Notre Dame Journal of Formal Logic
Volume 33, Number 1, Winter 1992

Interpretability of Robinson Arithmetic in the Ramified Second-Order Theory of Dense Linear Order

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Abstract After a description of the ways in which predicative higher-order logic is thought too weak to be of interest, it is shown to be in some ways surprisingly rich: dense linear order, which has a decidable first-order theory, has an essentially undecidable theory in ramified second-order logic. Extensions of the main result are described and their philosophical significance briefly discussed.

Although some claims have been made for the philosophical interest of predicative higher-order logics (Hacking [3], Hazen [4] and [5]), the general impression among mathematical logicians seems to be that systems like Ramified Type Theory are too complicated (in uninteresting ways) and too weak to be worth studying. It cannot be denied that these systems are, in some ways, extremely weak. As pointed out in Sundholm [8], Ramified Type Theory satisfies the hypotheses of Lindstrom's Theorem (these hypotheses could be summed up roughly as being that a completeness proof by Henkin's methods is possible for the logic, which was shown for Ramified Type Theory in Leblanc [6]), so that in one way its language has no more expressive power than that of first-order logic: if two possible worlds are not discriminated by any first-order sentence, then they will not be discriminated by sentences of a ramified higher-order language (based, in an appropriate sense, on the same primitive predicates) either. The significance of Lindstrom's Theorem, however, should not be overrated. There are other possible measures of the expressive power of a language. For example, first-order logic with predicate modifiers is, from the point of view of completeness theorems, only trivially different from ordinary first-order logic, but a predicate modifier language with a finite vocabulary may be able to define more subsets of the domain of a model than can be defined in any ordinary first-order language with finitely many predicates, all definable in the first language:

Received May 1, 1987

that is, on a measure of expressive power at least as significant intuitively as Lindstrom's, no pure predicate fragment of a predicate modifier language will exhaust the expressive power of the full language. Thus we may hope to find senses in which, despite Lindstrom's Theorem, a predicative higher-order language allows the expression of propositions that cannot be expressed in the corresponding first-order language.

In what follows I wish to show that even a very weak predicative logic, the "Ramified Functional Calculus of Second Order and Second Level" of Section 58 of Church [2], in which only the lowest order of propositional functions is quantified over and the ramification is pursued to only two levels, is surprisingly rich. In particular, although the first-order theory of dense linear order is well known to be decidable (Cantor's proof that, once we have decided whether or not to have endpoints, there is, up to isomorphism, only one denumerable dense order that can be syntacticalized into an elimination-of-quantifiers decision procedure), the essentially undecidable theory Q (Robinson Arithmetic) of Tarski [9] can be relatively interpreted in its ramified second-order theory. The ramified second-order theory of dense linear order, therefore, is a far more complicated affair, from a metamathematical point of view, than the corresponding first-order theory. If we think of propositions as essentially conceptual entities, it is plausible to stipulate that distinct sentences express the same proposition only if their equivalence is effectively recognizable. Under this stipulation our result shows that, despite the extreme weakness of ramified second-order logic, there are more propositions expressed in the language of the ramified second-order theory than in that of the first-order theory.

Our language will have the usual logical connectives and quantifier symbols. Lower case letters from the back of the alphabet will be individual variables. (Lowercase i, j, k, m, and n will be used later as defined numerical variables.) We will have two dyadic predicate constants, $\leq$ and $=$, expressing the order relation and the identity relation on individuals: these, from the point of view of the ramified logic, will be thought of as expressing propositional functions of the lowest level (predicative functions, in the language of either Whitehead and Russell [10] or [2]). (We take identity of individuals as primitive for the sake of convenience: if we defined it as the sharing of all lowest-level properties it would still be possible to interpret Q in the ramified second-order theory, though with somewhat more complicated definitions and using an additional level of ramification.) Uppercase A, B, C will be monadic predicate variables of the lowest level: variables which, in the notation of [2], would have the superscript $1/1$. Uppercase R will be a dyadic predicate variable of lowest level: one which, in [2], would have the superscript $2/1$. Uppercase F, G, H will be monadic predicate variables of second level: variables with superscript $1/2$. Where needed, primes will be used to create extra variables of the various sorts. Additional notation will be introduced by definition.

We adopt the (first-order) axiom that $\leq$ represents a dense linear order with neither top nor bottom. In detail: $\leq$ expresses a reflexive, transitive, antisymmetric, and connected relation; every individual has both $\leq$-successors and $\leq$-predecessors, and for any two distinct individuals there is some individual $\leq$-between them.

We now start defining things. First, set inclusion (a slight abuse of notation is involved here, since we take inclusion as a relation between monadic propositional functions rather than sets):

D1 $$F \subseteq G =_{df} \forall x(Fx \supset Gx).$$

(Here, and elsewhere when we define notation with free F, G, or H, we allow its use with A, B, or C as well.) Next we define a monadic propositional function as *standard* if the individuals of which it holds are discretely ordered by $\leq$ and either the function holds of nothing or there are top and bottom individuals of which it holds:

D2
$$\begin{aligned}
\mathbf{Std}(F) =_{df} (-\exists xFx \vee\ & \\
&(\exists x\forall y(Fx \;\&\; (Fy \supset y \leq x)) \;\& \\
&\exists x\forall y(Fx \;\&\; (Fy \supset x \leq y)) \;\& \\
&\forall x(Fx \supset (\forall y(Fy \supset y \leq x) \vee \\
&\qquad \exists y(Fy \;\&\; {\sim}x = y \;\&\; x \leq y \;\& \\
&\qquad \forall z((Fz \;\&\; x \leq z \;\&\; z \leq y) \supset (z = x \vee z = y))))) \;\& \\
&\forall x(Fx \supset (\forall y(Fy \supset x \leq y) \vee \\
&\qquad \exists y(Fy \;\&\; {\sim}x = y \;\&\; y \leq x \;\& \\
&\qquad \forall z((Fz \;\&\; y \leq z \;\&\; z \leq x) \supset (z = y \vee z = x))))))).
\end{aligned}$$

To illustrate what can be done in our theory, we have as a trivial theorem that:

$$\forall F(\mathbf{Std}(F) \supset \exists G(\mathbf{Std}(G) \;\&\; F \subseteq G \;\&\; {\sim}(G \subseteq F))).$$

In proof note that F is either "null" (our use of familiar set-theoretic terminology in connection with propositional functions should be self-explanatory) or else there is a topmost individual of which F holds. In the former case the "singleton" function of any individual will do; since $=$ is a predicate of level 1 we have:

$$\forall x\exists A\forall y(Ay \equiv y = x).$$

In the latter case there is, by our axiom, an individual above all those of which F holds, and the "union" of F with the singleton of any such individual will do. But we have:

$$\forall F\forall x\exists G\forall y(Gy \equiv (Fy \vee y = x)).$$

It is often possible to think of the predicate variables of a given level as ranging over the subsets of the domain of individuals which are first-order definable in terms of predicates defined by formulas in which only variables of lower level occur. (This is what lies behind the possibility of a *substitutional* interpretation of the higher-order quantification of predicative logics.) The principle illustrated here is that first-order definability must, in this connection, be understood as *parametric* first-order definability: the predicate variables range over subsets of the domain which would be first-order definable if all the elements of the domain had names.

Note that there are no bound predicate variables in the *definientia* for inclusion and standardness. These notions, therefore, may be used in formulas defining propositional functions of any level. This also holds for two conditions, that of being an injection and that of being a bijection, which we might impose on a relation:

D3 $$R\,\mathsf{Inj}F > G =_{df} \forall x \forall y(xRy \supset (Fx \,\&\, Gy)) \,\&\, \forall x(Fx \supset \exists y(Gy \,\&\, xRy)) \,\&\, \forall x \forall y \forall z((xRy \,\&\, xRz) \supset y = z) \,\&\, \forall x \forall y \forall z((xRz \,\&\, yRz) \supset x = y),$$

D4 $$R\,\mathsf{Bij}F > G =_{df} R\,\mathsf{Inj}F > G \,\&\, \forall x(Gx \supset \exists y(Fy \,\&\, yRx)).$$

We now come to notions whose definition does involve quantifying predicate variables, and which, therefore, cannot occur in formulas substituted for predicate variables of the first level. First, the notion of equipollence. (Notions of weak and strict cardinal inequality could be defined similarly.) As with inclusion, we will construe this notion as relating monadic propositional functions rather than sets:

D5 $$F = G =_{df} \exists R(R\,\mathsf{Bij}F > G).$$

(We have reused a symbol here, but there should be no confusion: = designates identity when it stands between individual variables, but it is an abbreviation for the formula defining equipollence when it stands between predicate variables.) Note that the bound variable here is of level 1, so this notion can occur in formulas defining propositional functions of level 2 (formulas substituted for variables of level 2). We have the

Proposition *Equipollence is an equivalence relation (and, given the obvious definition, cardinal inequality would be a quasi-order and equipollence a congruence for cardinal inequality).*

The proof is by the observation that if two relations are both injections, their composition is also an injection (and is a propositional function of level no higher than the given relations). By similarly trivial arguments we may prove that, e.g., if one monadic propositional function is included in another, it is cardinally less than or equal to it.

Finally, we adapt one of the standard definitions of a finite set to define the notion of a *finite* monadic propositional function (it will be convenient to restrict the definition so that only standard functions count as finite):

D6 $$\begin{aligned}\mathsf{Fin}(F) =_{df} \mathsf{Std}(F) \,\&\, &(\sim\exists xFx \vee \\ &\forall G(G \subseteq F \supset \\ &((\exists x(Fx \,\&\, \forall y(Fy \supset x \leq y) \,\&\, Gy) \supset \\ &(\forall x \forall y((Fx \,\&\, Gx \,\&\, Fy \,\&\, x \leq y \,\&\, \\ &\qquad \sim\exists z(Fz \,\&\, x \leq z \,\&\, \sim x = z \,\&\, \\ &\qquad\qquad z \leq y \,\&\, \sim z = y)) \supset Gy) \supset \\ &F \subseteq G))))).\end{aligned}$$

In words: F is finite just in case F is standard and either F is null or every G included in F which (i) holds of the bottom F individual and (ii) holds of the $\leq$-next F individual (if there is one) after any F individual it holds of, will hold of every F individual. Here the bound variable is of level 2, so the notion of finiteness cannot occur in any formula substituted for a variable of either level 1 or level 2. On the other hand, any formula containing free variables of level (at most) 2 and bound variables of level (at most) 1 may be substituted for the G in the *definiens*, giving us a "principle of induction" for proving things about finite propositional functions. Since this is the only definition containing a bound variable of level 2, all of our other defined notions may occur in the induction clauses of such arguments.

We can now prove a slightly surprising theorem:

T1 $$\forall F(\mathbf{Fin}(F) \supset \exists A \forall x(Fx \equiv Ax)).$$

That is, provided a monadic propositional function of second level is finite, there is a propositional function of first level that is coextensive with it. This gives us, for finite propositional functions, what the Axiom of Reducibility (which we do *not* assume: cf. Section 59 of [2]) would claim for all propositional functions of second level. (It is also, under the name *Proposition 89.12*, one of the correct assertions of the much criticized Appendix B to the second edition of [10].) To prove T1, note that the condition

$$\exists A \forall y(Ay \equiv (Fy \;\&\; y \leq x))$$

defines a propositional function of second level and so can be substituted for the bound variable G in D6: T1, in other words, may be proven by the form of induction mentioned at the end of the preceding paragraph.

By a similar argument (combined with our earlier observation that standard functions have "proper supersets"), we may prove:

T2′ $$\forall F \forall G((\mathbf{Std}(F) \;\&\; \mathbf{Fin}(G)) \supset$$
$$\exists A(A = G \;\&\; \forall x \forall y((Fx \;\&\; Ay) \supset (x \leq y \;\&\; {\sim} x = y)))).$$

Since the final clause,

$$\forall x \forall y((Fx \;\&\; Ay) \supset (x \leq y \;\&\; {\sim} x = y)),$$

implies that F and A are "disjoint", this tells us that, given two standard propositional functions with at least one finite, a function can be found which is equipollent with one and disjoint from the other. This allows us to form what is, as far as cardinality is concerned, a "disjoint union" of any two finite propositional functions. Further, the disjoint union of two *finite* propositional functions will itself be finite, allowing us in this case to strengthen this to:

T2 $$\forall F \forall G((\mathbf{Fin}(F) \;\&\; \mathbf{Fin}(G)) \supset$$
$$(\exists A(A = G \;\&\; \forall x \forall y((Fx \;\&\; Ay) \supset (x \leq y \;\&\; {\sim} x = y))) \;\&$$
$$\exists H(\mathbf{Fin}(H) \;\&\; \forall x(Hx \equiv (Fx \vee Ax))))).$$

If our system were based on an impredicative second-order logic we could argue that, since a propositional function holding of *one* more individual than a given finite function is easily shown finite, the added clause can be proven by induction on the individuals falling under G; but the added clause involves the notion of finitude, which in our logic may not occur in a formula substituted for the bound variable in D6. To prove T2, suppose that, for given finite F and G, we have found A and H satisfying the theorem except, perhaps, with respect to the finitude of H. It is clear from the way in which induction was used in proving T2′ that we may suppose that we have a bijection between G and A which is *order preserving*. Now let H' be an arbitrary propositional function (of level 1 or 2) which is included in H, holds of the $\leq$-first individual F holds of, and holds of the $\leq$-next H individual after any H individual it holds of. By induction on the F individuals we may show that $F \subseteq H'$; if G is null we are done. If there is a G individual, and so an A individual, the $\leq$-first A individual is the $\leq$-next H individual after the last F individual, so H' holds of it. The propositional function "x is the G individual correlated by our bijection with an A individual of which H' holds" is of the same level as G, so we may use induction on the G individuals to show that it holds of all G individuals, and so that H' holds of all A individuals. Since H' was arbitrary, we have shown that H is finite. This is fairly typical of the maneuvers involved in finding proofs in predicative higher-order logics: where, reasoning impredicatively, we would use induction with a quantified clause, in the ramified system we do the induction for an arbitrary propositional function of lower level and infer the result by quantifier logic.

Finally, a double application of the same technique (using induction both on the "members" of F and on those of G) allows us to interpolate copies of one finite function between the "members" (and after the last member) of another:

T3 $\forall F \forall G((\mathbf{Fin}(F) \ \&\ \mathbf{Fin}(G)) \supset$

$$\begin{aligned}
&\exists A(\mathbf{Fin}(A)\ \&\\
&\quad \forall x(Ax \supset \exists B(Bx\ \&\ B = G\ \&\\
&\qquad \exists y(Fy\ \&\ \forall z((Az\ \&\ y \leq z\ \&\\
&\qquad\quad \forall w((Fw\ \&\ y \leq w\ \&\ {\sim}y = w) \supset z \leq w)) \equiv Bz))))\ \&\\
&\quad \forall y(Fy \supset \exists B(B = G\ \&\ \forall z(Bz \supset y \leq z)\ \&\\
&\qquad \forall w((Fw\ \&\ y \leq w\ \&\ {\sim}y = w) \supset\\
&\qquad\quad \forall z(Bz \supset (z \leq w\ \&\ {\sim}z = w)))\ \&\ B \subseteq A)))).
\end{aligned}$$

(A divides up into "subsets" B, each B being equipollent to G and either between two members of F or after the last member of F, and there is such a B between any two members of F and another after the last member of F.) This gives us, for any two finite F and G, a finite A which is equipollent to the "Cartesian product" of F with G.

For simplicity, as we have not introduced any notation of abstracts to denote particular propositional functions, we choose a version of Robinson Arithmetic in which the only terms are variables and in which, therefore, zero, successor, sum, and product are represented by predicates. Writing $\mathbf{Ze}(n)$, $n\mathbf{Sc}m$, $i\mathbf{Sm}jk$, and $i\mathbf{Pr}jk$ for n is zero, n is the successor of m, i is the sum of j with k, and i is the product of j with k respectively, we will have fifteen axioms. Eight of these will simply assert the existence and uniqueness of zero, successors, etc. (In their presence, therefore, we may introduce the constant for zero and the usual symbols for the arithmetic functions in the familiar Russellian way.) The other seven are variants of the usual seven axioms of Q: zero is not a successor, numbers having the same successor are equal, recursion equations for sum and product, and every number is either zero or a successor. We provide a relative interpretation of these axioms by identifying numbers with finite propositional functions of second level. That is, we interpret numerical quantifications as restricted quantification over such propositional functions:

D7 $$\forall n(\ldots n \ldots) =_{df} \forall F(\mathsf{Fin}(F) \supset \ldots F \ldots),$$

D8 $$\exists n(\ldots n \ldots) =_{df} \exists F(\mathsf{Fin}(F) \,\&\ldots F \ldots).$$

Note that, since numerical variables are interpreted by monadic predicate variables, the identity sign between numerical variables is to be read as expressing equipollence. It remains to provide interpretants for the specific arithmetical predicates, to prove that equipollence is a congruence for the arithmetical predicates so interpreted, and to prove that the axioms of Robinson Arithmetic, as interpreted, are theorems of our system.

Zero: We take a propositional function to be zero iff it is not true of any individual:

D9 $$\underline{\mathsf{Ze}}(F) =_{df} \sim\exists x(Fx).$$

Anything equipollent to a null propositional function is null. The existence axiom is trivial:

A1 $\exists n(\underline{\mathsf{Ze}}(n))$,

and, since a null relation is a bijection between null sets, so is the uniqueness axiom:

A2 $(\underline{\mathsf{Ze}}(m) \,\&\, \underline{\mathsf{Ze}}(n)) \supset m = n.$

(In writing the axioms of Robinson Arithmetic we follow convention by omitting initial universal quantifiers.)

Successor: We define one propositional function as a successor of a second if the first holds of everything the second does, and holds of one extra individual, or if the first is equipollent to a function that does this:

D10 $$F\underline{\mathsf{Sc}}G =_{df} \exists B\exists A(A = F \,\&\, B = G \,\&$$
$$\exists x(\sim Bx \,\&\, \forall y(Ay \equiv (By \vee y = x)))).$$

T1 shows that there will always be first level functions as required, and by framing the definition in terms of such equipollent functions we ensure that equipollence will be a congruence for successor. The existence axiom,

A3 $\exists n(n\underline{\mathsf{Sc}}m)$,

follows from our trivial theorem about standard propositional functions having proper supersets. The uniqueness axiom,

A4 $(m\underline{\mathsf{Sc}}i \,\&\, n\underline{\mathsf{Sc}}i) \supset m = n$, is proven by noting that, where A and B each hold of the individuals and C holds of plus one extra, the "union" of the identity relation on C with the relation holding only between the two added individuals of A and B is a bijection between A and B.

A5 $\underline{\mathsf{Ze}}(m) \supset \sim(m\underline{\mathsf{Sc}}n)$,

zero is not a successor, follows from the fact that successors have to be true of *something*, whereas zeros cannot be. To prove

A6 $(i\underline{\mathsf{Sc}}m \;\&\; i\underline{\mathsf{Sc}}n) \supset m = n$,

numbers having the same successor are equal, suppose that F is a successor both to G and to H. Then there are A, A', both equipollent to F, with A holding of one more individual than some B equipollent to G and A' of one more individual than some C equipollent to H. Since compositions of bijections are bijections, this, with at most a bit of "fiddly work" to get the "extra" individuals falling under A and A' to "line up", gives us a bijection between G and H.

Sum: F is a sum of G and H if it is equipollent to a disjoint union of G and H:

D11
$$F\underline{\mathsf{Sm}}GH = \exists A \exists B \exists C(F = A \;\&\; G = B \;\&\; C = H \;\& \\ \sim\exists x(Bx \;\&\; Cx) \;\& \\ \forall x(Ax \equiv (Bx \vee Cx))).$$

Once again, framing the definition in terms of the existence of equipollent propositional functions ensures that equipollence is a congruence for sum. The existence axiom,

A7 $\exists i(i\mathsf{Sm}jk)$,

is an immediate consequence of T2, and the uniqueness axiom,

A8 $(m\underline{\mathsf{Sm}}ij \;\&\; n\underline{\mathsf{Sm}}ij) \supset m = n$,

is provable by an elementary argument. (We call an argument *elementary* if it goes by standard quantificational logic or, at worst, involves proving the existence of a propositional function—in the case of A8, a bijection—which is first-order definable in the data of the theorem.) The first recursion equation,

A9 $(\underline{\mathsf{Ze}}(j) \;\&\; n\underline{\mathsf{Sm}}ij) \supset n = i$,

follows from the fact that the "union" of any propositional function with a null one is equivalent to the original function. For the second recursion equation,

A10 $(k\underline{\mathsf{Sc}}j \;\&\; m\underline{\mathsf{Sm}}ij \;\&\; n\underline{\mathsf{Sm}}ik) \supset n\underline{\mathsf{Sc}}m$,

the elementary argument again turns on pairing off the "extra" individuals falling under certain "successor" propositional functions.

Product: We will again make sure that equipollence is a congruence for the defined notion by defining the arithmetic relation as holding between given propositional functions just in case there exist functions equipollent to them which satisfy a certain condition. The underlying idea, however, is that a function is the product of two functions just in case it is equipollent to their Cartesian product, or rather to the *ersatz* Cartesian product provided by T3. Thus:

D12 $$\begin{aligned} F\underline{\mathbf{Pr}}GH =_{df} \exists A \exists B \exists C(&A = F \,\&\, B = G \,\&\, C = H \,\&\\ &\forall x(Ax \supset \exists B'(B'x \,\&\, B' = B \,\&\\ &\quad \exists y(Cy \,\&\, \forall z(B'z \equiv\\ &\qquad (Az \,\&\, y \leq z \,\&\\ &\qquad \forall w((Cw \,\&\, y \leq w \,\&\, \sim y = w) \supset\\ &\qquad\quad (z \leq w \,\&\, \sim z = w)))))))\,\&\\ &\forall y(Cy \supset \exists B'(B' = B \,\&\, \forall z(B'z \supset y \leq z)\,\&\\ &\quad \forall w((Cw \,\&\, y \leq w \,\&\, \sim y = w) \supset\\ &\qquad \forall z(B'z \supset (z \leq w \,\&\, \sim z = w))\,\&\\ &\quad B' \subseteq A)))). \end{aligned}$$

Existence,

A11 $\exists i(i\underline{\mathbf{Pr}}jk)$,

is immediate from T3, and uniqueness,

A12 $(m\underline{\mathbf{Pr}}ij \,\&\, n\underline{\mathbf{Pr}}ij) \supset m = n$,

is proven elementarily by composing bijections. The first recursion equation,

A13 $(\underline{\mathbf{Ze}}(j) \,\&\, n\underline{\mathbf{Pr}}ij) \supset \underline{\mathbf{Ze}}(n)$,

follows from the requirement that if anything falls under the propositional function A in D12, there must exist an individual falling under C for it to $\leq$-follow. To prove the second recursion equation,

A14 $(k\underline{\mathbf{Sc}}j \,\&\, m\underline{\mathbf{Pr}}ij \,\&\, n\underline{\mathbf{Pr}}ik) \supset n\underline{\mathbf{Sm}}mi$,

suppose that $F\underline{\mathbf{Pr}}GH$, $F'\underline{\mathbf{Pr}}GH'$, and $H'\underline{\mathbf{Sc}}H$, with all the propositional functions finite. Then there will be $A = F$, dividing up into blocks of individuals, each block equipollent to G, with one block following each individual falling under a $C = H$, and an $A' = F'$, also dividing into blocks equipollent to G, with one block following each individual falling under a $C' = H'$. Since $H'\underline{\mathbf{Sc}}H$, $C'\underline{\mathbf{Sc}}C$, it follows elementarily that there is a one–one mapping, R, correlating the individuals falling under C with all but the $\leq$-last of the individuals falling under C'; since C is finite we may prove inductively that there is such an R which is order preserving. Since all the blocks of A and A' are equipollent to G and so to each other, we may use induction again to prove that there is a one–one mapping R' between the individuals falling under A and all but the last block of individuals falling under A'. (In the inductive step of this last argument we extend the R'

correlating the earlier blocks of A and A' individuals to a new block, defining the new R' in terms of the old and the given bijections between the blocks and G; R is used to specify *which* block of A' individuals to correlate with the new block of A individuals.) A' may now be viewed as the "union" of an initial part equipollent to A (and so to F) with a last block equipollent to G, showing that $F' \underline{\mathbf{Sm}} FG$.

Finally, the last axiom,

A15 $\underline{\mathbf{Ze}}(n) \vee \exists m(n\underline{\mathbf{Sc}}m)$,

amounts to the trivial observation that any non-null propositional function holds of some individual that some other function does not hold of.

Using the same interpretation of the language of arithmetic (including $\leq$ for numbers, defined as cardinal inequality of propositional functions), it is easy to verify the remainder of the usual ordered commutative semiring axioms. It is also possible to derive mathematical induction for arithmetic conditions containing only *bounded* numerical quantifiers, the trick being to treat the quantifiers as ranging over *standard* functions cardinally less than the bound rather than as over *finite* functions. Since it can be shown that any propositional function cardinally less than a finite one is itself finite, the intended meaning of bounded numerical quantification is successfully captured by this interpretation. For such further topics as the theory of the # function from Nelson [7], it appears that different definitions, involving higher levels of ramification, are needed; since the consistency of our system can be proven in arithmetic there are clearly limits to what can be achieved in this manner.

There is some interest in comparing our interpretation of a fragment of arithmetic with the treatment in [10]. Since we wished to work entirely in the Ramified Functional Calculus of Second Order, we were unable to use the "logicist" definitions of [10]. As suggested in Church [1], however, a treatment similar to ours might be more natural than the reliance on equivalence classes in [10] and just as much in keeping with the spirit of logicism. Certainly the treatment of numbers as propositional functions with a new "equality" relation is in the spirit of Russell's "definitions in use". The use of a ramified logic of higher order with something like the definitions of [10], however, might be more convenient in practice: intuitive combinatorial constructions can be formalized more naturally in a higher-order logic. The order relation on individuals and the axiom of dense linear order are convenient but can be replaced in a higher-order framework with much weaker axioms of infinity. Even in our second-order framework, the nonlogical predicate expressing the order relation could have been dispensed with (with some loss of simplicity and perspicuity) in favor of a second-order existential quantification stating that there is *some* relation ordering individuals in a dense linear order.

I conjecture that a predicative higher-order logic with a weak axiom of infinity would provide a natural and perspicuous framework for what Nelson in [7] calls *predicative arithmetic*. Since the criterion tentatively adopted in [7] of whether a piece of mathematics is predicative is whether it can be developed in a theory interpretable in Q, there is a sense in which this is trivially true. The substantive conjecture is about *naturalness* and (conceptual and practical) *perspicuity*.

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