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Concerted Instant-Interval Temporal Semantics II: Temporal Valuations and Logics of Change

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Abstract The general problem of the relationship between instant-based and interval-based temporal semantics is studied. The paper is in two parts. In the first part we specified conditions for the mutual definability of instant and interval temporal structures. In this second part we extend this 'area of agreement' for temporal semantics proper and consider some natural 'logics of change' generated by this correspondence.

In the first part of our paper we considered the possibilities of mutual definability for instant-based and interval-based temporal ontologies. Here we will try to extend this 'agreement area' on temporal semantics. We will consider various instant and interval valuations and will try to determine the conditions for their definability in terms of each other. It will be shown that such definable valuations give rise to natural logics, which could be regarded as logics of change.

1 Technical preliminaries We will be working in the frameworks of open dense linear instant-interval structures. Valuations on these structures will be represented as valuation predicates, that is, $[p]_i$ will denote a two-place valuation predicate $V(p,i)$ with respect to propositions and times (either instants or intervals). We will also be using the following notions:

Definition 1
(i) A point α is a *boundary point* of an interval t (in notation, $\alpha + t$) iff there is another point β such that t is bounded by these points.
(ii) A point α is an *internal point* of an interval t (in notation, $\alpha \lessdot t$) iff it belongs to t but is not a boundary point of it.

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In order to give a smooth description of the necessary notions we will introduce a number of auxiliary temporal operators, which will simplify the presentation of subsequent results. Not all of them are independent, some even coincide (cf. Up and U^{+}p below), but this is not essential for our purposes. But it is worth noting that they are sufficient for our descriptions, and this reflects the level of complexity required.

Interval operators:

$[\mathrm{T}p]_t \equiv (\exists s)(t \leq s \;\&\; [p]_s)$
$[\mathrm{S}p]_t \equiv (\exists s)(s \leq t \;\&\; [p]_s)$
$[\mathrm{L}p]_t \equiv (\exists \alpha)(\alpha < t \;\&\; \langle p \rangle_\alpha)$
$[\mathrm{L}^{\circ}p]_t \equiv (\exists\ \alpha)(\alpha <\cdot\ t \;\&\; \langle p \rangle_\alpha)$
$[\mathrm{L}^{+}p]_t \equiv (\exists \alpha)(\alpha + t \;\&\; \langle p \rangle_\alpha)$
$[\mathrm{I}p]_t \equiv (\alpha)(\alpha < t \rightarrow (\exists s)(\alpha < s \leq t \;\&\; [p]_s))$
$[\mathrm{I}^{\circ}p]_t \equiv (\alpha)(\alpha <\cdot\ t \rightarrow (\exists s)(\alpha <\cdot\ s \leq t \;\&\; [p]_s))$
$[\mathrm{I}^{+}p]_t \equiv (\alpha)(\alpha + t \rightarrow (\exists s)(\alpha + s \leq t \;\&\; [p]_s))$.

Instant operators:

$\langle \mathrm{A}p \rangle_\alpha \equiv (\exists t)(\alpha < t \;\&\; [p]_t)$
$\langle \mathrm{A}^{\circ}p \rangle_\alpha \equiv (\exists t)(\alpha <\cdot\ t \;\&\; [p]_t)$
$\langle \mathrm{A}^{+}p \rangle_\alpha \equiv (\exists t)(\alpha + t \;\&\; [p]_t)$
$\langle \mathrm{U}p \rangle_\alpha \equiv (t)(\alpha < t \rightarrow (\exists s)(\alpha < s \leq t \;\&\; [p]_s))$
$\langle \mathrm{U}^{\circ}p \rangle_\alpha \equiv (t)(\alpha <\cdot\ t \rightarrow (\exists s)(\alpha <\cdot\ s \leq t \;\&\; [p]_s))$
$\langle \mathrm{U}^{+}p \rangle_\alpha \equiv (t)(\alpha + t \rightarrow (\exists s)(\alpha + s \leq t \;\&\; [p]_s))$.

As can be seen, the operators L, A, and U depend on two valuations and hence their meaning is determined only if the 'destination' valuation is known, while the source valuation is not important.

2 Interval valuations We will begin with primary interval valuations, which we will call *at-valuations* (cf. Vlach [12]). However vague, the intuitive principle behind them is that an evaluation is made directly with respect to a given interval (and not, for example, by virtue of some subinterval of this interval). For such valuations we may expect that the distribution of truth values for a given proposition must depend mainly on this proposition and not on the properties of the valuation itself. Below we will describe the main types of interval-evaluated propositions (cf. Burgess [6]). Note that we give, in fact, pairs of dual properties. This duality will play an essential role in the following.

Definition 2

(i) A proposition p is called *persistent* if the formula (T$p \rightarrow p$) is universally valid.
(ii) A proposition is called *negatively persistent* if the formula ($p \rightarrow {\sim}\mathrm{T}{\sim}p$) is universally valid.

It is clear that a proposition is negatively persistent if and only if its (classical) negation is persistent and vice versa. It is easy to show that (negatively) persistent propositions are representable as having the form Tp (${\sim}\mathrm{T}{\sim}p$) or ${\sim}\mathrm{S}{\sim}p$ (Sp) for some proposition p. As an example of persistent propositions we will mention aspectual types of activities and states.

Definition 3

(i) A proposition p is *intervally convex* if the formula $(\mathrm{T}p \,\&\, \mathrm{S}p \rightarrow p)$ is universally valid.

(ii) A proposition p is *negatively intervally convex* if the formula $(p \rightarrow {\sim}\mathrm{T}{\sim}p \vee {\sim}\mathrm{S}{\sim}p)$ is universally valid.

Intervally convex propositions are representable as having the form $\mathrm{T}p \,\&\, \mathrm{S}p$ and thus are definable as conjunctions of persistent and negatively persistent propositions. In the same way negatively convex propositions are definable as disjunctions of persistent and dual persistent propositions. A proposition is convex and negatively convex if and only if it is persistent or negatively persistent. Note that interval convexity was proposed in fact in Taylor [10] as a characteristic property of activities.

Definition 4

(i) A proposition p is *cumulative* if $\mathrm{I}^{\circ}p \rightarrow p$ is universally valid.

(ii) A proposition p is *negatively cumulative* if $p \rightarrow {\sim}\mathrm{I}^{\circ}{\sim}p$ is universally valid (that is, if its negation is cumulative).

(Negatively) cumulative propositions are representable as having the form $\mathrm{I}^{\circ}p$ $({\sim}\mathrm{I}^{\circ}{\sim}p)$. Note also that persistent propositions are obviously negatively cumulative, while negatively persistent ones are cumulative.

It can be shown that propositions of all elementary aspectual types are in fact cumulative in at-valuations. Indeed, for activities, states, and generics this is obvious, while for achievements and accomplishments it is trivially true, because the former are false at intervals, whereas intervals at which an accomplishment is true are not overlapped.

Definition 5

(i) A proposition p is *strongly cumulative* if ${\sim}\mathrm{S}{\sim}\mathrm{S}p \rightarrow p$ is universally valid.

(ii) A proposition p is *strongly negatively cumulative* if $p \rightarrow \mathrm{S}{\sim}\mathrm{S}{\sim}p$ is universally valid.

Among aspectual types only states and generics are strongly cumulative, while activities are only simply cumulative, because two abutting temporal intervals of some activity with an instantaneous interruption between them (for example, an instantaneous stopping in motion) do not form an interval of an (uninterrupted) activity. It must be noted here that so-called 'mass analogy' for aspectual types corresponds to strong cumulativity rather than to simple cumulativity.

We will say that a proposition is *(negatively) homogeneous* if it is (negatively) persistent and (negatively) cumulative, and that it is *strongly (negatively) homogeneous* if it is (negatively) persistent and strongly (negatively) cumulative. It is easy to show that p is homogeneous iff ${\sim}\mathrm{L}^{\circ}{\sim}\mathrm{A}^{\circ}p \leftrightarrow p$ is valid and negatively homogeneous iff $p \leftrightarrow \mathrm{L}^{\circ}{\sim}\mathrm{A}^{\circ}{\sim}p$ is valid. A proposition p is strongly homogeneous iff $p \leftrightarrow {\sim}\mathrm{S}{\sim}\mathrm{S}p$ is valid, while p is strongly negatively homogeneous iff $p \leftrightarrow \mathrm{S}{\sim}\mathrm{S}{\sim}p$ is valid.[1]

In addition to at-valuations there are two other valuations common in temporal semantics for natural languages (cf. Vlach [12]). *For-valuations* are persistent for all propositions, while *in-valuations* are negatively persistent for all propositions. It is clear that the properties of persistence and negative persistence here characterize the valuations themselves, rather than the corresponding propositions. Note that any for-valuation could be represented as $(\exists s)(t \leq s \;\&\; \mathrm{V}(p,s))$ or as $(s)(s \leq t \rightarrow \mathrm{V}(p,s))$ for some valuation V, whereas a negatively persistent valuation is representable as $(\exists s)(s \leq t \;\&\; \mathrm{V}(p,s))$ or as $(s)(t \leq s \rightarrow \mathrm{V}(p,s))$.

What is important for the following is that since all aspectual types are cumulative in at-valuations, in the for-valuation, defined as $(\exists s)(t \leq s \;\&\; \mathrm{V}(p,s))$, where V is an at-valuation, they all will be homogeneous.

3 Instant valuations There are also plausible classifications for instant-evaluated propositions.

Definition 6
(i) A proposition p is *open* iff the following formula is universally valid: $p \leftrightarrow \mathrm{A}^\circ {\sim} \mathrm{L}^\circ {\sim} p$.
(ii) A proposition p is *closed* iff the formula $p \leftrightarrow {\sim}\mathrm{A}^\circ {\sim} \mathrm{L}^\circ p$ is universally valid.

According to Aristotle, propositions expressing motion are in fact open, while states are in general closed.

Definition 7
(i) A proposition p is *regular open* iff the formula $p \leftrightarrow \mathrm{A}^\circ {\sim} \mathrm{S} {\sim} \mathrm{L}^\circ p$ is universally valid.
(ii) A proposition p is *regular closed* iff the formula $p \leftrightarrow {\sim}\mathrm{A}^\circ {\sim} \mathrm{S} {\sim} \mathrm{L}^\circ {\sim} p$ is universally valid.

As will be shown, the above four types of propositions strongly correspond to interval types of (negative) homogeneity and strong homogeneity.

4 Finiteness requirements There are many reasons to think that empirically definable propositions and valuations cannot"change their minds" too often, or, to be more exact, corresponding truth values couldn't change an infinite number of times in a finite interval.[2] We now give a formal description of this requirement for instant-evaluated propositions:

Definition 8 An instant-evaluated proposition p is *finite* iff the following formula is universally valid:

$$(1) \qquad {\sim}\mathrm{U}^+{\sim}({\sim}\mathrm{L}^\circ{\sim}p \vee {\sim}\mathrm{L}^\circ p).$$

For compact intervals this means that they may be divided on a finite number of subintervals, such that for any of them p is either true throughout or false throughout. Another equivalent to condition (1) is that the truth-set of p has a discrete boundary in the topological sense. We will say that a valuation is finite if all evaluated propositions are finite.

The finiteness requirement (l) implies the universal interval validity of the following formula:

$$(2) \qquad \mathrm{S}({\sim}\mathrm{L}^\circ{\sim}p \vee {\sim}\mathrm{L}^\circ p).$$

This latter condition corresponds to von Wright's requirement about the possibility of dissecting any interval on subintervals in which p is either true throughout or false throughout.[3]

Now we will give corresponding finiteness requirements for interval-evaluated propositions. These requirements are less straightforward, but we may give for them a natural indirect description. For any interval valuation $\mathrm{V}(p,t)$ we may define the following 'two-indexed' instant valuation:

$$\mathrm{V}'(p,\alpha,\beta) \equiv (\exists t)(\alpha + t \;\&\; \beta + t \;\&\; \mathrm{V}(p,t)).$$

Note that if $\alpha \neq \beta$ then $\mathrm{V}'(p,\alpha,\beta) \leftrightarrow \mathrm{V}(p,[\alpha\beta])$. Now, for any fixed β, $\mathrm{V}'(p,\alpha,\beta)$ is an ordinary instant valuation, and it is natural to require that all these valuations be finite. But we will require more, namely that all these valuations be *jointly* finite:

$$(3)\quad \begin{aligned}&(\beta)(\beta + t \to (\exists s)(\beta + s \leq t \;\&\; (\alpha)((\gamma)(\gamma \neq \alpha \;\&\; \gamma \lessdot s \to \mathrm{V}(p,[\alpha\gamma])) \vee \\ &\qquad (\gamma)(\gamma \neq \alpha \;\&\; \gamma \lessdot s \to {\sim} V(p,[\alpha\gamma])))).\end{aligned}$$

The reason for this consists in an observation that in the opposite case we would have for a presumably finite intervally evaluated proposition a nonfinite differentiation of points for the derived instant-evaluated proposition. It will be shown in the next sections that there is a natural correspondence between conditions (1) and (3).

The formula (3) can be expressed in purely interval terms, but this is not essential for our purposes. Note, however, that it implies the universal interval validity of the following formula:

$$(4)\qquad \mathrm{S}({\sim}\mathrm{S}{\sim}p \vee {\sim}\mathrm{S}p).$$

Moreover, it is easy to show that for any proposition p and any instant α there are two nonoverlapping intervals t and s with a common boundary α, such that p has the same truth values in the following five classes of intervals:

(i) $\{u : \alpha \lessdot u \;\&\; u \leq t \nabla s\}$
(ii) $\{u : \alpha + u \;\&\; u \leq t\}$
(iii) $\{u : \alpha + u \;\&\; u \leq s\}$
(iv) $\{u : {\sim}\alpha < u \;\&\; u \leq t\}$
(v) $\{u : {\sim}\alpha < u \;\&\; u \leq s\}$

5 Instant-definable interval valuations We are approaching our main topic. The question is when interval valuations could be defined via some underlying instant valuations. One partial answer was given in Burgess [6], though in somewhat different terminology:

Theorem 1
(i) *Any homogeneous interval valuation $[p]_t$ is definable as*

$$(5)\qquad (\alpha)(\alpha \lessdot t \to \langle p\rangle_\alpha)$$

for some instant valuation $\langle p\rangle_\alpha$ and it may be required that $\langle p\rangle_\alpha$ be an open valuation.
(ii) *Any strongly homogeneous valuation is definable as* (5) *with the additional requirement that $\langle p\rangle_\alpha$ must be a regular open valuation (or as $[{\sim}\mathrm{S}{\sim}\mathrm{S}{\sim}\mathrm{L}^{\circ}{\sim}p]_t$ without such a requirement).*

Proof: Since for a homogeneous valuation p is equivalent to $\sim L^\circ \sim A^\circ p$, we may define the required instant valuation simply as $A^\circ p$ (and it is easy to see that it is an open valuation). Now, since any strongly homogeneous valuation is homogeneous, we have that it is also representable as (5). But it must satisfy strong cumulativity, that is, $\sim S \sim S \sim L^\circ \sim p \to \sim L^\circ \sim p$ must be valid, and hence the instant valuation must be regular.

Below we will give a couple of similar theorems.

Theorem 2

(i) *Any negatively homogeneous interval valuation* $[p]_t$ *is definable as*

$$(6) \qquad (\exists\alpha)(\alpha <\cdot\, t \;\&\; \langle p\rangle_\alpha)$$

for some instant valuation $\langle p\rangle_\alpha$*, and it may be required that* $\langle p\rangle_\alpha$ *be a closed valuation.*

(ii) *Any strongly negatively homogeneous valuation may be defined as* (6) *with the additional requirement that* $\langle p\rangle_\alpha$ *must be regular closed (or as* $[S\sim L^\circ \sim p]$ *without additional requirements).*

Proof: In any negatively homogeneous valuation p is equivalent to $L^\circ \sim A^\circ \sim p$ and hence we may define the required instant valuation as $\sim A^\circ \sim p$. In the case of strong negative homogeneity this valuation must satisfy $L^\circ p \to S \sim S \sim L^\circ p$ and hence it is regular closed. On the other hand, if we defined $\langle p\rangle_\alpha$ as $\langle A^\circ \sim S \sim p\rangle$, then we would have $p \leftrightarrow S \sim L^\circ \sim p$.

Theorem 3

(i) *Valuations that are convex, cumulative, and negatively cumulative are definable as*

$$(7) \qquad (\alpha)(\alpha <\cdot\, t \to \langle p\rangle_\alpha) \;\&\; (\exists\alpha)(\alpha <\cdot\, t \;\&\; (p)_\alpha)$$

where $\langle p\rangle_\alpha$ *and* $(p)_\alpha$ *are two instant valuations.*

(ii) *Valuations that are negatively convex, cumulative, and negatively cumulative are definable as*

$$(8) \qquad (\alpha)(\alpha <\cdot\, t \to \langle p\rangle_\alpha) \vee (\exists\alpha)(\alpha <\cdot\, t \;\&\; (p)_\alpha).$$

Proof: If we define $\langle p\rangle_\alpha$ as $[A^\circ p]_\alpha$ and $(p)_\alpha$ as $[\sim U^\circ \sim p]_\alpha$ then we will obtain that for any valuation which satisfies the condition from (i) above $[p]_t$ is equivalent to (7). For (ii) $\langle p\rangle_\alpha$ is definable as $[U^\circ p]_\alpha$ and $(p)_\alpha$ as $[\sim A^\circ \sim p]_\alpha$.

Although the above approach covers some important cases, we do not see for the time being how it could be generalized. Below we will try to employ a more straightforward approach.

5.1 Valuations as generalized quantifiers Suppose that we have an interval valuation $[p]_t$ and an instant valuation $\langle p\rangle_\alpha$ such that the following condition holds for any propositions p and q:

$$(*) \qquad \langle p\rangle = \langle q\rangle \to [p] = [q],$$

where $[p]$ and $\langle p \rangle$ are truth-sets of the proposition p in corresponding valuations. Then there must exist some relation Q between intervals and sets of instants such that for any p and any interval t:

$$(**) \qquad [p]_t \leftrightarrow Q(\langle p \rangle, t).$$

Thus, our interval valuation will be in fact definable in terms of the instant valuation $\langle p \rangle_\alpha$. On the other hand, the interval valuation in question could be considered as a *generalized quantifier* on instants and hence we may employ here a corresponding relatively well known theory (cf. especially van Benthem [1]–[2]).[4] Note, however, that in our case there are some important restrictions on possible arguments of the predicate Q: the first argument must be a truth-set of some proposition, while the second argument must be an interval. We will require that the set of truth-sets of propositions satisfy the following two conditions:

(i) It is closed with respect to finite intersections, taking complements, and automorphisms of the point-interval structure on itself.
(ii) It contains all convex sets of points.

It is clear that the set of all finite propositions and the set of all weakly finite propositions satisfy these conditions. The case of finite propositions will be considered separately below. Our sole unconditionally adopted postulate will be the following *Quality condition* (cf. van Benthem [2]):

(Quality) If f is an automorphism of a point-interval structure onto itself, then for any p and t:

$$Q(\langle p \rangle, t) \leftrightarrow Q(f(\langle p \rangle), f(t)).$$

Note that the exact force of this postulate depends on the class of possible automorphisms. We will presuppose that point-interval structures satisfy the principle of indistinguishability (cf. van Benthem [3]): For any two finite sets F and G of points and intervals having the same (first-order) type, there exists some automorphism sending F to G. Note that since our structure does not determine the direction of time, possible automorphisms must include *reversions* around some point.

Now we will give a list of plausible conditions for instant-definable interval valuations and explore their relationships. The plausability of these conditions stems from various sources, which will become clear in due course. The first such condition is a locality principle:

(Locality) $Q(\langle p \rangle, t)\ \&\ \langle p \rangle \cap \langle t \rangle^\circ = \langle q \rangle \cap \langle t \rangle^\circ \rightarrow Q(\langle q \rangle, t).$

(Here $\langle t \rangle^\circ$ denotes the set of internal points of t.) We may state this principle in a more 'logical' fashion, namely as the validity of the following interval formula:

$$\sim\mathrm{L}^\circ\sim(p \leftrightarrow q) \rightarrow (p \leftrightarrow q).$$

This principle states that the truth value of a proposition at an interval depends only on points of this interval.[5]

Our second plausible condition is the positivity principle:

(Positivity) $\langle p \rangle \subseteq \langle q \rangle \rightarrow [p] \subseteq [q].$

We will use also the following activity principles:

(Activity) $Q(\mathrm{P}, t)$ for any interval t (P denotes the set of all points)

(Negative Activity) $\sim Q(\varnothing, t)$ for any interval t

(Local Activity) $\langle t\rangle^{\circ} \subseteq \langle p\rangle \rightarrow Q(\langle p\rangle, t)$ (or $\sim \mathrm{L}^{\circ}\sim p \rightarrow p$)

(Local Negative Activity) $Q(\langle p\rangle, t) \rightarrow \langle p\rangle \cap \langle t\rangle^{\circ} \neq \varnothing$ (or $p \rightarrow \mathrm{L}^{\circ} p$).

And finally we will use some conditions of a logical kind:

(Consistency) $\sim(Q(\langle p\rangle, t) \;\&\; Q(\mathrm{P} - \langle p\rangle, t))$ (for any p and t)

(Completeness) $Q(\langle p\rangle, t) \vee Q(\mathrm{P} - \langle p\rangle, t)$ (for any p and t).

(Conjunctivity) $Q(\langle p\rangle, t) \;\&\; Q(\langle q\rangle, t) \rightarrow Q(\langle p\rangle \cap \langle q\rangle, t)$

(Disjunctivity) $Q(\langle p\rangle \cup \langle q\rangle, t) \rightarrow Q(\langle p\rangle, t) \vee Q(\langle q\rangle, t)$.

The logical role of these conditions will become clear in the next sections.

Nontriviality We will say that an interval valuation is nontrivial if it essentially depends both on propositions and on intervals of evaluation. To be more precise, a valuation is *intervally trivial* if $(t)(s)([p]_t \leftrightarrow [p]_s)$ for any proposition p, and it is *propositionally trivial* if for any p and q, $(t)([p]_t \leftrightarrow [q]_t)$. So a valuation will be nontrivial if it is both propositionally and intervally nontrivial.

Lemma 1 *An instant-definable valuation is propositionally trivial iff it is either a universally true or a universally false valuation.*

Proof: Since we may map any interval onto any other interval through some automorphism, we have that if some proposition is true for some interval, then any proposition will be true for any interval.

Lemma 2 *An instant-definable valuation is intervally trivial iff one of the following conditions holds*:
(i) *The valuation is both persistent and negatively persistent*
(ii) $(t)(s)((p)([p]_t \leftrightarrow [p]_s) \rightarrow t = s)$.

Proof: (i) is obvious, since for any two intervals there is always an interval containing them as parts. Suppose that two different intervals t and s have the same sets of propositions true relative to them:

$$(*) \qquad (p)([p]_t \leftrightarrow [p]_s).$$

It is clear that there are only five configurations for t and s, which have different (first-order) types: (a) $\sim(t * s)$; (b) $t \circ s \;\&\; \sim(t \leq s) \;\&\; \sim(s \leq t)$; (c) $t < s \;\&\; (\exists u)(u + t \;\&\; u + s)$; (d) $t < s \;\&\; \sim(\exists u)(u + t \;\&\; u + s)$; (e) $t + s$. Since we may map any pairs of intervals belonging to the same type on each other, we have by Quality that if some pair of intervals satisfies (*), then any pair of the same type will also satisfy (*). Suppose now that $\{t, s\}$ has the type (a) and $\{u, v\}$ is an arbitrary pair of intervals. Then there is always an interval w that does not touch u and v. Hence we have that both $\{u, w\}$ and $\{v, w\}$ have the type (a) and therefore $[p]_u \leftrightarrow [p]_w \leftrightarrow [p]_v$ for any p. Thus, in this case the valuation will be intervally trivial. But for any configuration (b)–(e) we may find some interval w such that w does not touch t and $\{w,s\}$ has the same type as $\{t,s\}$. Therefore we have for any p: $[p]_t \leftrightarrow [p]_s \leftrightarrow [p]_w$ and $\{w,t\}$ has the type (a).

As a consequence of the above lemmas we have that propositional triviality implies interval triviality and therefore a valuation is nontrivial if and only if it is intervally nontrivial. The following lemmas show that nontriviality imposes strong restrictions on the co-occurence of the conditions defined at the beginning of this section.

Lemma 3 *A valuation is propositionally trivial iff it is positive, conjunctive, and disjunctive.*

Proof: Given $[p]_t$ let α be an arbitrary internal point of t. Suppose that H is some open half-line, bounded by α, that is a set $\{\gamma : \mathrm{b}(\beta\alpha\gamma)\}$ for some fixed β. Since $\langle p\rangle = (\langle p\rangle \cap H) \cup (\langle p\rangle \cap (\mathrm{P}-H))$, we have by disjunctivity that either $Q(\langle p\rangle \cap H, t)$ or $Q(\langle p\rangle \cap (\mathrm{P}-H), t)$. Suppose that $Q(\langle p\rangle \cap H, t)$ and consider a reversion f of the whole point-interval structure around α such that $f(t)=t$. Now by conjunctivity we have $Q((\langle p\rangle \cap H) \cap f(\langle p\rangle \cap H), t)$ and hence $Q(\varnothing, t)$. (In the case $Q(\langle p\rangle \cap (\mathrm{P}-H), t)$ we must consider a reversion around some point from t, which does not belong to $\langle p\rangle \cap (\mathrm{P}-H)$.) Thus we have that if a valuation is conjunctive and disjunctive then for any t and p, $[p]_t \rightarrow [\varnothing]_t$. In a similar way it can be shown that for any p and t, $[\mathrm{P}]_t \rightarrow [p]_t$. Now if this valuation is positive, then we have also the reverse implications and therefore the valuation is propositionally trivial.

Lemma 4 *A valuation is trivially false iff one of the following conditions holds*:
(i) *It is conjunctive, disjunctive, and negatively active*
(ii) *It is persistent, disjunctive, and locally negatively active*
(iii) *It is negatively persistent, conjunctive, negatively active, and local.*

Proof: (i) is a consequence of the proof of Lemma 3. Suppose now that $[p]_t$ for some p and t in some persistent, disjunctive, and locally negatively active valuation. Then (just as in the proof of Lemma 3) for some half-line H, bounded by an internal point of t, $Q(\langle p\rangle \cap H, t)$. Therefore by persistence $\langle p\rangle \cap H$ must be true in some subinterval of t that does not contain points from $\langle p\rangle \cap H$ – contradicting local negative activity. Thus, (ii) implies that a valuation is trivially false. Consider (iii). If $[p]_t$, then by locality $Q(\langle p\rangle \cap \langle t\rangle^{\circ}, t)$. Therefore by negative persistence $\langle p\rangle \cap \langle t\rangle^{\circ}$ must hold in some interval s, such that t is a proper part of s. Let α be some internal point of s not belonging to t and consider a reversion f around α, such that $f(s)=s$. By Quality $Q(f(\langle p\rangle \cap \langle t\rangle^{\circ}), s)$ and since $\langle p\rangle \cap \langle t\rangle^{\circ}$ is not intersected with $f(\langle p\rangle \cap \langle t\rangle^{\circ})$ we will obtain by conjunctivity $Q(\varnothing, s)$ – contradictiing negative activity. Therefore $[p]_t$ is false for any p and t.

We now give without proof a lemma dual to Lemma 4. With the proper understanding of this duality the proof of it could be obtained from the proof of Lemma 4.

Lemma 5 *A valuation is trivially true iff one of the following conditions holds*:

(i) *It is conjunctive, disjunctive, and active*
(ii) *It is negatively persistent, conjunctive, and locally active*
(iii) *It is persistent, disjunctive, active, and local.*

The following three lemmas involve the properties of consistency and completeness.

Lemma 6 *A valuation cannot be simultaneously positive, consistent, and complete.*

Proof: Suppose that H is an open half-line bounded by some internal point of an interval t. By consistency and completeness we have $Q(H,t) \leftrightarrow \sim Q(\mathrm{P} - H, t)$. Suppose that $Q(\mathrm{P} - H, t)$ is true and let G be some open half-line, which is also bounded by some internal point of t such that $\mathrm{P} - H \subseteq G$. Then by positivity $Q(G,t)$. But it is obvious that we can send H to G through some reversion that retains t, and hence by Quality $Q(H,t) \leftrightarrow Q(G,t)$—a contradiction. On the other hand, if $Q(H,t)$ is true, then we will consider the closure of H, which we will denote by F. By positivity $Q(F,t)$, but we may map F on $(\mathrm{P} - H)$ through a reversion, retaining t, and therefore $Q(\mathrm{P} - H, t)$ also must be true, which is impossible.

The proofs of the following two lemmas are completely analogous to the proof of Lemmas 4(i)–(ii) and 5(i)–(ii) and so will be omitted here.

Lemma 7

(i) *A valuation cannot be simultaneously complete, conjunctive, and negatively active*
(ii) *A valuation cannot be simultaneously complete, persistent, and locally negatively active.*

Lemma 8

(i) *A valuation cannot be simultaneously consistent, disjunctive, and active*
(ii) *A valuation cannot be simultaneously consistent, negatively persistent, and locally active.*

Now we will turn to nontrivial valuations. The following two theorems provide a practically complete description of the logical relationships between our conditions.

Theorem 4

(i) *If a valuation is conjunctive, positive, and negatively cumulative it is persistent, and if it is also nontrivial it is locally negatively active*
(ii) *If a valuation is conjunctive, positive, and locally active it is local*
(iii) *If a valuation is conjunctive, negatively active, and locally active it is locally negatively active*
(iv) *If a valuation is negatively active and conjunctive it is consistent*
(v) *If a valuation is consistent and (locally) active it is (locally) negatively active*
(vi) *If a nontrivial valuation is positive it is active and negatively active.*

Proof: (i) If a valuation is not persistent, then for some t, s and proposition p we have $s \leq t$, $[p]_t$, and $\sim [p]_s$. Now, using negative cumulativity we may find an interval u, such that $[p]_u$ and $s \leq u$, $u \leq t$, and $(\exists v)(v + s \ \& \ v + u)$ (that is, s is an internally abutting part of u). Now consider a reversion f of the point-interval structure around some internal point of s such that $f(u) = u$. By conjunctivity we have $Q(\langle p \rangle \cap f(\langle p \rangle), u)$. On the other hand, by positivity $\langle p \rangle \cap f(\langle p \rangle)$ is false in both s and in $f(s)$, u is a union (fusion) of s and $f(s)$, and hence by negative cumulativity $\langle p \rangle \cap f(\langle p \rangle)$ is false in u—a contradiction.

Suppose now that this valuation is not locally negatively active, i.e., for some q and u, $[q]_u$ and $\langle q \rangle \cap \langle u \rangle^\circ = \varnothing$. We will show that in this case the valuation is negatively persistent and hence intervally trivial. Suppose that it is not so, i.e., for some s and t such that $s \leq t$ we have $[p]_s$ and $\sim [p]_t$ for some proposition p. Let v be such an interval, where t is a nonabutting part of it (that is, $t \leq v$ and $\sim (\exists w)(w + t \ \& \ w + v)$). Since we may map u on v, we have by Quality that for some proposition r, $[r]_v$ and $\langle r \rangle \cap \langle v \rangle^\circ = \varnothing$. Then by persistence $[r]_s$, and hence by conjunctivity $Q(\langle r \rangle \cap \langle p \rangle, s)$. Now consider an automorphism f that sends s to t while retaining v. Since $\langle r \rangle \cap \langle p \rangle \cap \langle v \rangle^\circ = \varnothing$, we have that $f(\langle r \rangle \cap \langle p \rangle) = \langle r \rangle \cap \langle p \rangle$ and therefore by Quality $Q(\langle r \rangle \cap \langle p \rangle, t)$. Hence, by positivity $Q(\langle p \rangle, t)$, which is impossible. Therefore the valuation is negatively persistent and thereby intervally trivial.

(ii) If $Q(\langle p \rangle \cap \langle t \rangle^\circ, t)$ is false then by conjunctivity either $Q(\langle p \rangle, t)$ or $Q(\langle t \rangle^\circ, t)$ is false and hence by local activity $Q(\langle p \rangle, t)$ is false. On the other hand, if $Q(\langle p \rangle \cap \langle t \rangle^\circ, t)$, then by positivity $Q(\langle p \rangle, t)$. Therefore the valuation is local.

(iii) Conjunctivity and local activity imply that if $Q(\langle p \rangle, t)$ then $Q(\langle p \rangle \cap \langle t \rangle^\circ, t)$. But if the valuation is not locally negatively active, then for some q and t, $Q(\langle q \rangle, t)$ and $\langle q \rangle \cap \langle t \rangle^\circ = \varnothing$. Therefore we have $Q(\varnothing, t)$ which contradicts negative activity.

(iv) If for some p and t we have both $Q(\langle p \rangle, t)$ and $Q(\mathrm{P} - \langle p \rangle, t)$ then by conjunctivity $Q(\varnothing, t)$—a contradiction with negative activity.

(v) If $Q(\mathrm{P}, t)$ then by consistency $\sim Q(\varnothing, t)$. If for some p and t, $[p]_t$ and $\langle p \rangle \cap \langle t \rangle^\circ = \varnothing$, then by consistency $\sim Q(\mathrm{P} - \langle p \rangle, t)$ and $\langle t \rangle \subseteq \mathrm{P} - \langle p \rangle$—a contradiction with local activity.

(vi) This follows from the fact that if $Q(\varnothing, t)$ or $\sim Q(\mathrm{P}, t)$ then positivity implies propositional triviality.

The following theorem is dual to Theorem 4. We omit the proof.

Theorem 5

(i) *If a valuation is disjunctive, positive, and cumulative it is negatively persistent, and if it is also nontrivial it is locally active*
(ii) *If a valuation is disjunctive, positive, and locally negatively active it is local*
(iii) *If a valuation is disjunctive, active, and locally negatively active it is locally active*
(iv) *If a valuation is active and disjunctive it is complete*
(v) *If a valuation is complete and (locally) negatively active it is (locally) active.*

The following theorem, which we also give without proof here, holds under some strong requirements on possible propositions and automorphisms:[6]

Theorem 6
(i) *If a valuation is conjunctive, positive, local, and cumulative it is persistent.*
(ii) *If a valuation is disjunctive, positive, local, and negatively cumulative it is negatively persistent.*

A number of examples could be produced in order to show, in effect, that the above lemmas and theorems practically exhaust the possible dependencies between the introduced notions in the general case. Below we will consider a special case of finite valuations.

5.1.1 Finite case Here we will consider the situation when the underlying instant valuation is finite, in accordance with Definition 8. It is easy to show that an interval valuation that is definable on the basis of a finite instant valuation is also finite. In this most simple of the natural cases we have some additional dependencies between our conditions and, moreover, an exact characterization of some sets of valuations.

We say that an instant-definable valuation is *locally dense* if the formula $(p \rightarrow {\sim}\mathrm{S}{\sim}\mathrm{L}^{\circ}p)$ is universally valid, and *locally negatively dense* if the formula $(\mathrm{S}{\sim}\mathrm{L}^{\circ}{\sim}p \rightarrow p)$ is universally valid. Then we have:

Lemma 9 *If a valuation is local, cumulative, negatively cumulative, and locally dense then it is either universally false or coincides with one of the following three valuations*: (1) $[{\sim}\mathrm{L}^{\circ}{\sim}p]_t$; (2) $[{\sim}\mathrm{S}{\sim}\mathrm{S}{\sim}\mathrm{L}^{\circ}{\sim}p \;\&\; \mathrm{L}^{\circ}{\sim}p]_t$; (3) $[{\sim}\mathrm{S}{\sim}\mathrm{S}{\sim}\mathrm{L}^{\circ}{\sim}p]_t$.

Proof: Suppose that the valuation is nontrivial and such that for some p and t, $[{\sim}\mathrm{S}{\sim}\mathrm{S}{\sim}\mathrm{L}^{\circ}{\sim}p]_t$ and p is false at t. Consider the following two cases:

(a): For some q and s, $[\mathrm{L}^{\circ}{\sim}q \;\&\; {\sim}\mathrm{S}{\sim}\mathrm{S}{\sim}\mathrm{L}^{\circ}{\sim}q]_s$ and q is false at s. Then by locality and cumulativity q must be false at some interval that contains only one point that does not belong to $\langle q \rangle$. But then by negative cumulativity and local density any proposition p must be false in all intervals that contain some points from $\mathrm{P} - \langle p \rangle$. In other words, the formula $(p \rightarrow {\sim}\mathrm{L}^{\circ}{\sim}p)$ is valid for this valuation. Since the valuation is not universally false, we have for some interval u and some proposition r, $[r]_u$ and $[{\sim}\mathrm{L}^{\circ}{\sim}r]_u$. Therefore by Quality and locality we have that $(p \leftrightarrow {\sim}\mathrm{L}^{\circ}{\sim}p)$ is a valid formula and hence the valuation coincides with $[{\sim}\mathrm{L}^{\circ}{\sim}p]$.

(b): The formula $(\mathrm{L}^{\circ}{\sim}q \;\&\; {\sim}\mathrm{S}{\sim}\mathrm{S}{\sim}\mathrm{L}^{\circ}{\sim}q \rightarrow q)$ is valid in our valuation. Then by cumulativity the interval t must contain a subinterval u such that $[{\sim}p \;\&\; {\sim}\mathrm{L}^{\circ}{\sim}p]$, and hence by Quality and locality the formula $({\sim}\mathrm{L}^{\circ}{\sim}p \rightarrow {\sim}p)$ is valid. But this implies that the valuation coincides with (2), q.e.d.

The following lemma is dual to Lemma 9:

Lemma 10 *If a valuation is local, cumulative, negatively cumulative, and locally negatively dense then it is either universally true or coincides with one of the following three valuations*: (1′) $[\mathrm{L}^{\circ}p]_t$; (2′) $[\mathrm{S}{\sim}\mathrm{L}^{\circ}{\sim}p \vee {\sim}\mathrm{L}^{\circ}p]_t$; (3′) $[\mathrm{S}{\sim}\mathrm{L}^{\circ}{\sim}p]_t$.

The above lemmas have the following consequences:

(i) Valuations that are local, cumulative, negatively cumulative, conjunctive, and negatively active are either trivially false or coincide with the valuations (1)–(3) above.

(i′) Local, cumulative, negatively cumulative, disjunctive, and active valuations are either trivially true or coincide with the valuations (1′)–(3′) above.

(ii) Local, homogeneous, and negatively active valuations are either trivially false or coincide with (1) or (3) (and hence they are conjunctive and positive).

(ii′) Local, negatively homogeneous, and active valuations are either trivially true or coincide with (1′) or (3′) (and hence they are disjunctive and positive).

(iii) Local, cumulative, negatively cumulative, conjunctive, active, and negatively active valuations coincide with (1) or (3) and hence are positive.

(iii′) Local, cumulative, negatively cumulative, disjunctive, active, and negatively active valuations coincide with (1′) or (3′) and hence are positive.

Lemma 11 *If a valuation is local, cumulative, negatively cumulative, and positive then it is either propositionally trivial or coincides with* (1), (1′), (3) *or* (3′).

Proof: It is sufficient to prove that the above conditions imply that the valuation is either locally dense or locally negatively dense. Suppose that this is not so. Then for some p and t, $[p \,\&\, \mathrm{S}{\sim}\mathrm{L}^{\circ}p]_t$ and by negative cumulativity we may find intervals u and s such that $[{\sim}\mathrm{L}^{\circ}p]_u$, $[p]_s$, and u is an internally abutting part of s. Then by positivity we have that for some p', $[{\sim}\mathrm{L}p']_u$, $[p']_s$, and $\langle p'\rangle$ covers $\langle s\rangle^{\circ}/\langle u\rangle^{\circ}$. On the other hand, for some q and t', $[\mathrm{S}{\sim}\mathrm{L}^{\circ}{\sim}q \,\&\, {\sim}q]_{t'}$, and in the same way we may obtain intervals u' and s' such that u' is an internally abutting part of s' and for some q', $[q']_{s'}$, $[{\sim}\mathrm{L}^{\circ}{\sim}q']_{u'}$ and $\langle q'\rangle$ is not intersected with $\langle s'\rangle^{\circ}/\langle u'\rangle^{\circ}$. But this is impossible, since we may map $\{u,s\}$ on $\{u',s'\}$ through some automorphism.

As a consequence of this lemma we have that any local, cumulative, negatively cumulative, and positive valuation is either conjunctive and persistent, or disjunctive and negatively persistent. Below we will give without proof some additional properties of finite instant-definable valuations.

Lemma 12 *If a valuation is local, persistent, and negatively active it is trivially false or coincides with one of the valuations*:

$$\{[t \text{ contains no more than } n \text{ points, not belonging to } \langle p\rangle]\}.$$

Note, that the above conditions imply that the valuation is locally dense. As a consequence we have that any local, persistent, and negatively active valuation is positive.

Lemma 13 *Any local, negatively cumulative, conjunctive, and negatively active valuation is either trivially false or coincides with one of* (1)–(3) *from Lemma* 9.

A consequence of this lemma is that all such valuations are cumulative, and if they are also active they are positive.

Lemma 14 *A valuation is local, conjunctive, positive, and negatively active if and only if it is locally dense or coincides with* $[\mathrm{I}^{+}(\sim\mathrm{L}^{\circ}\sim p)]$.

A consequence of this fact is that any such valuation is cumulative. And finally as a most elegant result of this kind we have:

Corollary

(i) *Any positive, conjunctive, and locally active valuation coincides with* $[\sim\mathrm{L}^{\circ}\sim p]$ *or with* $[\sim\mathrm{S}\sim\mathrm{S}\sim\mathrm{L}^{\circ}\sim p]$.

(ii) *Any positive, disjunctive, and locally negatively active valuation coincides with* $[\mathrm{L}^{\circ}p]$ *or with* $[\mathrm{S}\sim\mathrm{L}^{\circ}\sim p]$.

6 Interval-definable instant valuations Just as interval valuations can be defined in terms of some 'hidden' instant valuations, for many instant valuations it is also possible to give definitions in terms of interval valuations. And it seems that such definability should in fact be obligatory for those who consider instants themselves as definable constructs. Indeed, if we consider instants as certain constructions out of intervals, then all their properties and all events occuring in it must in fact correspond to some complex expressions about intervals, that are involved in their definition.

In fact, from a technical point of view a theory of interval-definable instant valuations could be developed along the same lines as a theory of instant-definable interval valuations. That is why we will restrict ourselves here to only some key points. We will show, first, that Theorems 1 and 2 from Section 5 can, in fact, be reversed.

Theorem 7

(i) *Any open instant valuation* $\langle p\rangle_{\alpha}$ *is definable as*

$$(\exists t)(\alpha \lessdot t \;\&\; [p]_t)$$

for some interval valuation $[p]_t$, *and it may be required that* $[p]_t$ *be a homogeneous valuation*

(ii) *Any open valuation is definable as* $\langle \mathrm{U}^{\circ}p\rangle$ *for some persistent interval valuation*

(iii) *Any regular open valuation is definable as* $\langle \mathrm{A}^{\circ}p\rangle$ *for some strongly homogeneous interval valuation.*

Proof: (i) Since an open valuation is characterized by the universal validity of the formula $(p \leftrightarrow \mathrm{A}^{\circ}\sim\mathrm{L}^{\circ}\sim p)$, we may define the required interval valuation simply as $[\sim\mathrm{L}^{\circ}\sim p]$; it is obvious that this valuation is homogeneous. On the other hand, an instant valuation of the form $\langle \mathrm{A}^{\circ}p\rangle$ is open for any underlying interval valuation.

(ii) For any open valuation the formula $(p \leftrightarrow \mathrm{U}^{\circ}\sim\mathrm{L}^{\circ}\sim p)$ is valid and hence we may define the required valuation as $[\sim\mathrm{L}^{\circ}\sim p]_t$. On the other hand, any valuation of the form $\langle \mathrm{U}^{\circ}p\rangle$ is open for some persistent interval valuation.

(iii) Since for open regular valuations $(p \leftrightarrow \mathrm{A}^{\circ}\sim\mathrm{S}\sim\mathrm{L}^{\circ}p)$ is a valid formula, we may define the required interval valuation as $[\sim\mathrm{S}\sim\mathrm{L}^{\circ}p]$ which is obviously a strongly homogeneous valuation.

The following theorem is dual to Theorem 7:

Theorem 8

(i) *Any closed instant valuation is definable as* $\langle\sim\mathrm{A}^{\circ}\sim p\rangle$ *for some underlying interval valuation and it may by required that this interval valuation be negatively homogeneous*

(ii) *Any closed valuation is definable as* $\langle\sim\mathrm{U}^{\circ}\sim p\rangle$ *for some negatively persistent interval valuation*

(iii) *Any regular closed valuation is definable as* $\langle\sim\mathrm{A}^{\circ}\sim p\rangle$ *for some strongly negatively homogeneous valuation.*

So far we have considered general representability results. But in this case the 'generalized quantifiers' approach is also applicable. Suppose that for an instant valuation $\langle p\rangle_{\alpha}$ and an interval valuation $[p]_t$ we have:

$$[p]=[q]\rightarrow\langle p\rangle=\langle q\rangle.$$

Then there must exist a relation R between instants and sets of intervals such that

$$\langle p\rangle_{\alpha}\leftrightarrow R([p],\alpha).$$

Just as for instant-definable interval valuations we will adopt the Quality principle:

(Quality) $\quad R([p],\alpha)\leftrightarrow R(f([p]),f(\alpha)).$

Now we may define for this generalized quantifier practically the same conditions as for instant-definable interval valuations, namely, Locality, Positivity, (Local) Activity, Consistency, Completeness, Conjunctivity, and Disjunctivity. The corresponding definitions are completely analogous to their point counterparts, except for the cases of Locality and Local Activity. For instant-definable valuations we have restricted ourselves to the condition of locality with respect to internal points of an interval. But in the case of instant valuations there are some equally plausible forms of locality, among which we will choose here the following two:

(Locality$_1$) $\quad \mathrm{A}^{\circ}\sim\mathrm{S}\sim(p\leftrightarrow q)\rightarrow(p\leftrightarrow q)$

(Locality$_2$) $\quad \sim\mathrm{A}\sim(p\leftrightarrow q)\rightarrow(p\leftrightarrow q).$

We have also two pairs of Local Activity conditions, which correspond in fact to the above kinds of locality:

(Local$_1$ Activity) $\quad \mathrm{A}^{\circ}\sim\mathrm{S}\sim p\rightarrow p$

(Local$_1$ Negative Activity) $\quad p\rightarrow\sim\mathrm{A}^{\circ}\sim\mathrm{S}p$

(Local$_2$ Activity) $\quad \sim\mathrm{A}\sim p\rightarrow p$

(Local$_2$ Negative Activity) $\quad p\rightarrow\mathrm{A}p.$

Just as for interval valuations it can be shown that positivity, conjunctivity, and local activity (as well as positivity, disjunctivity, and local negative activity) imply the corresponding locality. Among other dependencies between our conditions we will mention only a few. First, it can be shown that the valuations $\langle \sim A^{\circ} \sim p \rangle$, $\langle U^{\circ} p \rangle$, and $\langle A^{\circ} \sim S \sim p \rangle$ exemplify all the possible maximal compatible sets of conditions that include some form of locality. On the other hand, positive, conjunctive, and disjunctive valuations are trivial only in the nonfinite case (see below). However, such valuations must be both Local_1 and Local_2. Below we will consider in some detail one class of finite valuations.

6.1 Finite case From our definition of finiteness for interval valuations it follows that any instant valuation defined on a finite interval valuation is also finite. Now, if this instant valuation is Local_1, then it is determined by the distribution of truth values for the corresponding interval valuation in any neighborhood of an instant. But in finite interval valuations for any proposition p and any instant α there is an interval t, containing α, such that all intervals from t belong to one of the five classes with the same truth value for p (see Section 4). In fact, we have a situation in which the truth value of a proposition at an instant is determined, in effect, by five truth-valued parameters, corresponding to the above five classes. We have actually only twenty distributions of truth values for these parameters, which are different up to the mirror symmetry, and hence any interval-definable instant valuation of this kind is completely determined by those truth values which it assigns to these twenty configurations.

If the above instant valuation is also Local_2, then it depends only on three truth-valued parameters, which correspond, in fact, to three different point-filters, determining the same point. It is interesting to note that this situation could be reinterpreted as saying that our instant valuation is determined by some 'hidden' valuation on generalized points (see the last Section in Bochman [5]). It can be shown that all such valuations could be described as all the posible logical combinations of $U^{\circ}p$, $U^{+}p$, and $U^{+}\sim p$. Note that among these valuations there is one (namely, $\langle U^{\circ}p \rangle$), that is simultaneously conjunctive, disjunctive, and positive for finite propositions.

7 Internal logics of definable valuations Definable valuations (both instant and interval ones) give rise to natural logical connectives, which correspond to classical connectives in their defining counterparts. We first consider instant-definable interval valuations.

For any instant valuation $\langle p \rangle_{\alpha}$ we may define 'classical' connectives in the following natural way:

$$\langle p \,\hat{\&}\, q \rangle = \langle p \rangle \cap \langle q \rangle \qquad \text{(that is, for any } \alpha, \langle p \,\&\, q \rangle_{\alpha} \leftrightarrow \langle p \rangle_{\alpha} \,\&\, \langle q \rangle_{\alpha}, \text{ etc.)}$$

$$\langle p \,\hat{\vee}\, q \rangle = \langle p \rangle \cup \langle q \rangle$$

$$\langle -p \rangle = \mathrm{P} - \langle p \rangle$$

$$\langle p \Rightarrow q \rangle = (\mathrm{P} - \langle p \rangle) \cup \langle q \rangle.$$

Now, if an interval valuation $[p]_t$ is instant-definable as $Q(\langle p \rangle, t)$, then we have that the above complex propositions will obtain nonclassical definitions in this interval valuation:

$$[p \mathbin{\hat{\&}} q]_t \leftrightarrow Q(\langle p\rangle \cap \langle q\rangle, t)$$
$$[p \mathbin{\hat{\vee}} q]_t \leftrightarrow Q(\langle p\rangle \cup \langle q\rangle, t)$$
$$[-p]_t \leftrightarrow Q(\mathrm{P} - \langle p\rangle, t)$$
$$[p \Rightarrow q]_t \leftrightarrow Q((\mathrm{P} - \langle p\rangle) \cup \langle q\rangle, t).$$

The logic corresponding to the above semantics is, however, too weak in the general case. It lacks even tautologies (that is, universally valid formulas) and has, in fact, only 'reversible' entailments, e.g. commutativity, associativity, and distributivity of conjunction and disjunction and all reduction equivalences of classical propositional logic. But all additional properties depend on the particular quantifier Q. We have, in fact, the following remarkable correspondences between conditions on Q and some natural logical conditions:

Activity: The formula $(p \mathbin{\hat{\vee}} -p)$ is universally valid

Negative activity: The formula $(p \mathbin{\hat{\&}} -p)$ is universally false

Positivity: The entailments $[p \mathbin{\hat{\&}} q]_t \rightarrow [p]_t$ and $[p]_t \rightarrow [p \mathbin{\hat{\vee}} q]$

Consistency: $[p]_t$ and $[-p]_t$ cannot simultaneously hold

Completeness: Either $[p]_t$ or $[-p]_t$ is true

Conjunctivity: The entailment $[p]_t \;\&\; [q]_t \rightarrow [p \mathbin{\hat{\&}} q]_t$

Disjunctivity: The entailment $[p \mathbin{\hat{\vee}} q]_t \rightarrow [p]_t \vee [q]_t$.

If a valuation is active, then all tautologies of classical logic are universally valid formulas in such a semantics. A valuation is conjunctive and positive iff its 'internal' conjunction is classical:

$$[p \mathbin{\hat{\&}} q]_t \leftrightarrow [p]_t \;\&\; [q]_t,$$

whereas it is positive and disjunctive iff its internal disjunction is classical:

$$[p \mathbin{\hat{\vee}} q]_t \leftrightarrow [p]_t \vee [q]_t.$$

But as was shown in Section 5.1, nontrivial interval valuations cannot be simultaneously conjunctive, disjunctive, and positive and hence the corresponding semantics cannot be classical for all the above connectives.

To any definable valuation $[p]_t \leftrightarrow Q(\langle p\rangle, t)$ a *dual valuation* naturally corresponds, which is defined as follows:

$$]p[_t \equiv {\sim} Q(\mathrm{P} - \langle p\rangle, t).$$

This valuation has exactly those properties that are dual to the properties of the source valuation. In addition to this, $[-p]_t \leftrightarrow {\sim}]p[_t$ and $]-p[_t \leftrightarrow {\sim}[p]$. Note that a valuation $[p]_t$ is consistent iff $[p]_t \rightarrow]p[_t$ and complete iff $]p[_t \rightarrow [p]_t$.

The case of interval-definable instant valuations is completely analogous. Note, however, that in the finite case there is a valuation $(\langle \mathrm{U}^\circ p\rangle)$ which is classical with respect to all its internal connectives.

7.1 One 'unified' logic of change We will begin with an internal logic of homogeneous valuations. This choice can be justified by the fact that For-valuations, as mentioned above (cf. Section 2), are homogeneous, because they are (definitionally) persistent and (empirically) cumulative for the main types of temporal propositions. As has been shown, any homogeneous valuation is representable as $[\sim L^{\circ}\sim p]_t$ for some instant valuation. Suppose now that a certain homogeneous valuation $\|p\|_t$ has a 'hidden' definition

(1) $$\|p\|_t \equiv (\alpha)(\alpha \lessdot t \rightarrow (p)_\alpha)$$

for some instant valuation $(p)_\alpha$. Then our interval valuation has the following dual counterpart:

(2) $$|p|_t \equiv (\exists\alpha)(\alpha \lessdot t \;\&\; (p)_\alpha).$$

We also define the following dual pair of derived instant valuations:

(3) $$\langle\langle p\rangle\rangle_\alpha \equiv (A^{\circ}\sim L^{\circ}\sim p)_\alpha$$

(4) $$\langle p\rangle_\alpha \equiv (\sim A^{\circ}\sim L^{\circ} p)_\alpha.$$

It is clear that the valuation $\langle\langle p\rangle\rangle_\alpha$ is open, while $\langle p\rangle_\alpha$ is closed. Moreover, we have that these valuations also characterize our interval valuations:

(5) $$\|p\|_t \leftrightarrow (\alpha)(\alpha \lessdot t \rightarrow \langle\langle p\rangle\rangle_\alpha)$$

(6) $$|p|_t \leftrightarrow (\exists\alpha)(\alpha \lessdot t \;\&\; \langle p\rangle_\alpha).$$

Note also that the above definitions are in fact 'reversible', because the valuations $\langle\langle p\rangle\rangle_\alpha$ and $\langle p\rangle_\alpha$ are in turn definable in terms of the corresponding interval valuations:

(7) $$\langle\langle p\rangle\rangle_\alpha \leftrightarrow (\exists t)(\alpha \lessdot t \;\&\; \|p\|_t)$$

(8) $$\langle p\rangle_\alpha \leftrightarrow (t)(\alpha \lessdot t \rightarrow |p|_t).$$

Hence we have, in fact, two alternative ways of defining internal logical connectives for our interval valuations, depending on whether we use our source instant valuation $(p)_\alpha$ or its derivatives $\langle\langle p\rangle\rangle_\alpha$ and $\langle p\rangle_\alpha$.

We first consider 'derivative' internal connectives. There is, however, a problem here, because internal negation and implication do not preserve the property of openness (respectively, closedness) of corresponding instant valuations. Hence, the corresponding definitions for these connectives, given below, will be modified in order to preserve these properties:

(9) $$\begin{aligned} \langle\langle p \mathbin{\mathring{\&}} q\rangle\rangle &= \langle\langle p\rangle\rangle \cap \langle\langle q\rangle\rangle \\ \langle\langle p \mathbin{\mathring{\vee}} q\rangle\rangle &= \langle\langle p\rangle\rangle \cup \langle\langle q\rangle\rangle \\ \langle\langle \neg p\rangle\rangle &= \mathrm{Int}(\mathrm{P} - \langle\langle p\rangle\rangle) \\ \langle\langle p \Rightarrow q\rangle\rangle &= \mathrm{Int}((\mathrm{P} - \langle\langle p\rangle\rangle) \cup \langle\langle q\rangle\rangle). \end{aligned}$$

(10) $$\begin{aligned} \langle p \mathbin{\mathring{\&}} q\rangle &= \langle p\rangle \cap \langle q\rangle \\ \langle p \mathbin{\mathring{\vee}} q\rangle &= \langle p\rangle \cup \langle q\rangle \\ \langle \neg p\rangle &= \mathrm{Cl}(\mathrm{P} - \langle p\rangle) \\ \langle p \Rightarrow q\rangle &= \mathrm{Cl}((\mathrm{P} - \langle p\rangle) \cup \langle q\rangle). \end{aligned}$$

These definitions generate the following natural definitions in our interval valuations:

(11)
$$\|p \mathbin{\mathring{\&}} q\|_t \leftrightarrow (\alpha)(\alpha \mathrel{<\cdot} t \rightarrow (\langle\langle p\rangle\rangle_\alpha \mathbin{\&} \langle\langle q\rangle\rangle_\alpha))$$
$$\|p \mathbin{\mathring{\vee}} q\|_t \leftrightarrow (\alpha)(\alpha \mathrel{<\cdot} t \rightarrow (\langle\langle p\rangle\rangle_\alpha \vee \langle\langle q\rangle\rangle_\alpha))$$
$$\|\neg p\|_t \leftrightarrow (\alpha)(\alpha \mathrel{<\cdot} t \rightarrow \langle\langle p\rangle\rangle_\alpha)$$
$$\|p \Rightarrow q\|_t \leftrightarrow (\alpha)(\alpha \mathrel{<\cdot} t \rightarrow (\langle\langle p\rangle\rangle_\alpha \rightarrow \langle\langle q\rangle\rangle_\alpha)).$$

(12)
$$|p \mathbin{\mathring{\&}} q|_t \leftrightarrow (\exists\alpha)(\alpha \mathrel{<\cdot} t \mathbin{\&} (\langle p\rangle_\alpha \mathbin{\&} \langle q\rangle_\alpha))$$
$$|p \mathbin{\mathring{\vee}} q|_t \leftrightarrow (\exists\alpha)(\alpha \mathrel{<\cdot} t \mathbin{\&} (\langle p\rangle_\alpha \vee \langle q\rangle_\alpha))$$
$$|\neg p|_t \leftrightarrow (\exists\alpha)(\alpha \mathrel{<\cdot} t \mathbin{\&} \sim\langle p\rangle_\alpha)$$
$$|p \Rightarrow q|_t \leftrightarrow (\exists\alpha)(\alpha \mathrel{<\cdot} t \mathbin{\&} (\langle p\rangle_\alpha \rightarrow \langle q\rangle_\alpha)).$$

Now, using (7) and (8) we obtain the following equivalences:

(13)
$$\|p \mathbin{\mathring{\&}} q\|_t \leftrightarrow \|p\|_t \mathbin{\&} \|q\|_t$$
$$\|p \mathbin{\mathring{\vee}} q\|_t \leftrightarrow (\alpha)(\alpha \mathrel{<\cdot} t \rightarrow (\exists s)(\alpha \mathrel{<\cdot} s \mathbin{\&} (\|p\|_s \vee \|q\|_s))$$
$$\|\neg p\|_t \leftrightarrow (s)(s \leq t \rightarrow \sim\|p\|_s)$$
$$\|p \Rightarrow q\|_s \leftrightarrow (s)(s \leq t \rightarrow (\|p\|_s \rightarrow \|q\|_s)).$$

(14)
$$|p \mathbin{\mathring{\&}} q|_t \leftrightarrow (\exists\alpha)(\alpha \mathrel{<\cdot} t \mathbin{\&} (s)(\alpha \mathrel{<\cdot} s \rightarrow |p|_s \mathbin{\&} |q|_s))$$
$$|p \mathbin{\mathring{\vee}} q|_t \leftrightarrow |p|_t \vee |q|_t$$
$$|\neg p|_t \leftrightarrow (\exists s)(s \leq t \mathbin{\&} \sim|p|_s)$$
$$|p \Rightarrow q|_t \leftrightarrow |\neg p|_t \vee |q|_t.$$

In other words the above connectives are, in fact,'interval definable', that is, they are definable inside our interval valuations.

It can be seen that the semantic structure obtained for the valuation $\|p\|_t$ is in fact a modified version of Beth semantics for intuitionistic logic and hence the logic corresponding to it is exactly intuitionistic logic. Note also that the instant semantics (9) is a well-known topological representation of the same logic on a real line (cf. Scott [9]). Moreover, using (7) and (8) we may obtain interval definitions for instant-definable connectives (9) and (10) (similar to (12) and (13) above), and thereby will extend the mutual definability of corresponding instant and interval valuations on their associated semantics.

Now consider the logic associated with the dual interval valuation $|p|_t$. This logic is known as *dual intuitionistic logic* and this duality is naturally expressed in its axiomatics: while the intuitionistic logic is axiomatizable in a sequential calculus with classical definitions of the connectives, but with single consequents in sequents, the dual intuitionistic logic is axiomatizable with the same axioms, but in the sequential calculus with single antecedents.

Now we will add to our semantics a new negation connective, namely, an internal negation with respect to our source instant valuation $(p)_\alpha$:

(15) $$\|-p\|_t \leftrightarrow (\alpha)(\alpha <\cdot\, t \to \sim (p)_\alpha)$$
$$|-p|_t \leftrightarrow (\exists\alpha)(\alpha <\cdot\, t \;\&\; \sim (p)_\alpha).$$

We then have the following equivalences:

(16) $$\|-p\|_t \leftrightarrow (\alpha)(\alpha <\cdot\, t \to \sim \langle p\rangle_\alpha) \leftrightarrow \sim |p|_t$$
$$|-p|_t \leftrightarrow (\exists\alpha)(\alpha <\cdot\, t \;\&\; \sim \langle\langle p\rangle\rangle_\alpha) \leftrightarrow \sim \|p\|_t.$$

(17) $$\langle\langle -p\rangle\rangle_\alpha \leftrightarrow \sim \langle p\rangle_\alpha$$
$$\langle -p\rangle_\alpha \leftrightarrow \sim \langle\langle p\rangle\rangle_\alpha.$$

Thus, the above negation joins, in fact, our two interval valuations into a single semantic system.[7] We still have, however, two possible ways of defining semantic entailment, with respect to the valuation $\|p\|_t$ or with respect to the valuation $|p|_t$. It can be shown that, with some minor (and, it seems, unjustified) changes, the logic corresponding to semantic entailment relative to the valuation $\|p\|_t$ will coincide with intuitionistic logic with strong negation (cf. Gurevich [7] and Thomason [11]).[8] On the other hand, the logic generated by the semantic entailment relative to $|p|_t$ is a natural dual to the above logic. It is intersting to note that while the former logic is consistent (though not complete) with respect to 'strong' negation "$-$", its dual is complete but not consistent. The situation with contradiction is naturally interpreted in the following way: If some proposition p changes its truth value relative to the valuation $(\ )_\alpha$ at some instant β, then in all intervals containing β we will have both $|p|_t$ and $|-p|_t$. Moreover, the same logic is determined also by the instant semantics generated by definitions (9), (10), and (17) with respect to semantic entailment relative to the valuation $\langle p\rangle_\alpha$. In this semantics we will obtain contradictions (with respect to strong negation) at *any* instant of change. That is why this logic could be considered as a plausible candidate for the 'dialectical logic of change'.[9]

NOTES

1. As it turns out we may distinguish (though not characterize!) elementary aspectual types with the help of the above features in the following way:
   (i) States are strongly homogeneous
   (ii) Generics (iteratives) are strongly cumulative (though not in general persistent)
   (iii) Activities are homogeneous
   (iv) Accomplishments are neither persistent nor cumulative
   (v) Achievements are false at intervals.
2. This topic deserves special consideration. We may mention here Aristotle's dictum that any temporal interval may contain only a finite number of actual instants. We want, in fact, to restore (with minor changes) M. Black's original analysis of Zeno's paradoxes (cf. Black [4]), despite his later corrections (we particularly refer the reader to his notion of well-bounded change). It also seems that the finiteness requirement is one of the important conditions for distinguishing natural changes from the so-called 'Cambridge changes'.

3. Thus, the finiteness requirement precludes the existence of what was termed in von Wright [13] 'dialectical contradictions'. Nevertheless, as will be shown below, the possibility of some such contradictions still remains.

4. Note that we may identify an interval t with the set of its points.

5. There may also be other, weaker locality conditions, which we will not consider in this paper:
   (i) $\sim L \sim (p \leftrightarrow q) \rightarrow (p \leftrightarrow q)$
   (ii) $\sim L \sim A^{\circ}(\sim L^{\circ} \sim (p \leftrightarrow q)) \rightarrow (p \leftrightarrow q)$.

6. Namely, it requires the existence of at least weakly finite propositions and of some automorphisms between (countably) infinite sets of points and intervals of the same type.

7. In fact, we may dispense with one of these interval valuations altogether, if we consider negations of elementary propositions as also elementary propositions. Then de Morgan's laws and some other equivalences will yield corresponding definitions for complex formulas.

8. The required changes involve the definitions of implication and intuitionistic negation in the dual valuation $|p|_t$:

$$|p \Rightarrow q|_t \leftrightarrow \|p\|_t \rightarrow |q|_t$$

$$|\neg p|_t \leftrightarrow \sim \|p\|_t.$$

   Note that in both cases we have that $\neg p$ is strongly equivalent to $(p \Rightarrow -p)$. The characteristic additional axiom for intuitionistic negation in intuitionistic logic with strong negation is $p \Leftrightarrow -\neg p$, while in our case it is $-\neg p \Leftrightarrow \neg - p$.

9. Priest [8] has proposed a 'dialectical logic of change' based on what was called the 'Leibniz principle'. As can be seen, this principle is just the condition of closedness for the corresponding instant valuation and hence our logics have much in common.

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