α (e.g., for some i, π_i is $(x_1 = x_2 \,\&\, x_1 \neq x_3 \,\&\, x_1 = x_4 \,\&\ldots\&\, x_3 = x_5 \,\&\ldots\&\, x_{n-1} \neq x_n))$ and 'V_i' represents the disjunction for each i of the formulas $(\pi_i \,\&\, A^=(x_1) \ldots (x_n)(\pi_i \supset \alpha))$. For non-modal β it is easily shown that $sat((\pi_i \,\&\, A^=(x_1) \ldots (x_n)(\pi_i \supset \beta)), v, s, w)$ in $QA^=$ iff $sat((\pi_i \,\&\, A(x_1) \ldots (x_n)(\pi_i \supset \beta)), v, s, w)$ in QA. This establishes the link between $QA^=$ and QA, for any formula of $QA^=$ successively replaces subformulas $A^=\alpha$ with $V_i(\pi_i \,\&\, A(x_1) \ldots (x_n)(\pi_i \supset \alpha))$ from the inside out. A straightforward mathematical induction on the depth of nested modal formulas of $QA^=$ establishes that the formula resulting from the operation is satisfied by $\langle v, s, w\rangle$ in QA just in case its twin in $QA^=$ is satisfied by this same sequence in $QA^=$.

The sematical predicates for analytic truth (AT) and logical truth (LT) in QA may be extended in a straightforward manner to predicates for analytic consequence (AC) and logical consequence (LC): for any (possibly infinite) set of QA sentences Γ and any QA sentence α,

$$AC(\Gamma, \alpha, v) \text{ iff } \forall s \forall w(\forall \beta \in \Gamma\, sat(\beta, v, s, w) \text{ only if } sat(\alpha, v, s, w))$$

and

$$LC(\Gamma, \alpha) \text{ iff } \forall v AC(\Gamma, \alpha, v) \ .$$

Under a given meaning assignment v, the analytic consequences of an infinite QA theory Γ may exceed the analytic consequences of each of Γ's finite parts. For proposition letter α let $v(\alpha, w) = t$ just in case w contains an infinite number of things (i.e., $\psi(w)$ is not finite). Let Γ be the infinite set $\{(\exists x)(\exists y) \sim y = x,\ (\exists x)(\exists y)(\exists z)(\sim x = y \,\&\, \sim x = z \,\&\, \sim y = z), \ldots\}$ (where the i^{th} sentence in our specification of Γ says that there are at least $i + 1$ distinct things). Then $AC(\Gamma, \alpha, v)$ but for each finite $\Delta \subseteq \Gamma$, not $AC(\Delta, \alpha, v)$. So analytic consequence is not compact for some meaning assignments.

Logical consequence is compact for QA. That is, $\forall\Gamma\forall\alpha$ in QA $LC(\Gamma, \alpha)$ iff for some finite $\Delta \subseteq \Gamma$, $LC(\Delta, \alpha)$ (see the Appendix). Let β be a conjunction of all sentences in such a Δ. Then $LC(\Delta, \alpha)$ iff $LT(\beta \supset \alpha)$, so the QA axioms provide a recursive axiomatization of the general notion of logical consequence in QA.

3.3 PAN *PAN* is a system for propositional modal logic containing the operator A for analytic necessity and the operator N for (some kind of) nomic necessity. All well-formed sentences of *PA* are well formed for *PAN*, and the result of substituting 'N' for some (or all) of the occurrences of 'A' in a *PA* sentence is a sentence of PAN.

We will construct a semantics for *PAN* in which $N\alpha$ asserts the physical necessity of α. An analogous construction would be equally appropriate to any kind of law-like, or nomic, necessity operator.

A first attempt might be to extend the intended model structure for *PA* to include the class of all physically possible worlds. So the intended model structure becomes (G, H, K, ψ), where G, K, and ψ are as before, H is the class of all physically possible worlds, and $G \in H$. The semantics is that of *PA* with the following additional clause in the definition of *truth*:

(5) $T(N\alpha, v, w)$ iff $\forall w'(w' \in H$ only if $T(\alpha, v, w'))$.

This approach is unsatisfactory. In this system $LT(N\alpha \supset AN\alpha)$, since for any meaning assignment v, $\exists wT(N\alpha, v, w)$ only if $AT(N\alpha, v)$. To illustrate the difficulty, let α be a proposition letter and v be a meaning assignment such that $\forall w(v(\alpha, w) = t$ iff w is a world in which the nonmodal sentences which state "the laws of Newtonian mechanics" hold true). *So* $N\alpha$ says, under v, that the laws of Newtonian mechanics are physically necessary. Then $\exists wT(N\alpha, v, w)$ iff $\forall w'(w' \in H$ only if $v(\alpha, w') = t)$. That is, $N\alpha$ is true under v at a world just in case all physically possible worlds satisfy the laws of Newtonian mechanics. But current physical theory indicates that the actual world, G, is not Newtonian; the laws, being false, are certainly not physically necessary. Thus, not $\exists wT(N\alpha, v, w)$; so $AT(\sim N\alpha, v)$. It is analytically true in this system that Newtonian mechanics is not physically necessary.

Newtonian mechanics is not physically necessary. But surely this is only contingently true, not analytic. Physical necessity is a "contingent necessity"; it is not a function of meaning alone. Our first attempt doesn't capture this feature. For modal systems generally the contingent element in the truth of a sentence under a meaning assignment is represented by the relativity of truth to a possible world. If the truth of a sentence of form $N\alpha$ is to be contingent, then truth under a meaning assignment must be relativized both to a possible world and a possible class of physically possible worlds. There is only one class of all possible worlds. But many of its subclasses are analytically possible candidates for the actual class of all physically possible worlds (just as any of its members is a candidate for the actual world).

The intended model structure for *PAN* is $(\langle G, H\rangle, J, K, \psi)$ where G is the actual world, H is the actual class of physically possible worlds, and K and ψ are as before (ψ plays no role until quantification is introduced). J is a class of pairs $\langle w, W\rangle$ in which $w \in W$ and W is a set of worlds physically possible with respect to each other. Although K continues to be the class of all possible worlds, we think of J as the class of all possible physical states of affairs, and we define truth relative to these rather than to possible worlds. $\langle G, H\rangle \in J$, and we assume J has the following properties: if $\langle w, W\rangle \in J$, then $\forall w'(w' \in W$ only if $\langle w', W\rangle \in J)$; $\forall w \in K\ \exists W \subseteq K$ such that $\langle w, W\rangle \in J$ (since what is true in any world must be physically possible there). Perhaps not every subset of K is a set of mutually physically possible worlds. But for *PAN* we will assume that there is a W of each nonzero countable cardinality such that for some $w \in W$, $\langle w, W\rangle \in J$. For it ought to be analytically possible that the number of physically possible worlds may be any finite or denumerable cardinality (or larger, though we won't assume so). Finally, we assume that at least denumerably many of the W such that for some $w \in W$, $\langle w, W\rangle \in J$, are themselves at least denumerable. (These last two assumptions are used in the completeness proof for *PAN* given in the Appendix, although in the presence of the latter assumption the former is not necessary.)

The meaning assignments v are now functions from elements $\langle w, W\rangle$ of J and proposition letters into $\{t, f\}$ (i.e., $v(\alpha, \langle w, W\rangle) \in \{t, f\}$). Truth is defined relative to meaning assignments and members of J:

(1) for proposition letter α, $T(\alpha, v, \langle w, W\rangle)$ iff $v(\alpha, \langle w, W\rangle) = t$
(2) for sentence α, $T({\sim}\alpha, v, \langle w, W\rangle)$ iff not $T(\alpha, v, \langle w, W\rangle)$
(3) for sentences α and β, $T((\alpha \& \beta), v, \langle w, W\rangle)$ iff $T(\alpha, v, \langle w, W\rangle)$ and $T(\beta, v, \langle w, W\rangle)$
(4) $T(N\alpha, v, \langle w, W\rangle)$ iff $\forall w'(w' \in W$ only if $T(\alpha, v, \langle w', W\rangle))$
(5) $T(A\alpha, v, \langle w, W\rangle)$ iff $\forall w' \forall W'(\langle w', W'\rangle \in J$ only if $T(\alpha, v, \langle w', W'\rangle))$.

'U' will abbreviate analytic possibility, as in *PA*, and 'P' will abbreviate physical possibility (i.e., $P\alpha$ abbreviates ${\sim}N{\sim}\alpha$).

PAN treats J and its members in much the same way that *PA* treats K and its members. For example, truth-simpliciter (relative to a meaning assignment only) in *PA* is defined as follows: $T(v, \alpha)$ iff $T(\alpha, v, G)$. So in *PA*, $T(U\alpha, v)$ iff $\exists w(w \in K$ and $T(\alpha, v, w))$. Truth-simpliciter in *PAN* is truth (relative to a meaning assignment) at $\langle G, H\rangle$: $T(\alpha, v)$ iff $T(\alpha, v, \langle G, H\rangle)$. So in *PAN*, $T(U\alpha, v)$ iff $\exists w \exists W(\langle w, W\rangle \in J$ and $T(\alpha, v, \langle w, W\rangle))$. Thus in a sense the ordered pairs $\langle w, W\rangle$ in J represent the "possible worlds" or "possible states of affairs" in *PAN*. But these states of affairs have both a nomic component (W) and a non-nomic component (w). It is because of these two components of a possible physical state of affairs that a sentence (under a meaning assignment) can be true at a world in K relative to one nomic component and false at the same world relative to another. Semantical predicates for analytic truth relative to a meaning assignment and for logical truth in *PAN* are defined as follows:

$AT(\alpha, v)$ iff $\forall W \forall w(\langle w, W\rangle \in J$ only if $T(\alpha, v, \langle w, W\rangle))$
$LT(\alpha)$ iff $\forall v AT(\alpha, v)$.

Let *PN* be *PAN* restricted to sentences in which '*A*' and '*U*' don't occur. *PN* is a propositional modal system for physical necessity alone. Its semantics is obtained by dropping all mention of '*A*' from the semantics of *PAN*. In particular, logical truth for *PN* is defined exactly as for *PAN*. Logical truth in *PN* closely resembles the notion of validity$_2$ discussed in Section 2. Think of $\langle w, W\rangle$ as a model structure ($\langle G, H\rangle$ is the intended model structure) and each v as defining a model on that model structure. Now call a sentence of *PN* valid$_2$ just in case it is true in every model on every model structure. Then the set of valid$_2$ sentences of *PN* coincides with its set of logically true sentences. Results of investigations of validity$_2$ for *S*5-type propositional model systems are easily carried over to *PN*. The *S*5 axioms and inference rules (i.e., those for *PL* with '*L*' replaced by '*N*' everywhere) are sound and complete for *PN*.

Our disappointment with studies involving validity$_2$ is, primarily, that little thoughtful attention is given to what it represents. For most purposes the interesting notion of validity is the one which coincides with the notion of logical truth. For modal systems where the modal operator is supposed to represent logical or analytic necessity, this is validity$_1$. For systems in which the modal operator represents some sort of "logically contingent" necessity (e.g., some type of nomic necessity) a notion resembling validity$_2$ coincides with the notion of logical truth. *PAN* contains operators for both kinds of modality. In this mixed system logical truth exhibits features of both validity$_1$ and validity$_2$.

The axioms and inference rules of *S*5 are sound and complete for *PA*, and they are sound and complete for *PN*. The *S*5 axioms and inference rules applied to *PAN* sentences for '*A*' (in place of '*L*') and again for '*N*' (in place of '*L*'), together with the following A-N axiom and inference rule, form a sound and complete characterization of logical truth in *PAN* (see the Appendix):

A-N $\vdash (A\alpha \supset N\alpha)$.
U-N *If α contains no occurrences of 'A' or 'U' then $\vdash |\alpha|$ only if $\vdash U\alpha$, where $|\alpha|$ is the result of erasing all occurrences of 'N' and 'P' in α.*

Axiom A-N is clearly sound for *PAN*. The soundness of (*U-N*) derives from the fact that J contains some members of form $\langle w, \{w\}\rangle$ because we assumed it analytically possible that there is only one physically possible world. Suppose α contains no occurrences of '*A*' or '*U*' and $|\alpha|$ is a (non-modal) propositional logic tautology. Then for any v and for all $\langle w, \{w\}\rangle \in J$ $T(\alpha, v, \langle w, \{w\}\rangle)$ (since for any subformula of form $N\beta$ or $P\beta$ in α, $T(N\beta, v, \langle w, \{w\}\rangle)$ iff $T(\beta, v, \langle w, \{w\}\rangle)$ iff $T(P\beta, v, \langle w, \{w\}\rangle)$. It follows that $\forall v \exists \langle w, W\rangle \in J\ T(\alpha, v, \langle w, W\rangle)$; so $\forall v \forall \langle w, W\rangle \in J\ T(U\alpha, v, \langle w, W\rangle)$; thus, $LT(U\alpha)$.

We won't spell out a quantified version of *PAN* here. Doing so would require a philosophical investigation of the individuation of physically possible objects across worlds within a set of physically possible worlds. Semantical theories resulting from various positions on the nature of cross-world objects may impose different degrees of *de re* to *de dicto* reduction for quantification across the nomic necessity operator. In general, a quantified modal logic for a contingent necessity operator 'N' and analyticity operator 'A' will have the following characteristics (for pairs $\langle w, W\rangle$ satisfying appropriate constraints resulting from a metaphysical view on the nature of physical necessity and the cross-world identity of objects):

(1) for n-ary predicate symbols α^n, $v(\alpha^n, \langle w, W\rangle) \subseteq \left(\bigcup_{w' \in W} \psi(w')\right)^n$
(2) for individual variables x, $s(x, \langle w, W\rangle) \in \bigcup_{w' \in W} \psi(w')$
(3) $sat((x)\alpha, v, s, \langle w, W\rangle)$ iff $\forall s'((s'$ agrees with s except on x and $s'(x, \langle w, W\rangle) \in \psi(w))$, only if $sat(\alpha, v, s', \langle w, W\rangle)$)
(4) $sat(N\alpha, v, s, \langle w, W\rangle)$ iff $\forall w'(w' \in W$ only if $sat(\alpha, v, s, \langle w', W\rangle))$
(5) $sat(A\alpha, v, s, \langle w, W\rangle)$ iff $\forall s' \forall w' \forall W'(\langle w', W'\rangle$ satisfies the appropriate constraints only if $sat(\alpha, v, s', \langle w', W'\rangle))$
(6) $LT(\alpha)$ iff $\forall v \forall s \forall w \forall W(\langle w, W\rangle$ satisfies the appropriate constraints only if $sat(\alpha, v, s, \langle w, W\rangle))$.

It should be apparent from the method by which completeness for *QA* and *PAN* are proven in the Appendix that if there is a sound and complete axiomatization of the set of logical truths (validity$_2$ style) which doesn't contain 'A', then the logical truths of the full system (with A) will have a sound and complete axiomatization.

Appendix In Section 3 we claimed soundness and completeness for *PA*, *QA*, and *PAN*, and compactness for *QA*. Soundness proofs for these systems are straightforward, so this Appendix is devoted to completeness and compactness.

The completeness of *PA* is easy to establish since the axioms and rules of *PA* are just those of *S5* and the semantics of *PA* is just the Kripke semantics for *S5* restricted to the single model structure $\mathcal{S} = (G, K, \psi)$. Since we require K to be at least denumerably infinite, the completeness of *PA* is immediate from Kripke's result in [13] that *S5* has a denumerable universal model structure.

Completeness proofs for *QA* and *PAN* have the same general form and rely on the completeness of the underlying systems to which 'A' and 'U' are appended. Consider a language $\mathcal{L}$ containing 'A' and 'U' and perhaps other modal operators (e.g., 'N' and 'P'). Let S be the semantics for $\mathcal{L}$, and let Ax be a set of axioms and rules on $\mathcal{L}$. A sentence β of $\mathcal{L}$ is said to be in *ACNF* (i.e., 'A'-conjunctive normal form) if it is of the form

$$\beta_1 \& \ldots \& \beta_m \ ,$$

where each β_j is called a *major conjunct* of β, and has the form

$$(A\phi_1 \vee \ldots \vee A\phi_n \vee U\Delta \vee \Pi)_j \ ,$$

where the ϕ_i, Δ, and Π contain no occurrences of 'A' or 'U'. A completeness proof can be obtained if $\mathcal{L}$ satisfies the following three conditions:

(I) (i) For each sentence α of $\mathcal{L}$ there is a sentence β in *ACNF* such that

$$LT_s(\alpha) \text{ iff } LT_s(\beta)$$

and

$$\vdash_{\overline{Ax}} \alpha \text{ iff } \vdash_{\overline{Ax}} \beta \ .$$

(ii) For any sentence β is *ACNF* with major conjuncts β_j

$$LT_s(\beta) \text{ iff } LT_s(\beta_j), \text{ for each } j$$

and

$$\vdash_{\overline{Ax}} \beta \text{ iff } \vdash_{\overline{Ax}} \beta_j, \text{ for each } j \ .$$

(II) For each *ACNF* major conjunct β_j there are sentences $\gamma_{j1}, \gamma_{j2}, \ldots$ of $\mathcal{L}$ containing no occurrences of 'A' or 'U' such that:

(i) $LT_s(\beta_j)$ only if $LT_s(\gamma_{ji})$, for some i; and
(ii) $\vdash_{\overline{Ax}} \gamma_{ji}$, for some i, only if $\vdash_{\overline{Ax}} \beta_j$.

(III) Ax is complete on S restricted to the set of sentences of $\mathcal{L}$ containing no occurrences of 'A' or 'U' (i.e., for any sentence δ of S not containing 'A' or 'U', $LT_s(\delta)$ only if $\vdash_{\overline{Ax}} \delta$).

Given (I)–(III) it is easy to show that for any sentence α in $\mathcal{L}$, $LT_s(\alpha)$ only if $\vdash_{\overline{Ax}} \alpha$. For suppose that $LT_s(\alpha)$. Then by (I)(i) there is a sentence β in *ACNF* such that $LT_s(\beta)$, and by (I)(ii) $LT_s(\beta_j)$ for each j. It follows by (II)(i) that for each β_j there is a γ_{ji} such that $LT_s(\gamma_{ji})$. Hence $\vdash_{\overline{Ax}} \gamma_{ji}$ by (III), since γ_{ji} contains no 'A' or 'U'. But (II)(ii) now guarantees that $\vdash_{\overline{Ax}} \beta_j$, for each j. Finally, by (I)(ii) we have $\vdash_{\overline{Ax}} \beta$, and by (I)(i) $\vdash_{\overline{Ax}} \alpha$.

We now go about showing how to establish (I)–(III) for *QA* and *PAN*.

QA

(I) (i) Using Axiom 12 it can be shown that every sentence α is provably equivalent to a sentence α', where α' is like α except that each well-formed part $A\beta$ of α has been replaced by $A\beta'$, and β' contains no free occurrences of any variable. Well-known quantificational theorems (e.g., $(x)(\gamma \vee \delta) \equiv (\gamma \vee (x)\delta)$, if x is not free in γ) can then be used to transform each $A\beta'$ into $A\beta''$, where β'' is either nonmodal or molecular, and to assure that no quantifier in β'' has any A or U in its scope. Finally, $S5$ modal reduction theorems will allow further transformations so that eventually we get $\vdash \alpha \equiv \theta$, where θ is in *ACNF*. Hence $LT(\alpha \equiv \theta)$, by the soundness of *QA*. The two parts of (I)(i) are now immediate.

(ii) The semantics of *QA* makes a conjunction *LT* just in case each conjunct is, and the axioms make a conjunction provable just in case each conjunct is.

(II) (i) Let C_i be a nonmodal first-order sentence which says that there are exactly i things. For each major conjunct β_j, take the sentences $\gamma_{j1}, \gamma_{j2}, \ldots$, to be $(\phi_1 \vee \Delta), \ldots, (\phi_n \vee \Delta)$, $(\Pi \vee \Delta)$, and $(C_i \supset \Delta)$, for each nonzero positive integer i. It is easy to verify that

$LT(A\phi_1 \vee \ldots \vee A\phi_n \vee U\Delta \vee \Pi)$ iff
$LT(A\phi_1 \vee \ldots \vee A\phi_n \vee U\Delta \vee A\Pi)$.

So it suffices to consider only major conjuncts that contain no unmodalized disjuncts. We show that

$LT(A\phi_1 \vee A\phi_2 \vee U\Delta)$ only if
$LT(\phi_1 \vee \Delta)$ or $LT(\phi_2 \vee \Delta)$ or $LT(C_i \supset \Delta)$, for some i.

Generalization to cases where $n > 2$ is straightforward.

Suppose neither $(\phi_1 \vee \Delta)$ nor $(\phi_2 \vee \Delta)$ nor any of the $(C_i \supset \Delta)$ are LT. Then there are worlds w_1, w_2, and valuations v_1, v_2, such that $T(\sim\phi_1, v_1, w_1)$, $T(\sim\Delta, v_1, w_1)$, $T(\sim\phi_2, v_2, w_2)$, and $T(\sim\Delta, v_2, w_2)$. And since we assume that for each nonzero countable cardinal τ there are at least denumerably many $w \in K$ such that $\psi(w)$ is of cardinality τ, we can choose $w_1 \neq w_2$. (For the proof to generalize to cases where n is arbitrarily large, there can be no finite limit on the cardinality of K.) Furthermore, for each nonzero positive integer i, there is a $w_i \in K$ and a valuation v_i such that $T(C_i, v_i, w_i)$ and $T(\sim\Delta, v_i, w_i)$. Since Δ and the C_i are completely nonmodal, we can think of the pairs $\langle\psi(w_i), v_i\rangle$ as nonmodal first-order models, all of which verify $\sim\Delta$. So $\sim\Delta$ has models of arbitrarily large finite cardinality, and hence by the upward Löwenheim-Skolem theorem it has a model $\langle\psi(w_\tau), v_\tau\rangle$ of each cardinality τ.

We now specify a valuation v^* which combines the features of v_1, v_2, and all the v_τ that are important for our purposes. Choose v^* so that for any proposition letter or predicate letter α, $v^*(\alpha, w_1) = v_1(\alpha, w_1)$, and $v^*(\alpha, w_2) = v_2(\alpha, w_2)$.

For each w such that $\psi(w)$ is of cardinality τ, $w_1 \neq w \neq w_2$, there is a valuation v such that $\langle\psi(w), v\rangle$ is isomorphic and hence elementarily equivalent to $\langle\psi(w_\tau), v_\tau\rangle$. For each such w let $v^*(\alpha, w) = v_\tau(\alpha, w_\tau)$, modulo the isomorphism.

It is now easy to verify that $T(\sim\phi_1, v^*, w_1)$, $T(\sim\phi_2, v^*, w_2)$, and $T(\sim\Delta, v^*, w)$, for all $w \in K$.

Hence for any $w \in K$, $T(\sim A\phi_1, v^*, w)$, $T(\sim A\phi_2, v^*, w)$, and $T(\sim U\Delta, v^*, w)$. So $(A\phi_1 \vee A\phi_2 \vee U\Delta)$ is not LT.

(ii) Since

$\vdash(A\phi_1 \vee \ldots \vee A\phi_n \vee U\Delta \vee \Pi)$ iff
$\vdash(A\phi_1 \vee \ldots \vee A\phi_n \vee U\Delta \vee A\Pi)$,

it suffices to show that if $\vdash(\phi \vee \Delta)$ then $\vdash(A\phi \vee U\Delta)$, and if $\vdash(C_i \supset \Delta)$, for any nonzero positive integer i, then $\vdash U\Delta$. The former is an easily verified feature for $S5$, and the latter follows from familiar features of $S5$ and the fact that $\vdash UC_i$, for all i.

(III) Standard first-order logic is complete with respect to the semantics of QA. The nonmodal axioms of QA constitute standard first-order logic.

PAN

(I) (i) For each sentence α of PAN there is a sentence β in $ACNF$ such that $\vdash \alpha \equiv \beta$. This, together with the soundness of PAN, is sufficient for both parts of (I)(i). That $\vdash \alpha \equiv \beta$ is indeed the case can be shown by appeal to provable biconditionals which allow all occurrences of 'N' and 'P' to be driven into subformulas of α until no occurrences of 'A' or 'U' occur in their scopes. Since 'A' and 'N' are both $S5$ necessity operators, these biconditionals are the usual ones for $S5$ (see [5], pp. 51–55) and the following that are theorems of PAN:

$$\begin{aligned}
&NA\delta \equiv A\delta\\
&PA\delta \equiv A\delta\\
&N(\delta \,\&\, A\gamma) \equiv (N\delta \,\&\, A\gamma)\\
&N(\delta \,\&\, U\gamma) \equiv (N\delta \,\&\, U\gamma)\\
&N(\delta \vee A\gamma) \equiv (N\delta \vee A\gamma)\\
&N(\delta \vee U\gamma) \equiv (N\delta \vee U\gamma).
\end{aligned}$$

Once we have a sentence in which no 'A' or 'U' occurs within the scope of an 'N' or 'P', standard methods of $S5$ can be used to obtain an $ACNF$.

(ii) Trivial, as for QA.

(II) (i) For each major conjunct β_j, take the sentences $\gamma_{j1}, \gamma_{j2}, \ldots, \gamma_{jn+2}$, to be $(\phi_1 \vee P\Delta), \ldots, (\phi_n \vee P\Delta)$, $(\Pi \vee P\Delta)$, $|\Delta|$. (Notice that here, unlike the proof for QA, we need only finitely many such sentences.) It is easy to verify that $LT(U\Delta \equiv UP\Delta)$ and hence that

$$\begin{aligned}
&LT(A\phi_1 \vee \ldots \vee A\phi_n \vee U\Delta \vee \Pi) \text{ iff}\\
&LT(A\phi_1 \vee \ldots \vee A\phi_n \vee UP\Delta \vee A\Pi).
\end{aligned}$$

So it suffices to show that

$$\begin{aligned}
&LT(A\phi_1 \vee A\phi_2 \vee UP\Delta) \text{ only if}\\
&LT(\phi_1 \vee P\Delta) \text{ or } LT(\phi_2 \vee P\Delta) \text{ or } LT(|\Delta|),
\end{aligned}$$

since generalization to cases where $n > 2$ is straightforward.

Suppose that neither $(\phi_1 \vee P\Delta)$ nor $(\phi_2 \vee P\Delta)$ nor $|\Delta|$ is LT. Then there are possible physical states of affairs (i.e., members of J) $\langle w_1, W_1\rangle$, $\langle w_2, W_2\rangle$, $\langle w_3, W_3\rangle$, and valuations v_1, v_2, v_3 such that $T({\sim}\phi_1, v_1, \langle w_1, W_1\rangle)$, $\forall w \in W_1 T({\sim}\Delta, v_1, \langle w, W_1\rangle)$, $T({\sim}\phi_2, v_2, \langle w_2, W_2\rangle)$, $\forall w \in W_2 T({\sim}\Delta, v_2, \langle w, W_2\rangle)$, and $T({\sim}|\Delta|, v_3, \langle w_3, W_3\rangle)$. And since we assume that there are at least denumerably many W_i such that for some $w \in W_i$, $\langle w, W_i\rangle \in J$, and that at least denumerably many of these W_i are at least denumerable, we can choose W_1, W_2, W_3 so that they are distinct from each other. Furthermore, since $|\Delta|$ is entirely nonmodal, W_3 can be $\{w_3\}$, and since N and P

are vacuous (i.e., they are the identity function on truth values) at any one-membered W_i, $T(\sim\Delta, v_3, \langle w_3, W_3\rangle)$.

We now define a valuation v^* such that for any proposition letter α, $\forall w \in W_1 v^*(\alpha, \langle w, W_1\rangle) = v_1(\alpha, \langle w, W_1\rangle)$, $\forall w \in W_2 v^*(\alpha, \langle w, W_2\rangle) = v_2(\alpha, \langle w, W_2\rangle)$, and $\forall\langle w, W\rangle \in J$ such that $W_1 \neq W \neq W_2$, $v^*(\alpha, \langle w, W\rangle) = v_3(\alpha, \langle w_3, W_3\rangle)$. Hence $T(\sim\phi_1, v^*, \langle w_1, W_1\rangle)$, $T(\sim\phi_2, v^*, \langle w_2, W_2\rangle)$, and $T(\sim\Delta, v^*, \langle w, W\rangle)$, for all $\langle w, W\rangle \in J$. So for any $\langle w, W\rangle \in J$, $T(\sim A\phi_1, v^*, \langle w, W\rangle)$, $T(\sim A\phi_2, v^*, \langle w, W\rangle)$, and $T(\sim UP\Delta, v^*, \langle w, W\rangle)$. Hence $(A\phi_1 \vee A\phi_2 \vee UP\Delta)$ is not LT.

(ii) Since

$$\vdash (A\phi_1 \vee \ldots \vee A\phi_n \vee U\Delta \vee \Pi) \text{ iff}$$
$$\vdash (A\phi_1 \vee \ldots \vee A\phi_n \vee UP\Delta \vee A\Pi),$$

it suffices to show that if $\vdash(\phi \vee \psi)$ then $\vdash(A\phi \vee U\psi)$, and if $\vdash|\Delta|$ then $\vdash U\Delta$. The former holds because of the $S5$ axioms for 'A', and the latter holds by Rule (U-N).

(III) Consider that fragment of PAN that contains no occurrences of 'A' or 'U'. The operative axioms are just those of $S5$, and, in view of our assumption that there is a W_i of each nonzero countable cardinality, the semantics is just standard validity$_2$-style semantics for $S5$. Hence the LT sentences of this fragment coincide with its theorems.

The foregoing proof of completeness for PAN also provides a decision procedure for validity. Given any sentences of PAN, there is an effective method for finding a logically equivalent sentence in $ACNF$. This $ACNF$ is valid iff each of its major conjuncts, β_j, is, and β_j is valid iff at least one of the sentences γ_{ji} is. But these sentences contain no occurrences of 'A' or 'U' and are decidable by $S5$ methods.

Finally, we indicate briefly how compactness may be proven for QA; i.e., we want to show that if α is a logical consequence of an infinite set of QA sentences Γ', $LC(\Gamma', \alpha)$, then there is a finite $\Delta' \subseteq \Gamma'$ such that $LC(\Delta', \alpha)$. It suffices to show that if every finite subset Δ of an infinite set Γ is satisfied by a QA interpretation, then Γ is satisfied by a QA interpretation.

Let Γ be an infinite set of sentences of QA and suppose every finite subset of Γ has a QA interpretation. Define Γ^* as the union of Γ with the QA axioms. Then every finite subset Δ of Γ^* has a QA interpretation, $\langle v, w\rangle$. An interpretation satisfying a Δ is defined on the intended model structure $\mathrm{S} = (G, K, \psi)$, so each Δ has a Kripke-type $S5$ model of form $\langle v', (w', W', \psi')\rangle$ in which (w', W', ψ') is a model structure with set of worlds W'. There is a compactness theorem for the Kripke-type $S5$ semantics for quantified modal logic [26], and a Löwenheim-Skolem-like result which says that any satisfiable set of sentences has a model structure with no more than a denumerable set of worlds [27]. So Γ^* has a Kripke-type model $\langle v''(w'', W'', \psi'')\rangle$ for which W'' is denumerable. Also, $\exists w \in W''$ such that the cardinality of $\psi''(w)$ is τ, for each countable cardinality τ, as the QA axioms in Γ^* require.

Let f map K onto W'' such that for all $w \in K$ the cardinality of $\psi''(f(w))$ is the same as the cardinality of $\psi(w)$. Define v^* for the intended QA model structure so that for every proposition letter or n-ary predicate letter α, $\forall w \in K\, v^*(\alpha, w) = v''(\alpha, f(w))$ modulo some 1-1 mapping of $\psi(w)$ onto $\psi''(f(w))$.

Then $\exists w \in K$ such that the QA interpretation $\langle v^*, w\rangle$ satisfies Γ^*. So $\langle v^*, w\rangle$ satisfies Γ, for some w in K.

NOTES

1. For some purposes an intensional language may be interpreted on a model structure other than $\mathcal{S}$. In a tense logic the intended model structure may be as follows: the indices in K are moments of time in the actual world, G is the present moment, $\psi(w)$ is the set of things that exist in the actual world at moment w, and $w R w'$ holds just in case w is before w' in the actual world. Such a model structure may be of some interest, but it cannot furnish a general theory of meaning for nonlogical terms.

2. A related point is discussed in Section 2 in connection with what is there called the structural approach to validity$_2$.

3. See, for example, Kripke [13] and [14], Montague [19], Kaplan [10], Lewis [15], Adams [1], and Plantinga [21].

4. In view of the early work of Carnap and Kanger, our approach to semantics for modal logic and our definition of validity are not really new. Carnap studied an analogue of validity$_1$, using state-descriptions, in the 1940s. Validity$_1$ as we define it here was first proposed and studied by Kanger in 1957. But since the publication of completeness theorems for validity$_2$ in Kripke [12], little attention has been paid to the earlier, more intuitive notion. See Carnap [2], Section 41, Kanger [6], [7], [8], and [9].

5. The question of the philosophical interest of validity$_2$ is one that has received surprisingly little attention in the literature. The only discussions with which we are familiar are Pollock [22] and [23], Makinson [17] and [18], and Plantinga [20], pp. 126–128.

6. Thomason [25], p. 127. It is not at all clear that Thomason advocates what we are here calling the structural approach, but the passage quoted does suggest it.

7. Cresswell [4] has shown that even truth-functional connectives may be treated so that their interpretation varies from one model structure to another. However, we do not consider this an appropriate treatment for any connectives in the languages with which we are dealing.

8. See Montague [19], pp. 105–106, for a detailed account of these two approaches to tense logic.

9. See Hughes and Cresswell [5], pp. 51–55, for the S5 normal form theorem.

10. See Church [3], pp. 192–193, for an account of uniform substitution appropriate to preserve validity in first-order logic. The relevant notion for the axioms and rules for QL is Church's supplemented to treat all variables in the scope of a modal operator as Church treats (other) bound variables.

11. Thus QA satisfies a requirement suggested by Pollock [22].

REFERENCES

[1] Adams, R., "Theories of actuality," *Nous*, vol. 8 (1974), pp. 211-231; reprinted in [16].

[2] Carnap, R., *Meaning and Necessity*, The University of Chicago Press, Chicago, Illinois, 1947.

[3] Church, A., *Introduction to Mathematical Logic*, Vol. 1, Princeton University Press, Princeton, New Jersey, 1956.

[4] Cresswell, M. J., "Classical intensional logic," *Theoria*, vol. 26 (1970), pp. 347-371.

[5] Hughes, G. E. and M. J. Cresswell, *An Introduction to Modal Logic*, Methuen and Co., London, 1968.

[6] Kanger, S., *Provability in Logic*, Almqvist and Wiksell, Stockholm, 1957.

[7] Kanger, S., "The morning star paradox," *Theoria*, vol. 23 (1957), pp. 1-11.

[8] Kanger, S., "On the characterization of modalities," *Theoria*, vol. 23 (1957), pp. 152-155.

[9] Kanger, S. "New foundations for ethical theory," in *Deontic Logic: Introductory and Systematic Readings*, ed., R. Hilpinen, D. Reidel, Dordrecht, 1970.

[10] Kaplan, D. "Transworld heir lines," presented in 1967 to a joint symposium of the APA and ASL, and published for the first time in [16].

[11] Kaplan, D., "On the logic of demonstratives," *Journal of Philosophical Logic*, vol. 8 (1978), pp. 81-98.

[12] Kripke, S. A., "A completeness theorem in modal logic," *The Journal of Symbolic Logic*, vol. 24 (1959), pp. 1-14.

[13] Kripke, S. A., "Semantical analysis of modal logic I," *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, vol. 9 (1963), pp. 67-96.

[14] Kripke, S. A., "Semantical considerations on modal logic,"*Acta Philosophica Fennica*, vol. 16 (1963), pp. 83-94.

[15] Lewis, D., *Counterfactuals*, Basil Blackwell, London, 1973.

[16] Loux, M., ed., *The Possible and the Actual: Readings in the Metaphysics of Modality*, Cornell University Press, Ithaca, New York, 1979.

[17] Makinson, D., "Rules of truth for modal logic," D. Phil. Thesis, Oxford, 1965.

[18] Makinson, D., "How meaningful are modal operators?" *Australasian Journal of Philosophy*, vol. 44 (1966), pp. 331-337.

[19] Montague, R., "Pragmatics," in *Contemporary Philosophy: A Survey*, ed., R. Klibansky, La Nuova Italia Editrice 1968; reprinted as pp. 93-118 in *Formal Philosophy: Selected Papers of Richard Montague*, ed., R. H. Thomason, Yale University Press, New Haven, Connecticut, 1974.

[20] Plantinga, A., *The Nature of Necessity*, Oxford University Press, Oxford, 1974.

[21] Plantinga, A., "Actualism and possible worlds," *Theoria*, vol. 42 (1976), pp. 139-160. Reprinted in [16].

[22] Pollock, J. L., "Model theory and modal logic," *Logique et Analyse*, no. 35-36 (1966), pp. 313-317.

[23] Pollock, J. L., "Logical validity in modal logic," *The Monist*, vol. 51 (1967), pp. 128-135.

[24] Stalnaker, R., "Possible worlds," *Nous*, vol. 10 (1976), pp. 65-75. Reprinted in [16].

[25] Thomason, R. H., "Modal logic and metaphysics," in *The Logical Way of Doing Things*, ed., Karel Lambert, Yale University Press, New Haven, Connecticut, 1969, pp. 119-146.

[26] van Fraassen, B., "Compactness and Löwenheim-Skolem proofs in modal logic," *Logique et Analyse*, vol. 12 (1969), pp. 167-178.

[27] van Fraassen, B., *Formal Semantics and Logic*, Macmillian, New York, 1971.

Department of Philosophy
University of Minnesota
Minneapolis, Minnesota 55455