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Notre Dame Journal of Formal Logic
Volume 33, Number 1, Winter 1992

A Model in Which Every Kurepa Tree Is Thick

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Abstract In this paper we show that, assuming the existence of two strongly inaccessible cardinals, it is consistent with CH (or $\neg CH$) plus $2^{\omega_1} > \omega_2$ that there exists a Kurepa tree with 2^{ω_1}-many branches and no ω_1-trees have λ-many branches for some λ strictly between ω_1 and 2^{ω_1}.

A *tree* is a partially ordered set $(T,<_T)$ such that for every $t \in T$, the set $\{s \in T : s <_T t\}$ is well-ordered. $(T',<_{T'})$ is a subtree of $(T,<_T)$ if $T' \subseteq T$ and $<_{T'} = <_T \cap T' \times T'$. We shall not distinguish a tree $(T,<_T)$ from its domain T. Let $ht_T(t)$, the *height* of t in T, be the order type of $\{s \in T : s <_T t\}$, let T_α, the α-th level of T, be the set $\{t \in T : ht_T(t) = \alpha\}$, and let $ht(T)$, the height of T, be the smallest ordinal α such that $T_\alpha = \emptyset$. By a *branch of* T we mean a linearly ordered subset of T which intersects every nonempty level of T. Let $\mathcal{B}(T)$ be the set of all branches of T.

T is called an *κ-tree* for some regular cardinal κ if $|T| = \kappa$ and $ht(T) = \kappa$. An ω_1-tree is called a Kurepa tree if $|T_\alpha| < \omega_1$ for every $\alpha < \omega_1$ and $|\mathcal{B}(T)| > \omega_1$. A Kurepa tree T is called thick if $|\mathcal{B}(T)| = 2^{\omega_1}$. An ω_1-tree is called a Jech–Kunen tree if $\omega_1 < |\mathcal{B}(T)| < 2^{\omega_1}$.

It is obvious that under CH plus $2^{\omega_1} > \omega_2$, (1) a Jech–Kunen tree T is a Kurepa tree if $|T_\alpha| < \omega_1$ for every $\alpha < \omega_1$; (2) a Kurepa tree T is a Jech–Kunen tree if it is not thick.

The independence of the existence of a Kurepa tree was proved by Silver (see Kunen [7]). In [3], Jech constructed by forcing a model of CH plus $2^{\omega_1} > \omega_2$, in which there is a Jech–Kunen tree. In fact, it is a Kurepa tree with less than 2^{ω_1}-many branches. The independence of the existence of a Jech–Kunen tree (in terms of a compact Hausdorff space) under CH plus $2^{\omega_1} > \omega_2$ was given by Kunen [6]. The detailed proof can be found in Juhász [5], Theorem 4.8. In Kunen's model all Kurepa trees, including those with 2^{ω_1}-many branches, are also killed. Is it necessary to kill all Kurepa trees when we kill all Jech–Kunen trees? In Jin [4], Kunen proved that it is consistent with CH plus $2^{\omega_1} > \omega_2$ that there is a thick Kurepa tree which has no Jech–Kunen subtrees. So it is natural to ask

Received July 15, 1991; revised September 23, 1991

whether it is consistent with *CH* plus $2^{\omega_1} > \omega_2$ that there exists a thick Kurepa tree and there are no Jech–Kunen trees. Next we will give a positive answer by assuming the existence of two strongly inaccessible cardinals. (Note that the assumption of one strongly inaccessible cardinal is necessary for killing all Jech–Kunen trees.)

Theorem 1 *Assuming the existence of two strongly inaccessible cardinals, it is consistent with CH plus $2^{\omega_1} > \omega_2$ that there exists a thick Kurepa tree and there are no Jech–Kunen trees.*

In order to prove the theorem we need some notation and a lemma from Devlin [2] which plays a key role in our proofs. By a *poset* we mean a partially ordered set with a largest element. We always let $1_{\mathbf{P}}$ be the largest element of a poset $\mathbf{P}$. Let I, J be two sets and λ be a cardinal.

$$Fn(I,J,\lambda) = \{f : f \text{ is a function}, f \subseteq I \times J \text{ and } |f| < \lambda\}$$

is a poset ordered by reverse inclusion. We omit λ if $\lambda = \omega$. Let I be a subset of an ordinal κ and λ be a cardinal.

$$Lv(I,\lambda) = \{f : f \text{ is a function}, f \subseteq (I \times \lambda) \times \kappa,\ |f| < \lambda \text{ and } \forall\langle\alpha,\beta\rangle \in \mathrm{dom}(f)(f(\alpha,\beta) \in \alpha)\}$$

is a poset ordered by reverse inclusion.

Let 2^α be the set of all functions from α to 2 and $2^{<\kappa} = \bigcup_{\alpha<\kappa} 2^\alpha$. Then $2^{<\kappa}$ is a tree ordered by inclusion.

In forcing arguments we let $\dot{a}$ be a name for a and $\ddot{a}$ be a name for $\dot{a}$. We always assume the consistency of *ZFC* and let M denote a countable transitive model of *ZFC*. The author refers to [7] for background in forcing and refers to Todorčević [9] for background in trees.

Lemma 2 *Let $\mathbf{P}, \mathbf{P}'$ be two posets in M such that $\mathbf{P}$ has κ-c.c. and $\mathbf{P}'$ is κ-closed in M, where κ is a regular cardinal in M. Let $G_{\mathbf{P}}$ be a $\mathbf{P}$-generic filter over M and $G_{\mathbf{P}'}$ be a $\mathbf{P}'$-generic filter over $M[G_{\mathbf{P}}]$. Let T be a κ-tree in $M[G_{\mathbf{P}}]$. If T has a new branch B in $M[G_{\mathbf{P}}][G_{\mathbf{P}'}] \setminus M[G_{\mathbf{P}}]$, then T has a subtree T' in $M[G_{\mathbf{P}}]$, which is isomorphic to the tree $\langle 2^{<\kappa} \cap M, \subseteq\rangle$.*

Proof: First we work within M. In the proof we always let $i = 0, 1$. Without loss of generality we can assume that $|T_0| = 1$ and

$$1_{\mathbf{P}} \Vdash_{\mathbf{P}} (1_{\mathbf{P}'} \Vdash_{\mathbf{P}'} (\ddot{B} \text{ is a branch of } \dot{T})).$$

Claim 1 *Let $\alpha < \kappa$ and $q \in \mathbf{P}'$. Then there is a $q' \leq_{\mathbf{P}'} q$ such that*

$$1_{\mathbf{P}} \Vdash_{\mathbf{P}} (\Phi(\alpha, q', \dot{T}, \ddot{B})),$$

where

$$\Phi(\alpha, q, \dot{T}, \ddot{B}) =_{df} (\exists y \in \dot{T}_\alpha)(q \Vdash_{\mathbf{P}'} (y \in \ddot{B})).$$

Proof of Claim 1: Replace ω_1 by κ in the proof of Lemma 3.6 (in [2]).

Claim 2 *Let $\alpha < \kappa$, $q \in \mathbf{P}'$ and $1_{\mathbf{P}} \Vdash_{\mathbf{P}} (\Phi(\alpha, q, \dot{T}, \ddot{B}))$. Then there is a $\beta < \kappa$, $\beta > \alpha$, and $q^i \leq_{\mathbf{P}'} q$ such that*

$$1_{\mathbf{P}} \Vdash_{\mathbf{P}} (\Psi(\alpha, \beta, q, q^0, q^1, \dot{T}, \ddot{B})),$$

where

$$\Psi(\alpha,\beta,q,q^0,q^1,\dot{T},\ddot{B}) =_{df} [\textit{if } x\in \dot{T}_\alpha \textit{ and } q \Vdash_{\mathbf{P}'} (x\in\ddot{B}), \textit{ then there are } x^i\in\dot{T}_\beta,\ x^0\neq x^1 \textit{ and } x<_T x^i \textit{ such that } q^i \Vdash_{\mathbf{P}'} (x^i\in\ddot{B})].$$

Proof of Claim 2: Replace ω_1 by κ in the proof of Lemma 3.6 (in [2]).

Claim 3 *Let δ be an ordinal below κ. Let $\langle q_\gamma:\gamma<\delta\rangle$ be a decreasing sequence in $\mathbf{P}'$ and $\langle\alpha_\gamma:\gamma<\delta\rangle$ be an increasing sequence in κ such that*

$$1_{\mathbf{P}} \Vdash_{\mathbf{P}} (\Phi(\alpha_\gamma,q_\gamma,\dot{T},\ddot{B}))$$

for all $\gamma<\delta$. Let $\alpha_\delta=\sup\{\alpha_\gamma:\gamma<\delta\}$. Then there is a $q\leq_{\mathbf{P}'} q_\gamma$ for all $\gamma<\delta$ such that

$$1_{\mathbf{P}} \Vdash_{\mathbf{P}} (\Phi(\alpha_\delta,q,\dot{T},\ddot{B})).$$

Proof of Claim 3: Since $\mathbf{P}'$ is κ-closed in M, there is a $q'\in\mathbf{P}'$ such that $q'\leq_{\mathbf{P}'} q_\gamma$ for all $\gamma<\delta$. By Claim 1 there is a $q\leq_{\mathbf{P}'} q'$ such that

$$1_{\mathbf{P}} \Vdash_{\mathbf{P}} (\Phi(\alpha_\delta,q,\dot{T},\ddot{B})).$$

This ends the proof of Claim 3.

We now prove the lemma. We construct a subset $\bar{\mathbf{P}}=\{p_s:s\in 2^{<\kappa}\}$ of $\mathbf{P}'$ and a subset $O=\{\alpha_s:s\in 2^{<\kappa}\}$ of κ in M such that

(1) the map $s\mapsto p_s$ is an isomorphic imbedding from $\langle 2^{<\kappa},\subseteq\rangle$ to $\mathbf{P}'$ in M.
(2) $\forall s,t\in 2^{<\kappa}$ $(s\subseteq t$ and $s\neq t\to\alpha_s<\alpha_t)$.
(3) $\alpha_{s\hat{}\langle 0\rangle}=\alpha_{s\hat{}\langle 1\rangle}$ for all $s\in 2^{<\kappa}$.
(4) $1_{\mathbf{P}} \Vdash_{\mathbf{P}} (\Phi(\alpha_s,p_s,\dot{T},\ddot{B}))$ for all $s\in 2^{<\kappa}$.
(5) $1_{\mathbf{P}} \Vdash_{\mathbf{P}} (\Psi(\alpha_s,\alpha_{s\hat{}\langle 0\rangle},p_s,p_{s\hat{}\langle 0\rangle},p_{s\hat{}\langle 1\rangle},\dot{T},\ddot{B}))$ for all $s\in 2^{<\kappa}$.

Let $\alpha_{\langle\ \rangle}=0$ and $p_{\langle\ \rangle}=1_{\mathbf{P}'}$. Assume that we have α_s and p_s for all $s\in 2^{<\kappa}$.

Case 1. $\alpha=\gamma+1$.
Let $s\in 2^\gamma$. Since

$$1_{\mathbf{P}} \Vdash_{\mathbf{P}} (\Phi(\alpha_s,p_s,\dot{T},\ddot{B})),$$

then there is a $\beta<\kappa$, $\beta>\alpha_s$, and $q^i\leq_{\mathbf{P}'} p_s$ such that

$$1_{\mathbf{P}} \Vdash_{\mathbf{P}} (\Psi(\alpha_s,\beta,p_s,q^0,q^1,\dot{T},\ddot{B}))$$

by Claim 2. Let $\alpha_{s\hat{}\langle i\rangle}=\beta$ and $p_{s\hat{}\langle i\rangle}=q^i$. (Note that q^0,q^1 are incompatible by Claim 2.)

Let G be any $\mathbf{P}$-generic filter over M. Then

$$M[G]\models[\Phi(\alpha_s,p_s,T,\dot{B})].$$

Hence in $M[G]$ there is an $x\in T_{\alpha_s}$ such that $p_s \Vdash_{\mathbf{P}'} (x\in\dot{B})$. Since

$$M[G]\models[\Psi(\alpha_s,\alpha_{s\hat{}\langle 0\rangle},p_s,p_{s\hat{}\langle 0\rangle},p_{s\hat{}\langle 1\rangle},T,\dot{B}) \text{ and } x\in T_{\alpha_s}],$$

then there are $x^i\in T_{\alpha_{s\hat{}\langle i\rangle}}$ such that

$$M[G]\models[p_{s\hat{}\langle i\rangle} \Vdash_{\mathbf{P}'} (x^i\in\dot{B})].$$

This implies that

$$1_{\mathbf{P}} \Vdash_{\mathbf{P}} (\Phi(\alpha_{s^\frown\langle i\rangle}, p_{s^\frown\langle i\rangle}, \dot{T}, \ddot{B})).$$

Case 2. α is a limit ordinal below κ.

Let $s \in 2^\alpha$. Since $\langle \alpha_{s\restriction\beta} : \beta < \alpha \rangle$ is increasing in κ, $\langle p_{s\restriction\beta} : \beta < \alpha \rangle$ is decreasing in $\mathbf{P}'$ and

$$1_{\mathbf{P}} \Vdash_{\mathbf{P}} (\Phi(\alpha_{s\restriction\beta}, p_{s\restriction\beta}, \dot{T}, \ddot{B}))$$

for all $\beta < \alpha$, then there is an

$$\alpha_s = \sup\{\alpha_{s\restriction\beta} : \beta < \alpha\}$$

and a $p_s \leq_{\mathbf{P}'} p_{s\restriction\beta}$ for all $\beta < \alpha$ such that

$$1_{\mathbf{P}} \Vdash_P (\Phi(\alpha_s, p_s, \dot{T}, \ddot{B}))$$

by Claim 3.

We now work within $M[G_{\mathbf{P}}]$ to construct a subtree $T' = \{t_s : s \in 2^{<\kappa} \cap M\}$ of T such that

(1) the map $s \mapsto t_s$ is an isomorphic imbedding from $\langle 2^{<\kappa} \cap M, \subseteq\rangle$ to T.
(2) $t_s \in T_{\alpha_s}$ and $p_s \Vdash_{\mathbf{P}'} (t_s \in \dot{B})$ for all $s \in 2^{<\kappa} \cap M$.

Let $t_{\langle\ \rangle}$ be the element in T_0. Assume that we have t_s for all $s \in 2^{<\alpha} \cap M$.

Case 1. $\alpha = \beta + 1$.

Let $s \in 2^\beta \cap M$. Since $p_s \Vdash_{\mathbf{P}'} (t_s \in \dot{B})$ and $\Psi(\alpha_s, \alpha_{s^\frown\langle 0\rangle}, p_s, p_{s^\frown\langle 0\rangle}, p_{s^\frown\langle 1\rangle}, T, \dot{B})$ is true, there are $t^i \in T_{\alpha_{s^\frown\langle 0\rangle}}$ such that $t <_T t^i$, $t^0 \neq t^1$, and $p_{s^\frown\langle i\rangle} \Vdash_{\mathbf{P}'} (t^i \in \dot{B})$.

Let $t_{s^\frown\langle i\rangle} = t^i$ for $i = 0, 1$.

Case 2. α is a limit ordinal below κ.

Let $s \in 2^\alpha \cap M$. Since $\Phi(\alpha_s, p_s, T, \dot{B})$ is true, there is an $x \in T_{\alpha_s}$, such that $p_s \Vdash_{\mathbf{P}'} (x \in \dot{B})$. Since $\forall\beta < \alpha\ (p_s \leq p_{s\restriction\beta})$, then $p_s \Vdash_{\mathbf{P}'} (t_{s\restriction\beta} \in \dot{B})$. Now $t_{s\restriction\beta} <_T x$ because $\alpha_s > \alpha_{s\restriction\beta}$ for all $\beta < \alpha$.

Let $t_s = x$.

We have now finished construction and T' is a desired subtree of T.

Proof of Theorem 1: Let $\kappa_1 < \kappa_2$ be two inaccessible cardinals in M. Let $\mathbf{P}_1 = Lv(\kappa_2, \kappa_1)$, $\mathbf{P}_2 = Fn(\kappa_2^+, 2, \kappa_1)$, and $\mathbf{P}_3 = Lv(\kappa_1, \omega)$ in M. Let G_1 be a $\mathbf{P}_1$-generic filter over M, $M' = M[G_1]$, G_2 be a $\mathbf{P}_2$-generic filter over M', $M'' = M'[G_2]$, G_3 be a $\mathbf{P}_3$-generic filter over M'' and $M''' = M''[G_3]$. We want to show that $M''' \models [CH, 2^{\omega_1} = \omega_3$, there exists a thick Kurepa tree and there exist no Jech–Kunen trees].

We list some simple facts first:

(1) $M' \models [2^{\kappa_1} = \kappa_1^+ = \kappa_2]$.
(2) $M'' \models [2^{\kappa_1} = \kappa_1^{++} = \kappa_2^+]$.
(3) $M''' \models [CH, \kappa_1 = \omega_1, 2^{\omega_1} = \omega_3 = \kappa_1^{++}$ and $T = \langle 2^{<\kappa} \cap M'', \subseteq\rangle$ is a thick Kurepa tree.]

See [7], p. 232 for the proof of this.

We now show that in M''' there are no Jech–Kunen trees.

Suppose that T is a Jech–Kunen tree in M'''. Since the cardinality of T is $\omega_1 = \kappa_1$, there exists a $\theta < \kappa_2$ and a subset $I \subseteq \kappa_2^+$ of power κ_1 such that

$$T \in M[G_1 \cap Lv(\theta, \kappa_1)][G_2 \cap Fn(I, 2, \kappa_1)][G_3].$$

Let $G_1' = G_1 \cap Lv(\theta, \kappa_1)$, $G_1'' = G_1 \cap Lv(\kappa_2 \backslash \theta, \kappa_1)$, $G_2' = G_2 \cap Fn(I, 2, \kappa_1)$ and $G_2'' = G_2 \cap Fn(\kappa_2^+ \backslash I, 2, \kappa_1)$. Then the cardinality of $\mathcal{B}(T)$ in $M[G_1'][G_2'][G_3]$ is less than κ_2. Since the cardinality of $\mathcal{B}(T)$ in M''' is at least $\omega_2 = \kappa_2$, there exists a new branch of T in $M''' \backslash M[G_1'][G_2'][G_3]$.

$\mathbf{P}_3$ has κ_1-*c.c.* and $Lv(\kappa_2 \backslash \theta, \kappa_1) \times Fn(\kappa_2^+ \backslash I, 2, \kappa_1)$ is κ_1-closed. By Lemma 2, there exists a subtree T' of T in $M[G_1'][G_2'][G_3]$, which is isomorphic to the tree $\langle 2^{<\kappa_1} \cap M[G_1'][G_2'], \subseteq \rangle$.

Now we have that $M''' \models [|\mathcal{B}(T')| = 2^{\kappa_1} = \kappa_2^+ = 2^{\omega_1}]$. Since

$$M''' \models [|\mathcal{B}(T)| \geq |\mathcal{B}(T')| = 2^{\omega_1}],$$

T cannot be a Jech–Kunen tree in M'''. A contradiction.

Remark In the proof above $\mathbf{P}_2$ can be $Fn(\lambda, 2, \kappa_1)$ for any regular cardinal $\lambda > \kappa_1$. As a result 2^{ω_1} can be very large in the final model.

Corollary 3 *Assuming the existence of two strongly inaccessible cardinals, it is consistent with CH plus* $2^{\omega_1} > \omega_2$ *that every Kurepa tree is thick.*

Remark: We call that a Kurepa tree T is thin if $|\mathcal{B}(T)| = \omega_2$. If we start from M, a model of *GCH*, let $\mathbf{P} = Fn(\kappa, 2, \omega_1)$ for some regular cardinal $\kappa > \omega_2$ in M and G be a $\mathbf{P}$-generic filter over M, then $M[G]$ is a model of *CH* plus $2^{\omega_1} > \omega_2$ in which every Kurepa tree is thin. It is interesting to compare this with the above corollary.

Under $\neg CH$, an ω_1-tree is called a *Canadian* tree (Baumgartner [1]) (or a weak Kurepa tree—see Todorčević [8]) if $|\mathcal{B}(T)| > \omega_1$.

Corollary 4 *Assuming the existence of two strongly inaccessible cardinals, it is consistent with* $\neg CH$ *plus* $2^{\omega_1} > \omega_2$ *that there exists a thick Kurepa tree and every Canadian tree has* 2^{ω_1}*-many branches.*

Proof: Let M, $\mathbf{P}_1$, $\mathbf{P}_2$, G_1, G_2, M', and M'' be the same as in the proof of Theorem 1. Let

$$\mathbf{P}_3 = Lv(\kappa_1, \omega) \times Fn(\kappa_2^+, 2),$$

G_3 be a $\mathbf{P}_3$-generic filter over M'' and $M''' = M''[G_3]$. Then

$$M''' \models [2^{\omega} = 2^{\omega_1} = \omega_3 \text{ and there exists a thick Kurepa tree.}]$$

Let T be a Canadian tree in M'''. Then there exists a subset I of κ_2^+ with $|I| \leq \kappa_1$ such that

$$T \in M''[G_3 \cap Lv(\kappa_1, \omega) \times Fn(I, 2)].$$

Let $G_3' = G_3 \cap Lv(\kappa_1, \omega) \times Fn(I, 2)$. Since $Fn(\kappa_2^+ \backslash I, 2)$ is σ-centered, every branch of T in M''' is already in $M''[G_3']$. Since $Lv(\kappa_1, \omega) \times Fn(I, 2)$ is also κ_1-*c.c.*, then by the same argument as in the proof of Theorem 1 we can show that T has $2^{\omega_1} = \kappa_2^+ = \omega_3$-many branches in $M''[G_3']$. Hence T has $2^{\omega_1} = \omega_3$-many branches in M'''.

We would like to end this paper by asking some questions.

(1) Can we find a model of CH plus $2^{\omega_1} > \omega_2$ in which there exists a Jech-Kunen tree but there are no Kurepa trees?

The author found [4], by assuming the existence of one inaccessible cardinal, a model of CH plus $2^{\omega_1} > \omega_2$ in which there exists a Jech-Kunen tree which has no Kurepa subtrees.

(2) Can we assume the existence of only one inaccessible cardinal in Theorem 1?
(3) Can we add Martin's Axiom to the model in Corollary 4?
(4) Can we find a model of CH plus $2^{\omega_1} = \omega_4$ in which only Kurepa trees with ω_3-many branches exist?

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