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Some Failures of Interpolation in Modal Logic

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Offered here is a relatively simple test for failure of interpolation in propositional modal logic. Additional, and quite important, negative results of this same general sort have recently found their way into the literature. Fine [2] has shown that interpolation fails in quantified *S5* and in a broad class of quantified modal logics when strengthened with the Barcan formula and its converse. And on the propositional level, Maksimova [5], [6] has identified twenty-four normal extensions of *S4* in which interpolation holds but has proven that there exist only thirteen other possible candidates from within that class of logics. The present result, though partially overlapping that of Maksimova, takes us beyond the extensions of *S4* and covers nonnormal logics as well.

The key is found in Lemmon's classical characterization of Halldén-incomplete logics (see [3]).

Theorem 1 *L is Halldén-incomplete if and only if L is the intersection of two logics neither of which contains the other.*

The proof proceeds by showing, in effect, that if α and β share no variables, $\alpha \vee \beta \in L$, $\alpha \notin L$ and $\beta \notin L$, then α and β are theorems of consistent extensions of L. But this fact also yields

Theorem 2 *If L has only one Post-complete extension and is Halldén-incomplete, then interpolation fails in L.*

Proof: Suppose L has only one Post-complete extension C and that $\alpha \vee \beta \in L$, $\alpha \notin L$ and $\beta \notin L$ for some α, β having no variables in common. If p is any variable foreign to α and β, then $\sim\alpha \wedge (p \to p)$ and $\beta \wedge (p \to p)$ have a single variable in common and

$$\sim\alpha \wedge (p \to p) \to \beta \wedge (p \to p) \in L.$$

Assume for *reductio* that $\sim\alpha \wedge (p \to p) \to \gamma$, $\gamma \to \beta \wedge (p \to p) \in L$ for some γ containing only the variable p. Then $\alpha \vee \gamma$, $\sim\gamma \vee \beta \in L$, $\gamma \notin L$ and $\sim\gamma \notin L$, where α and γ share no variables, nor do $\sim\gamma$ and β. It follows that γ and $\sim\gamma$

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are both theorems of consistent extensions of L, and hence theorems of C, an impossibility.

It would be nice if one could turn Theorem 2 around to show that failure of interpolation implies Halldén-incompleteness, at least for some tolerably interesting class of logics. Unfortunately, this possibility is fairly well foreclosed by Maksimova's result and the fact that *S4*—indeed, even *S5*—has infinitely many Halldén-complete normal extensions.

Lest the theorem itself should seem a bit parochial, we might note that there exist some very weak modal logics with only one Post-complete extension. For example, let L be the smallest normal logic closed under the rule to infer $\sim\Box\alpha$ from $\Box\sim\alpha$. An induction on the number of occurrences of $p_0 \vee \sim p_0$ in α reveals that if a formula α is variable-free in the sense of [4], then $\alpha \in L$ or $\sim\alpha \in L$. The consistent extensions of L therefore all agree on their variable-free theorems, whence it follows by Lemmas 1 and 2 of [4] that L has only one Post-complete extension. This expands considerably, of course, Sobociński's well-known result that T has only one such extension.

It often happens, when one seeks to illustrate this theorem or that, that the many systems constructed by Sobociński provide a rich source of nice examples. No less so here. Since *S4* has only one Post-complete extension, interpolation fails in *S4.1.1*, *S4.1.2* and in every member of his family $\mathcal{Z}$, each of which is Halldén-incomplete by Theorem 1 and the intersection results of [7]:

$$\begin{aligned} S4.1.1 &= S4.2.1 \cap K1.1, \\ S4.1.2 &= S4.4 \cap K1.2, \end{aligned}$$

and each logic in $\mathcal{Z}$ is the intersection of *S5* and a member of Sobociński's family $\mathcal{K}$. Indeed, though interpolation holds in *S4.4* and *S5*, as shown in [8], it fails in all logics intermediate between the two. This follows at once from Theorem 2 and the characterization given in [7] of the normal extensions of *S4.4*, together with the fact that every extension of that system is normal (see [9]).

As a final application, we obtain a quick proof of the following theorem due originally to Maksimova.

Theorem 3 *Interpolation fails in* $2^{\aleph_0}$ *extensions of S4.*

Proof: Fine [1] shows how to construct reflexive transitive frames $\mathfrak{F}_i$ and formulas α_j such that $\mathfrak{F}_i$ validates α_j if and only if $i \neq j$. Let L be the logic determined by $\mathfrak{F}_0$ and Ω be the class of logics determined by nonempty subsets of $\{\mathfrak{F}_i | i > 0\}$. If $S \in \Omega$, S neither contains nor is contained in L, so $L \cap S$ is a Halldén-incomplete extension of *S4*. Moreover, if M and N are distinct members of Ω, then $\alpha_i \in M - N$ or $\alpha_i \in N - M$ for some $i > 0$. But now $\alpha_i \in L$, so $L \cap M \neq L \cap N$. It follows that there are $2^{\aleph_0}$ Halldén-incomplete extensions of *S4*. By Theorem 2, interpolation fails in each of those logics.

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