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Remarks on Special Lattices and Related Constructive Logics with Strong Negation

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Abstract The main purposes of this paper are to provide an algebraic analysis of a certain class of constructive logics with strong negation, and to investigate algebraically the relations between strong and nonconstructible negations definable in these logics. The tools used in this analysis are varieties of algebraic structures of ordered pairs called *special N-lattices* which were first introduced as algebraic models for Nelson's *constructive logic with strong negation* (CLSN). Via suitable restrictions of the domains, algebras of this type are shown to be algebraic models for the propositional fragments of E^0, CLSN, Intuitionistic Logic, and E^0_+. The differences between E^0 and CLSN are then studied, via the interrelations that the related algebras exhibit among strong and nonconstructible negations, properties of filters, and their behavior with respect to classically valid formulas.

Introduction Constructive Logic with Strong Negation (CLSN) was introduced to correct certain nonconstructive properties of Intuitionistic negation (see Nelson [5], Thomason [11], and Vorob'ev [13]). The logics E^0, E^0_+, and E^+ (henceforth *Effective Logics* (EL)) were introduced in Miglioli et al. [4] in order to study the concept of effectiveness in Computer Science. Though thought to be independent, CLSN and EL are in fact very close. As a matter of fact, the propositional fragment of E^0 has the same axioms as CLSN (see Rasiowa [9], Chapter XII) without the axioms for weak negation and strong implication, plus the following rules:

$$(\neg\mathrm{T\ Int})\quad \frac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix} \qquad \begin{matrix}[A] \\ \vdots \\ \neg B\end{matrix}}{\neg \mathbf{T}A} \qquad\qquad (\mathrm{T\ Int})\quad \frac{\begin{matrix}[\neg A] \\ \vdots \\ \neg B\end{matrix} \qquad \begin{matrix}[\neg A] \\ \vdots \\ \neg B\end{matrix}}{\mathbf{T}A}$$

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Roughly speaking, the operator $\mathbf{T}$ has been introduced to capture classical truth, in the sense that if CLASS is a classical propositional calculus to which $\mathbf{T}$ as the identical truth function ($\mathbf{T}(1) = 1$; $\mathbf{T}(0) = 0$) is added, then the following holds, for every formula P:

$$\text{CLASS} \vdash P \qquad \text{iff} \qquad \mathrm{E}^0 \vdash \mathbf{T}P.$$

The propositional fragment of E^0_+ is equal to E^0 plus the following rules for any atomic formula a:

$$\text{(R1)}\ \frac{\begin{matrix}[\neg a] \\ \vdots \\ B\end{matrix} \qquad \begin{matrix}[\neg a] \\ \vdots \\ \neg B\end{matrix}}{a} \qquad\qquad \text{(R2)}\ \frac{\begin{matrix}[a] \\ \vdots \\ \neg B\end{matrix} \qquad \begin{matrix}[a] \\ \vdots \\ \neg B\end{matrix}}{\neg a}$$

The propositional fragment of E^+ is equal to E^0_+ when the following rule is added:

$$\text{(E)}\ \frac{\mathbf{T}A \to (B \vee C)}{(\mathbf{T}A \to B) \vee (\mathbf{T}A \to C)}$$

CLSN was studied from an algebraic point of view by Białynicki-Birula and Rasiowa (see [1] and [7]–[9]) by means of algebras called *N-lattices*. While in any topological pseudo-Boolean algebra $\mathbf{H}$ any element a univocally determines its pseudocomplement $\sim a$ by set-theoretical and topological operations, such means are not sufficient for strong negation in constructive logics; for, in a certain sense, any element a is determined by its own strong negation $\neg a$. Białynicki-Birula and Rasiowa bypassed this problem using an involution g from the algebraic domain D to itself, through a "specular" domain D'. Intuitively speaking, the involution g simply links any element of D to its own strong negation. Nevertheless, this algebraic construction is rather complicated, and its intuitive mechanism and its relation to Kripke-style models are hidden.

A more natural procedure for representing the interdetermination between a and $\neg a$ appears to be the following: take as elements of the algebraic domain ordered pairs (of open subsets of a topological space) and consider the first element in the pair as the "positive value" and the second element as the "negative value" of the pair. Vakarelov [12], starting from an analysis of strong negation in CLSN, interpreted the exhibition of a "counterexample" as the refutation of a sentence, and introduced a semantics by ordered pairs called special N-lattices (SNL). EL and CLSN (as well as connections with Heyting algebras) were algebraically studied in Pagliani [6], where a semantics by ordered pairs, called $\mathrm{P}^0\mathrm{T}$ algebras, was independently used. $\mathrm{P}^0\mathrm{T}$ algebras were defined over topological T_0 spaces in order to preserve a natural correspondence with Kripke-style models for the logics under discussion.

In Sendlewski [10] a variety of N-lattices adequate for E^0 were independently defined while investigating the connections between N-lattices and Heyting algebras. In the present paper some remarks about this kind of lattice will be pointed out.

1 Basic definitions

(b1) Let X be a finite or denumerable set.
(b2) Let $\tau(X)$ be a T_0 space on X and $\mathrm{OP}\tau(X)$ the set of open subsets of $\tau(X)$.
(b3) Let $D_i(X)$ be a set of ordered pairs $\alpha = \langle x, \rho(x)\rangle$ in which $x \cap \rho(x) = \varnothing$ and $x, \rho(x) \in \mathrm{OP}\tau(X)$.

Intuitively, x will denote the "positive value" of α and $\rho(x)$ will denote its "negative value". We will call the first element of an ordered pair the *term* and the second element the *index*.

The following situations can be distinguished, if we do not set particular conditions over $\tau(X)$ (**I** denotes the interior operator, **C** denotes the closure operator, and $-$ denotes the set-theoretical complement):

(N1)	$\mathbf{IC}\rho(x) \subseteq \mathbf{I} - x$	forming the domain $D_1(X)$	(The general case)
(N2)	$\mathbf{IC}\rho(x) = \mathbf{I} - x$	forming the domain $D_2(X)$	
(N3)	$\rho(x) = \mathbf{I} - x$	forming the domain $D_3(X)$.	

2 Algebraic definitions We now present a general set of operations for each $a, b \in D_{1,2,3}$ such that $a = \langle a_1, a_2\rangle$, $b = \langle b_1, b_2\rangle$ (we will not distinguish algebraic symbols used on ordered pairs from set-theoretical symbols used on the elements of ordered pairs):

(A1) $\langle X, \varnothing\rangle = 1$; $\langle \varnothing, X\rangle = 0$
(A2) $a \cup b = \langle a_1 \cup b_1, a_2 \cap b_2\rangle$
(A3) $a \cap b = \langle a_1 \cap b_1, a_2 \cup b_2\rangle$
(A4) $a \to b = \langle \mathbf{I}(-a_1 \cup b_1), a_1 \cap b_2\rangle$ (weak relative pseudo-complementation)
(A5) $\mathbf{T}a = \langle \mathbf{IC}a_1, -\mathbf{C}a_1\rangle$ (classical truth operator)
(A6) $\neg a = \langle a_2, a_1\rangle$. (strong negation).

From (A1), (A2), and (A3) the following are immediate: $a \cup 1 = 1$, $a \cap 1 = a$, $a \cup 0 = a$, and $a \cap 0 = 0$.

We can now define the following quasi-ordering relation:

(QO) $a \leq b$ iff $a \to b = 1$.

More precisely, because the equation $a \to b = 1$ depends only on the terms a and b and not on their indices (see (A4)), the relation $\leq$ is indeed a quasi-ordering for domains of type D_1 and D_2. That is, the antisymmetry property does not hold: we can have $a, b \in D_1$ (D_2) such that $a \leftrightarrow b = 1$ but $\neg a \leftrightarrow \neg b \leq 1$.

If $a \leftrightarrow b = 1$ we say that a and b are *weakly equivalent*. So we have that in D_1 and D_2 substitution of weak equivalents and the weak counterpositional law do not hold.

However we can define a stronger relative pseudo-complementation:

(A7) $a \Rightarrow b = (a \to b) \cap (\neg b \to \neg a)$
(PO) $a \leq b$ iff $a \to b = 1$.

It is easily verified that $\leq$ is a partial ordering.

3 Constructive and nonconstructive negations Given this base, if we want an operation for nonconstructive negation we have to introduce, following Zeman [14], an operation ψ such that the following hold:

(NC1) $(a \to \psi a) \to \psi a = 1$
(NC2) $\neg a \to \psi a = 1$
(NC3) $\psi a \to \neg a \leq 1$.

So let us introduce the following two operations.

(A8) $\sim a = a \to 0$
(A9) $\approx a = a \to (b \cap \neg b)$.

Proposition 1 *Let ψ be $\sim$ or $\approx$. Then* (NC1), (NC2), *and* (NC3) *are satisfied.*

Proof: $\sim\langle a_1,a_2\rangle = \langle a_1,a_2\rangle \to \langle \varnothing,X\rangle = \langle \mathbf{I}(-a_1 \cup \varnothing), a_1 \cap X\rangle = \langle -\mathbf{C}a_1, a_1\rangle$. But $a_2 \leq -\mathbf{C}a_1$, so $\neg\langle a_1,a_2\rangle \to \sim\langle a_1,a_2\rangle = 1$ and $\sim\langle a_1,a_2\rangle \to \neg\langle a_1,a_2\rangle \leq 1$.

$\approx\langle a_1,a_2\rangle = \langle a_1,a_2\rangle \to \langle b_1 \cap b_2, b_2 \cup b_1\rangle$. But $b_1 \cap b_2 = \varnothing$ (from (b3)), so $\approx\langle a_1,a_2\rangle = \langle -\mathbf{C}a_1, a_1 \cap (b_2 \cup b_1)\rangle$. Then $\neg\langle a_1,a_2\rangle \to \approx\langle a_1,a_2\rangle = 1$ and $\approx\langle a_1,a_2\rangle \to \neg\langle a_1,a_2\rangle \leq 1$.

$\langle a_1,a_2\rangle \to \langle -\mathbf{C}a_1, x\rangle = \langle \mathbf{I}(-a_1 \cup -\mathbf{C}a_1), (a_1 \cap x)\rangle$. But $-a_1 \supseteq -\mathbf{C}a_1$, so $\langle a_1,a_2\rangle \to \langle -\mathbf{C}a_1, x\rangle = \langle -\mathbf{C}a_1, (a_1 \cap x)\rangle$, whence straightforwardly $(a \to \sim a) \to \sim a = 1$ and $(a \to \approx a) \to \approx a = 1$.

4 Special lattices and logics with strong negations

Proposition 2.1 $\boldsymbol{\theta}_1 = \langle D_1(X), \cap, \cup, \to, \sim, \approx, \neg, 1, 0\rangle$ *is an algebra* (which we shall call a "special Nelson lattice" (SNL)).

Proposition 2.2 $\boldsymbol{\theta}_2 = \langle D_2(X), \cap, \cup, \to, \sim, \approx, \neg, 1, 0\rangle$ *is an algebra* (which we shall call a "special effective lattice" (SEL)).

Let $\zeta(\mathrm{e})$ be the following translation from formulas of the language of E^0, $\mathrm{L}(\mathrm{E}^0)$, to SEL: For any propositional variable x and any formulas P, P':

$$\zeta(\mathrm{e})(x) = p \qquad (\text{for } p \in D_2)$$
$$\zeta(\mathrm{e})(P \wedge P') = \zeta(\mathrm{e})(P) \cap \zeta(\mathrm{e})(P')$$
$$\zeta(\mathrm{e})(P \vee P') = \zeta(\mathrm{e})(P) \cup \zeta(\mathrm{e})(P')$$
$$\zeta(\mathrm{e})(P \to P') = \zeta(\mathrm{e})(P) \to \zeta(\mathrm{e})(P')$$
$$\zeta(\mathrm{e})(\neg P) = \neg\zeta(\mathrm{e})(P)$$
$$\zeta(\mathrm{e})(TP) = \mathrm{T}\zeta(\mathrm{e})(P).$$

Proposition 3 $E^0 \vdash P$ *iff* SEL $\models \zeta(\mathrm{e})(P)$.

Outline of Proof: For any SEL $\mathbf{A}_2(\mathbf{X})$ let us denote by val^0 a valuation of formulas from the language $\mathrm{L}(\mathrm{E}^0)$ $\mathbf{A}_2(\mathbf{X})$ based on $\zeta(\mathrm{e})$. Let $\mathbf{K} = \langle K, \leq, \gg\rangle$ be a Kripke model for E^0 (see [4]), where $\gg$ denotes the forcing relation. Considering the topological space $\tau(K)$ of the cones of K, let $\mathbf{A}_2(\mathbf{K})$ be the SEL built over $\tau(K)$. $\forall \alpha, \alpha' \in K$ we define the following relation:

(1) $\alpha \gg A$ iff $\alpha \in (\mathrm{val}^0(A))_1$
(2) $\alpha' \gg \neg A$ iff $\alpha' \in (\mathrm{val}^0(A))_2$.

Then for any formula P and any $\alpha \in K$ the forcing relations of K are preserved by val^0 and vice versa, and via the completeness theorem for Kripke models for E^0 (see [4]) we have: If $\mathrm{E}^0 \nvdash$ then SEL $\nvDash \zeta(\mathrm{e})(A)$ for any wff $A \in \mathrm{L}(\mathrm{E}^0)$.

Definition 3.1 Following Rasiowa [9] (Chapter 5, par. 3) let $G(X_1)$ be a pseudo-field of some open subsets constituting a base for the topological space $\tau(X_1)$; let $g(x)$ be the involution defined on $X = X_1 \cup X_2$, $X_1 \cap X_2 = \varnothing$ (for simplification), $\mathrm{card}(X_1) = \mathrm{card}(X_2)$; and let $B(X)$ be the class of all the subsets $Y \subseteq X$ satisfying conditions (b1), (b2), and (b3) of [9].

Theorem 3.2 (Rasiowa) *Let* $\mathbf{QP}(\mathbf{X}) = \langle B(X), X, \cap, \cup, \rightarrow, \sim, \neg \rangle$, *where* $\cap$ *and* $\cup$ *are set-theoretical operations, and* $\forall a, b \in B(X)$:
(3.2.1) $a \rightarrow b = (\mathbf{I} - ((a \cap X_1) \cup (b \cap X_1))) \cup ((X_2 \cap -g(X_1 \cap a)) \cup X_2 \cap b)$
(3.2.2) $\neg a = X - g(a)$
(3.2.3) $\sim a = a \rightarrow \neg X$.

Then $\mathbf{QP}(\mathbf{X})$ *is a quasi-pseudo-Boolean algebra of open subsets of* X *connected with the pseudo-field of sets* $G(X_1)$.

Theorem 3.3 (Rasiowa) *Any quasi-pseudo-Boolean algebra is isomorphic to a quasi-pseudo-Boolean algebra of sets. More precisely, it is isomorphic to a quasi-pseudo-Boolean algebra of open subsets of a compact* T_0 *topological space.* (This latter type of algebra is called QPBT.)

Theorem 3.4 (Rasiowa) *For any formula* P *of* L(CLSN):

$$\mathrm{CLSN} \vdash P \textit{ iff } \mathrm{QPBT} \models P.$$

Proposition 4 *Let* $\mathbf{A}_1(\mathbf{X}_1) = \langle D_1(X_1), \cap, \cup, \rightarrow, \sim, \neg, 1, 0 \rangle$ *be an* SNL *built over the topological space* $\tau(X_1)$. *Let* $\mathbf{QB}(\mathbf{X}_1) = \langle B(X_1), \cap, \cup, \rightarrow, \sim, \neg, 1, 0 \rangle$ *be a* QPBA *connected with* $G(X_1)$. *Then* $\mathbf{A}_1(\mathbf{X}_1)$ *and* $\mathbf{QB}(\mathbf{X}_1)$ *are isomorphic.*

Outline of Proof: Let us consider the following mapping from $D_1(X_1)$ to $B(X_1)$:

$$f(\langle a_1, a_2 \rangle) = a_1 \cup g(\div a_2)$$

where g is the involution on $X = X_1 \cup X_2$ and $\div a = X_1 \cap -a$. We have to verify, for instance, that $f(a \rightarrow b) = f(a) \rightarrow f(b)$.

$$\begin{aligned}
f(a \rightarrow b) &= f(\langle a_1, a_2 \rangle \rightarrow \langle b_1, b_2 \rangle) \\
&= f(\mathbf{I}(-a_1 \cup b_1), a_1 \cap b_2) \\
&= (\mathbf{I}(-a_1 \cup b_1)) \cup g(\div (a_1 \cap b_2)) \\
f(a) \rightarrow f(b) &= (a_1 \cup g(\div a_2)) \rightarrow (b_1 \cup g(\div b_2)) \\
&= (\mathbf{I}(-(X_1 \cap (a_1 \cup g(\div a_2))) \cup (X_1 \cap b_1 \cup g(\div b_2)))) \\
&\quad \cup (X_2 \cap -g(X_1 \cap (a_1 \cup g(\div a_2))) \cup (X_2 \cap (b_1 \cup g(\div b_2)))) \\
&= (\mathbf{I}(-a_1 \cup b_1)) \cup (X_2 \cap -g(a_1)) \cup g(\div b_2).
\end{aligned}$$

Now all we have to show is that $g(\div(a_1 \cap b_2)) = (X_2 \cap -g(a_1)) \cup g(\div b_2)$. But

$$\begin{aligned}(X_2 \cap -g(a_1)) \cup g(\div b_2) &= (X_2 \cap -g(a_1)) \cup g(\div b_2)\\ &= g(\div a_1) \cup g(\div b_2)\\ &= g(\div(a_1 \cap b_2)).\end{aligned}$$

Let $\zeta(\mathrm{n})$ be a translation from formulas in L(CLSN) to SNL such that, for any propositional variable x and $p \in D_1$, $\zeta(\mathrm{n})(x) = p$, and for any formula P, $\zeta(\mathrm{n})(P)$ is $\zeta(\mathrm{e})(P)$ without $\zeta(\mathrm{e})(\mathbf{T}P)$, plus the following additional translation rule: $\zeta(\mathrm{n})(\sim P) = \sim\zeta(\mathrm{n})(P)$.

Via the isomorphism f we have a Kripke-style interpretation of QPBT algebra $\mathbf{QB}(\mathbf{X}_1)$: let val* be a valuation of formulas from the language L(CLSN) to SNL based on $\zeta(\mathrm{n})$; thus if val*$(A) = x$ then $f^{-1}(x) = \langle X_1 \cap x, \div g(X_2 \cap x)\rangle$ (it is easily seen that $f^{-1}(x) \in D_1(X_1)$), and so $x = \{\alpha : \alpha \gg A\} \cup \{\beta : \beta \not\gg \neg A\}$, where α, β belong to a Kripke-style Thomason model isomorphic to $\mathbf{A}_1(\mathbf{X}_1)$.

Proposition 5 (Vakarelov) (Corollary to Proposition 4) *For any formula A of* L(CLSN):

$$\mathrm{CLSN} \vdash A \textit{ iff } \mathrm{SNL} \vDash \zeta(\mathrm{n})(A).$$

Proposition 6 *If we add the following condition to conditions* (b1)–(b3) *of* Definition 3.1:

(b4) $\mathbf{IC}g(Y \cap X_2) = \mathbf{IC}(Y \cap X_1)$,

and let $D'(X)$ be the new domain, then $D'(X)$ and the domain $D_2(X)$ defined on the same topological space coincide.

Outline of Proof: We will prove that the function f defined as in Proposition 4 is biunivocal between $D_2(X)$ and $D'(X)$. Remembering that $f^{-1}(a) = \langle X_1 \cap a, \div g(X_2 \cap a)\rangle$ for $a \in D'(x)$, we have to prove for every Y that $\mathbf{IC}(\div g(X_2 \cap Y) = \div\mathbf{C}(X_1 \cap Y)$. From (b4) we have $\div\mathbf{IC}g(Y \cap X_2) = \div\mathbf{IC}(Y \cap X_1)$ and so $\mathbf{CI} \div g(Y \cap X_2) = \mathbf{CI} \div (Y \cap X_1)$. From (b2) we have that $g(Y \cap X_2)$ is closed, and from (b1) that $Y \cap X_1$ is open. Thus $\mathbf{IC} \div g(Y \cap X_2) = \mathbf{ICI} \div g(Y \cap X_2) = \mathbf{ICIC} \div (Y \cap X_1)$ and so $\mathbf{IC} \div g(Y \cap X_2) = \div\, \mathbf{CI}(Y \cap X_1) = \div\mathbf{C}(Y \cap X_1)$.

5 Properties of special N-lattices and special effective lattices

Proposition 7.1 *In any* SEL *or* SNL:
(7.1.1) $\mathbf{T}a = \sim\sim a$
(7.1.2) $\neg\mathbf{T}a \leq \mathbf{T}\neg a$.

Proof: (7.1.1) from (B1). (7.1.2) from (C2), (C4), and the fact that $\mathbf{IC}x \subseteq -\mathbf{C}x$ under the general condition.

Proposition 7.2 *In any* SEL:
(7.2.1) $\neg\mathbf{T}a = \mathbf{T}\neg a$
(7.2.2) $-\mathbf{C}a_1 \cap -\mathbf{C}a_2 = \varnothing$
(7.2.3) $\mathbf{T}a = \langle -\mathbf{C}a_2, -\mathbf{C}a_1\rangle$
(7.2.4) $0 \leftrightarrow (a \wedge \neg a) \Leftrightarrow (a \rightarrow \neg a) \wedge (\neg a \rightarrow a)$.

Proof: (7.2.1) from (C2), (C4) and the fact that, under the restriction (N2), $\mathbf{IC}a_2 = -\mathbf{C}a_1$. (7.2.2) and (7.2.3) from (N2). For (7.2.4), consider that $a \rightarrow \neg a = \langle \mathbf{I}(-a_1 \cup a_2), a_1 \cap a_1 \rangle = \langle -\mathbf{C}a_1, a_1 \rangle$ and $\neg a \rightarrow a = \langle \mathbf{I}(-a_2 \cup a_1), a_2 \cap a_2 \rangle = \langle -\mathbf{C}a_2, a_2 \rangle$. Together with (7.2.2) these equivalences yield (7.2.4).

Proposition 7.3 *In any* SNL:
(7.3.1) $-\mathbf{C}a_1 \cap -\mathbf{C}a_2 \supseteq \varnothing$
(7.3.2) $0 \leftrightarrow (a \wedge \neg a) \Rightarrow (a \rightarrow \neg a) \wedge (\neg a \rightarrow a)$.

Proof: (7.3.1) follows from (N1), and (7.3.2) follows from (7.3.1) according to the proof of (7.2.4).

The latter proposition makes manifest the main topo-algebraic differences between the general case (N1) and the restriction (N2).

We call any (eventually void) even sequence of symbols from the set $\{\sim, \neg\}$ a *positive modality*, and any odd sequence a *negative modality*. We say that two modalities $\mathbf{M}$ and $\mathbf{M}'$ are *weakly equivalent* (*strongly equivalent*) iff $\mathbf{M}a \rightarrow \mathbf{M}'a$ and $\mathbf{M}'a \rightarrow \mathbf{M}a$ ($\mathbf{M}a \Rightarrow \mathbf{M}'a$ and $\mathbf{M}'a \Rightarrow \mathbf{M}a$). We say that two modalities $\mathbf{M}$ and $\mathbf{M}'$ are *weakly distinct* (*strongly distinct*) iff they are not weakly equivalent (strongly equivalent).

Proposition 8 *Up to* (*strong*) *equivalences, the following are all and only the distinct modalities for* SEL's *and* SNL's:

SEL
positive weakly distinct modalities: $\varnothing, \sim\neg$
negative weakly distinct modalities: $\neg, \sim$
positive strongly distinct modalities: $\varnothing, \neg\sim, \sim\neg, \sim\sim$
negative strongly distinct modalities: $\neg, \sim, \neg\sim\neg, \neg\sim\sim$.

SNL
positive weakly distinct modalities: $\varnothing, \sim\neg, \sim\sim$
negative weakly distinct modalities: $\neg, \sim, \sim\sim\neg$
positive strongly distinct modalities: $\varnothing, \neg\sim, \sim\sim, \sim\neg, \neg\sim\sim\neg$
negative strongly distinct modalities: $\neg, \sim, \sim\sim\neg, \neg\sim\sim, \sim\neg\sim$.

Proof: The proof is given by the following reductions of the modalities based on terms and indices of the ordered pairs:

(A) Negative modalities of length 1:
  (1) $\neg\langle x,y \rangle = \langle y,x \rangle$
  (2) $\sim\langle x,y \rangle = \langle -\mathbf{C}x, x \rangle$.
  From Proposition 1.
(B) Positive modalities of length 2:
  (1) $\neg\neg\langle x,y \rangle = \langle x,y \rangle$
  (2) $\sim\sim\langle x,y \rangle = \langle \mathbf{IC}x, -\mathbf{C}x \rangle$
  (3) $\sim\neg\langle x,y \rangle = \langle -\mathbf{C}y, y \rangle$
  (4) $\neg\sim\langle x,y \rangle = \langle x, -\mathbf{C}x \rangle$
  From (A) and the fact that $-\mathbf{C} - \mathbf{C}z = \mathbf{IC}z$ for any (open) subset z.

(C) Negative modalities of length 3:
(1) $\neg\neg\neg\langle x,y\rangle = \neg\langle x,y\rangle$
(2) $\sim\sim\sim\langle x,y\rangle = \neg\sim\sim\langle x,y\rangle = \langle -\mathbf{C}x,\mathbf{IC}x\rangle$
(3) $\neg\sim\neg\langle x,y\rangle = \langle y, -\mathbf{C}y\rangle$
(4) $\sim\sim\neg\langle x,y\rangle = \langle \mathbf{IC}y,-\mathbf{C}y\rangle$
(5) $\neg\neg\sim\langle x,y\rangle = \sim\neg\sim\langle x,y\rangle = \sim\neg\neg(x,y) = \sim\langle x,y\rangle = \langle -\mathbf{C}x,x\rangle$.
From (A), (B), and the fact that $\mathbf{IC}-\mathbf{C}z = -\mathbf{CIC}z = -\mathbf{C}z$ for any open subset z.

(D) Positive modalities of length 4:
(1) $\neg\neg\neg\neg\langle x,y\rangle = \langle x,y\rangle$
(2) $\sim\sim\sim\sim\langle x,y\rangle = \neg\sim\sim\sim\langle x,y\rangle = \sim\sim\neg\sim\langle x,y\rangle = \sim\neg\sim\sim\langle x,y\rangle = \neg\neg\sim\sim\langle x,y\rangle = \sim\neg\neg\sim\langle x,y\rangle = \sim\sim\neg\neg\langle x,y\rangle = \sim\sim\langle x,y\rangle$
(3) $\neg\neg\sim\neg\langle x,y\rangle = \sim\neg\sim\neg\langle x,y\rangle = \sim\neg\neg\neg\langle x,y\rangle = \sim\neg\langle x,y\rangle$
(4) $\neg\neg\neg\sim\langle x,y\rangle = \neg\sim\neg\sim\langle x,y\rangle = \neg\sim\langle x,y\rangle$
(5) $\neg\sim\sim\neg\langle x,y\rangle = \sim\sim\sim\neg\langle x,y\rangle = \langle -\mathbf{C}y,\mathbf{IC}y\rangle$.
From (A), (B), and (C).

(E) We now increase the length of the irreducible modalities of length 4:
(1) $\neg\neg\sim\sim\neg\langle x,y\rangle = \sim\sim\neg\langle x,y\rangle$ (From (B)(1))
(2) $\sim\neg\sim\sim\neg\langle x,y\rangle = \sim\sim\neg\langle x,y\rangle$ (From (D)(2))
(3) $\neg\sim\sim\sim\neg\langle x,y\rangle = \sim\sim\neg\langle x,y\rangle$ (From (D)(2))
(4) $\sim\sim\sim\sim\neg\langle x,y\rangle = \sim\sim\neg\langle x,y\rangle$ (From (D)(2))

(F) Now note that in any SEL we have by (N2) that:

$$-\mathbf{C}a_1 = \mathbf{IC}a_2 \quad\text{and}\quad -\mathbf{C}a_2 = \mathbf{IC}a_1.$$

Proposition 9 *In an* SEL, $\forall a,b \in D_2(X)$:
(9.1) *a has the form $\langle x,\varnothing\rangle$ iff x is dense in $\tau(X)$*
(9.2) $\mathbf{T}\langle x,\varnothing\rangle = 1$
(9.3) $a \cup \neg a \leq 1$; *$a \cup \neg a$ has the form $\langle x,\varnothing\rangle$*
(9.4) $\mathbf{T}(a \cup \neg a) = 1$
(9.5) $a \to b = 1$ *iff* $a_1 \subseteq b_1$
(9.6) *If $a \to b$ then $\neg b \to \neg a$ is a dense element (that is, it has the form $\langle x,\varnothing\rangle$)*
(9.7) *If $\neg a \to \neg b$ then $b \to a$ is a dense element*
(9.8) *In any* SEL, *a Special Filter of the First Kind*, ∇ (see [9], Chapter 5, par. 4) *is maximal iff exactly a or $\neg a$ belongs to ∇, for any a*
(9.9) *The counterpositional law fails in the* SEL's
(9.10) $b \cap \neg b \leftrightarrow 0$
(9.11) *If $a = \langle\varnothing,x\rangle$ then for each b, $a \to b = 1$*
(9.12) $\neg(b \cap \neg b) \leq 1$; $\mathbf{T}\neg(b \cap \neg b) = 1$
(9.13) $\mathbf{T}a \Leftrightarrow \sim\approx a \leftrightarrow \approx\sim a \Leftrightarrow \approx\approx a \leftrightarrow \sim\neg a \leftrightarrow \approx\neg a$
(9.14) SEL $\models \mathbf{T}\zeta(\mathrm{e})(A)$ *iff* CLASS $\vdash A$
(9.15) *More generally, for any positive modality $\mathbf{M} = \mathbf{M}_1\mathbf{M}_2$ such that $\mathbf{M}_1 = \sim$ or $\mathbf{M}_1 = \approx$,* SEL $\models \mathbf{M}\zeta(\mathrm{e})(A)$ *iff* CLASS $\vdash A$.

Proof:
(9.1) Since $\varnothing = \mathbf{IC}\varnothing = -\mathbf{C}X$, $\mathbf{C}x = X$.
(9.2) From (A5) and (9.1).
(9.3) $a \cup \neg a = \langle a_1 \cup \rho(a_1),\rho(a_1) \cap a_1\rangle$. But from (b3), $\rho(a_1) \cap a_1 = \varnothing$ and

so from (9.1) $a_1 \cup \rho(a_1)$ is dense in $\tau(X)$. For since $\tau(X)$ need not be totally disconnected, in general $(a_1 \cup \rho(a_1)) \subseteq X$.

(9.4) From (9.2) and (9.3).

(9.5) From (A4).

(9.6) If $a \to b = \langle x,y \rangle$ then $\neg b \to \neg a = \langle w,y \rangle$ for some w. So if $a \to b = 1$, $y = \varnothing$ and w is dense in $\tau(X)$.

(9.7) Similar to the proof of (9.6).

(9.8) If $a_1 \cap b_1 = \varnothing$ then $a_2 \cup b_2$ is dense and thus $a_1 \cap b_2 \neq \varnothing$. So let us consider an element a such that neither $a \in \nabla$ nor $\neg a \in \nabla$. If there is a $y \in \nabla$ such that $y \cap a = \langle \varnothing, x \rangle$ for some x, then $y \cap \neg a \neq \varnothing$, and we set $\nabla' = \nabla \cup \{\neg a\}$. Otherwise we set $\nabla' = \nabla \cup \{a\}$. In either case, ∇' is proper and $\nabla \subseteq \nabla'$. So ∇ is not maximal. The opposite side is straightforwardly verified.

(9.9) From (9.5).

(9.10) From (9.5) and the fact that $b \cap \neg b = \langle \varnothing, x \rangle$.

(9.11) From (9.5).

(9.12) From $b \cap \neg b = \langle \varnothing, x \rangle$, (9.2), and (9.3).

(9.13) First of all, notice that from (N2), $-\mathbf{C}a_2 = \mathbf{IC}a_1$. For some $x \subseteq a_2$ and $w \subseteq -\mathbf{C}a_1$ we have the following: $\approx\neg a = \langle -\mathbf{C}a_2, x \rangle$; $\approx\approx a = \langle \mathbf{IC}a_1, w \rangle$; $\approx\sim a = \langle \mathbf{IC}a_1, w \rangle$; and $\sim\approx a = \langle \mathbf{IC}a_1, -\mathbf{C}a_1 \rangle$. The proof for $\sim\sim a$ and $\sim\neg a$ follows from 8(B).

(9.14)–(9.15) (Outline) If P is classically valid, then for each val^0, $\mathrm{val}^0(P)$ has the form $\langle x, \varnothing \rangle$. The result then follows from (9.2) and (9.13).

Proposition 10 *In any* SNL, $\forall a,b \in D_1(X)$:

(10.1) *a has the form $\langle x, \varnothing \rangle$ if x is dense in $\tau(X)$. (The converse does not hold)*

(10.2) $\mathbf{T}(a \cup \neg a) \leq 1$

(10.3) (9.5), (9.9), (9.10), and (9.11) *hold in any* SNL

(10.4) (9.6), (9.7), and (9.8) *fail in any* SNL

(10.5) $\mathbf{T}(a \cup \sim a) = 1$

(10.6) $\sim\neg(a \cup \neg a) = 1$

(10.7) $\sim\approx a \Leftrightarrow \mathbf{T}a \leftrightarrow \approx\sim a \Leftrightarrow \approx\approx a \leq \approx\neg a \leftrightarrow \sim\neg a$

(10.8) *Let ζ' be a translation from the classical connectives in to the operations of* SNL *such that the classical "$\neg$" is interpreted over the* SNL "$\neg$", *and let ζ'' be a similar translation such that the classical "$\neg$" is interpreted over the* SNL "$\sim$". *Let the other connectives be interpreted by $\zeta(e)$. Then*

(10.8.1) *If* SNL $\vdash \mathbf{T}\zeta'(A)$ *then* CLASS $\vdash A$. *More generally*:

(10.8.2) *For any positive modality* $\mathbf{M}$, *if* SNL $\models \mathbf{M}\zeta'(A)$ *then* CLASS $\vdash A$.

(10.8.3) SNL $\models \mathbf{T}\zeta''(A)$ *iff* CLASS $\vdash A$. *More generally*:

(10.8.4) *For any positive modality* $\mathbf{M} = \mathbf{M}_1\mathbf{M}_2$ *such that* $\mathbf{M}_1 = \sim$ *or* $\mathbf{M}_1 = \approx$, SNL $\models \mathbf{M}\zeta''(A)$ *iff* CLASS $\vdash A$.

Proof:

(10.1) From (N1).

(10.2) $a \cup \neg a$ has the form $\langle y, \varnothing \rangle$, but for (N1) y may not be dense in $\tau(X)$, so $\mathrm{T}a = \langle \mathbf{IC}y, -\mathbf{C}y \rangle \leq \langle X, \varnothing \rangle$.

(10.4) From (10.1).

(10.5) $a \cup \sim a = \langle a_1 \cup -\mathbf{C}a_1, a_1 \cap a_2 \rangle = \langle d, \varnothing \rangle$ with d dense in $\tau(X)$. The proof then follows from (A5).

(10.6) From Proposition 8(B)(3).

(10.7) From the proof of (9.13) and (N1), $(\mathbf{IC}a_1 \subseteq -\mathbf{C}a_2, \mathbf{IC}a_2 \subseteq -\mathbf{C}a_1)$.

(10.8) Any classical theorem under the translation ζ'' is a dense element while not every classical theorem under the translation ζ' has the form $\langle x,\varnothing\rangle$. So from (10.2), (10.5)-(10.7), and Proposition 9 the result follows.

6 *Special lattices and intuitionistic logic* In any topological space $\mathbf{I}(a \cap b) = \mathbf{I}a \cap \mathbf{I}b$. So in any domain D_3 of type (N3) we have, for $\forall a,b \in D_3$:

$$a \cup b \in D_3; \qquad \mathbf{T}a \in D_3.$$

But this does not hold for the operations $\rightarrow$, $\neg$, $\approx$, $\sim$, and $\cap$. By Proposition 8 we have, for $\forall a \in D_3$:

$$\neg{\sim}a \in D_3; \qquad \neg{\approx}a \in D_3; \qquad {\sim}{\sim}a \in D_3.$$

But note that only "$\neg\sim$" leaves the terms of the ordered pairs unchanged. So we can define on D_3 the following operations:

(A.10) $a\eta b = \neg{\sim}(a \cap b)$
(A.11) $a\pi b = \neg{\sim}(a \rightarrow b)$
(A.12) $\lambda a = \neg{\sim}\neg a$
(A.13) $\gamma a = \neg{\sim}{\sim}a = \neg{\sim}{\approx}a$.

Proposition 11 $\Theta_3 = \langle D_3(X), \eta, \cup, \pi, \lambda, \gamma, 1, 0\rangle$ *is an algebra.* (We will call it a "special Heyting lattice" (SHL).)

Proposition 12 *Let* $\mathbf{A}_2(\mathbf{X})$ *be an* SEL *with domain* $D_2(X)$ *and let* $D_2(X)^*$ *be the set of terms of the ordered pairs in* $D_2(X)$. *For any operation* ζ *in* $\mathbf{A}_2(\mathbf{X})$ *let* ζ_1 *be* ζ *restricted to* $D_2(X)^*$. *Then the algebra* $\mathbf{HA}_2(\mathbf{X}) = \langle D_2,(X)^*, \sim_1, \cap_1, \cup_1, \rightarrow_1, 1_1, 0_1\rangle$ *is isomorphic to the pseudo-Boolean algebra* $\mathbf{H}(\mathbf{X}) = \langle D_2(X)^*, \sim, \cap, \cup, \Rightarrow, 1, 0\rangle$ *(where the operations are the usual ones for topological pseudo-Boolean algebras). Moreover, let* $\mathbf{A}_3(\mathbf{X})$ *be an* SHL *with domain* $D_3(X)$; *if the pseudo-complementation* $\sim$ *is interpreted as the* SHL λ, *then* $\mathbf{HA}_2(\mathbf{X})$ *is isomorphic to* $\mathbf{A}_3(\mathbf{X})$.

Outline of Proof: For any $a \in D_2(X)^*$ we define the following mapping:

$$f(a) = \langle a, -\mathbf{C}a\rangle.$$

We now prove the last isomorphism for the case ($\rightarrow$):

$$f(a \rightarrow_1 b) = \langle \mathbf{I}(-a \cup b), -\mathbf{CI}(-a \cup b)\rangle.$$

But from Proposition 8

$$\begin{aligned} f(a)\pi f(b) &= \langle a, -\mathbf{C}a\rangle \pi \langle b, -\mathbf{C}b\rangle \\ &= \langle \mathbf{I}(-a \cup b), -\mathbf{CI}(-a \cup b)\rangle. \end{aligned}$$

Moreover, for any subset $x, -\mathbf{C}x$ is unique, so f is $1-1$.

Let ζ(h) be the following translation from any propositional variable x and any formulas P, P' of the propositional intuitionistic calculus INT to SHL:

$$\zeta(\mathrm{h})(x) = p \qquad \text{for } p \in D_3$$
$$\zeta(\mathrm{h})(P \wedge P') = \zeta(\mathrm{h})(P)\eta\zeta(\mathrm{h})(P')$$
$$\zeta(\mathrm{h})(P \vee P') = \zeta(\mathrm{h})(P) \cup \zeta(\mathrm{h})(P')$$
$$\zeta(\mathrm{h})(P \Rightarrow P') = \zeta(\mathrm{h})(P)\pi\zeta(\mathrm{h})(P')$$
$$\zeta(\mathrm{h})(\neg P) = \lambda\zeta(\mathrm{h})(P).$$

Then:

Corollary 13 *INT* $\vdash P$ *iff* SHL $\models \zeta(\mathrm{h})(P)$.

Concluding remarks We would now like to underscore some of the syntactical differences between the logics E^0 and CLSN. In E^0 there is no symbol for nonconstructive negation, and for each logical constant ψ there is a rule for the strong negation of ψ (including strong negation itself and **T**). The two rules $\neg$**T**-Introduction and **T**-Introduction taken together match our definition (A5), even if we have not literally translated them algebraically. As a matter of fact, algebraically speaking, $\mathbf{T}a = (a \to 0) \to 0 = {\sim}a \to 0$, and thus we have that $\neg\mathbf{T}a$ is different from ${\sim}a = a \to 0$:

$$\neg\mathbf{T}a = \langle -\mathbf{C}a_1, \mathbf{IC}a_1\rangle; \qquad {\sim}a = \langle -\mathbf{C}a_1, a_1\rangle$$

and so

$$\neg{\sim}a \leq \neg\neg\mathbf{T}a = \mathbf{T}a.$$

The literal translation of $\neg$**T**-Introduction is $\approx$. But again, in order to conserve the properties of **T** defined by natural deduction in E^0 we have to translate **T** (algebraically) in a different way. Let us now illustrate these particular relations between the calculus and the algebraic interpretation, paying special attention to the syntactical introduction of a nonconstructive negation in E^0.

From (9.13) and (10.7) we have:

In SNL

(a) $\approx\approx a \leq \approx\neg a$
(b) $\sim\sim \mathrm{a} \leq \sim\neg \mathrm{a}$.

In SEL

(a′) $\approx\approx a \leftrightarrow \approx\neg a$
(b′) $\sim\sim a \leftrightarrow \sim\neg a$.

From the algebraic point of view, the differences between (a)–(b) and (a′)–(b′) derive from the difference between (N1) and (N2). Now we are going to see that it is possible to introduce syntactically into E^0 more than one plausible nonconstructive negation (pairwise constructively incompatible).

Using the expressive capabilities of the calculus, let us introduce into E^0 a sign for nonconstructive negation, ψ, in one of the following weakly equivalent ways:

(D1) $\psi A \leftrightarrow \neg \mathbf{T} A$
(D2) $\psi A \leftrightarrow \mathbf{T} \neg A$
(D3) $\psi A \leftrightarrow (A \rightarrow p \wedge \neg p)$.

Assume that we have introduced such a ψ as a new rule into E^0. We can now prove in E^0:

$$(\psi^*) \qquad \psi\psi A \leftrightarrow \psi\neg A.$$

From left-to-right we need not use the **T**-rules (which are the specific rules of E^0), but this implication is provable in SNL (see point (b) above). But from right-to-left the **T**-rules are required. So, in view of (b′), the operation "$\sim$" defined by (A8) ought to be a good algebraic candidate for ψ.

When we have to specify the corresponding rule for $\neg\psi$, despite the previous supposition we have several choices for a definiens:

$(\neg\psi_1)$ $\neg\psi A \leftrightarrow \neg\neg A$

$(\neg\psi_2)$ $\neg\psi A \leftrightarrow \psi\psi A$

$(\neg\psi_3)$ $\neg\psi A \rightarrow \neg\neg A$.

Note that from Proposition (8)(B)(4) we have $\neg{\sim}a \rightarrow {\sim}{\sim}a$ but not ${\sim}{\sim}a \rightarrow \neg{\sim}a$. This fact together with (ψ^*) induces us to consider $(\neg\psi_1)$ as the most natural choice. But it remains nevertheless a choice.

If we do adopt $(\neg\psi_1)$ the algebraic translation ζ of ψ is:

$$\zeta(\psi A) = {\sim}\zeta(A).$$

If we adopt $(\neg\psi_2)$, the algebraic translation ζ of ψ is:

$$\zeta(\psi A) = \neg\mathbf{T}\zeta(A).$$

And if we adopt $(\neg\psi_3)$ the algebraic translation ζ of ψ is:

$$\zeta(\psi A) = {\approx}\zeta(A).$$

In any of these cases, in view of (D1)–(D3) the domain of the algebraic interpretation is D_2. So E^0 is a rather peculiar logic, and the nonconstructive negation of CLSN differs from any of the possible choices for ψ in E^0, as can be seen by Propositions (9.13) and (10.7).

If we were to adopt simultaneously either $(\neg\psi_1)$ and $(\neg\psi_2)$ or $(\neg\psi_2)$ and $(\neg\psi_3)$ the extension would collapse into a nonconstructive logic.

A close but still constructive extension is given by the logic E^0_+, as we shall now see.

Proposition 14 *An open set x of a topological space $\tau(X)$ is called* regular *if* $\mathbf{IC}x = x$. *We say that an element $a = \langle a_1, a_2\rangle$ of $D_1(X), D_2(X), D_3(X)$ is* regular *if both a_1 and a_2 are regular. Then for any regular elements $a, b \in D_2(X)$ the following hold*:
(14.1) $\neg a \Leftrightarrow \mathbf{T}\neg a$
(14.2) $(a \rightarrow b) \Leftrightarrow \mathbf{T}(a \rightarrow b)$
(14.3) $\neg(a \rightarrow b) \Leftrightarrow \mathbf{T}\neg(a \rightarrow b)$
(14.4) $(a \cap b) \Leftrightarrow \mathbf{T}(a \cap b)$

(14.5) $\neg(a \cup b) \Leftrightarrow \mathbf{T}\neg(a \cup b)$
(14.6) $\neg(a \cap b) \leq \mathbf{T}\neg(a \cap b)$
(14.7) $(a \cup b) \leq \mathbf{T}(a \cup b)$.

Outline of Proof: Let us prove (14.2). $(a \to b) = \langle \mathbf{I}(-a_1 \cup b_1), a_1 \cap b_2 \rangle$. Instead of a direct topological proof note that all we have to prove is that $\mathbf{I}(-\mathbf{IC}a_1 \cup \mathbf{IC}b_1) = \mathbf{ICI}(-\mathbf{IC}a_1 \cup \mathbf{IC}b_1)$. But this is equivalent to proving that $({\sim\sim}a \to {\sim\sim}b) = {\sim\sim}({\sim\sim}a \to {\sim\sim}b)$ in a pseudo-Boolean algebra.

The other propositions follow directly from Proposition 8 ((D2) and (C2)) and the fact that $\mathbf{IC}(a \cap b) = \mathbf{IC}a \cap \mathbf{IC}b$.

Proposition 15 *For any formula P of* $L(E_+^0)$ *that does not contain any subformula of the form $\neg(A \wedge B)$ or $A \vee B$, and for any* SEL $\mathbf{A}_2(\mathbf{X})$, *we have*: $\mathbf{A}_2(\mathbf{X}) \models \mathrm{val}^+(P) \leftrightarrow \mathrm{val}^+(\mathbf{T}P)$, *where* val^+ *is a univocal assignment from the set of propositional variables of* $L(E_+^0)$ *to the set of regular elements of* $\mathbf{A}_2(\mathbf{X})$, *and the operations are interpreted as for $\zeta(e)$ in* Proposition 3.

Proof: By induction from Proposition 14.

Proposition 16 *For any formula A and for every evaluation* val^+ *of* $L(E_+^0)$

$$E_+^0 \vdash A \textit{ iff } \mathrm{SEL} \models \mathrm{val}^+A.$$

Proof: Similar to the proof of Proposition 3.

Finally, we note that a proper semantics for E^+ is the semantics of "evaluation forms" presented in [4], which is not a Kripke-type semantics. Nevertheless, we can see that because rule (E) involves essentially the validity of an implication, by using (9.5) we can reduce the problem of looking for an algebraic model for E^+ to the problem of finding a structure in which

$$(\mathbf{T}_1 a \to_1 (b \cup_1 c)) \to_1 ((\mathbf{T}_1 a \to_1 b) \cup_1 (\mathbf{T}_1 a \to_1 c))$$

holds. By Proposition (7.1.1) the latter is reduced in its turn to

$$({\sim\sim}a \to (b \cup c)) \to (({\sim\sim}a \to b) \cup ({\sim\sim}a \to c)),$$

and since in any topological pseudo-Boolean algebra ${\sim\sim}a = \mathbf{I} - x$, for some x, everything is reduced to the problem of looking for a Kripke model for

$$({\sim}a \to (b \cup c)) \to (({\sim}a \to b) \cup ({\sim}a \to c))$$

(which is the Kreisel-Putnam principle, see [3]). So we could apply the results of Gabbay [2] together with the definitions of $\cap$, $\cup$, $\sim$, $\neg$, $\mathbf{T}$, and $\to$ to find an algebraic model for E^+. But note that in this case we have to impose a particular structure over the topological space underlying these SEL's.

REFERENCES

[1] Białynicki-Birula, A. and H. Rasiowa, "On representation of quasi-Boolean algebras," *Bulletin of Logic, Polish Academy of Sciences*, vol. 5 (1957), pp. 259–261.

[2] Gabbay, D. M., "The decidability of the Kreisel-Putnam system," *The Journal of Symbolic Logic*, vol. 35 (1970), pp. 431–437.

[3] Kreisel, G. and H. Putnam, "Eine Unableitsbarkeitsbeweismethode für den intuitionistischen Aussagenkalkul," *Archiv für Mathematische Logik und Grundlagenforschung*, vol. 3 (1957), pp. 35–57.

[4] Miglioli, P. A., U. Mosacato, M. Ornaghi, and G. Usberti, "A constructivism based on classical truth," *Notre Dame Journal of Formal Logic*, vol. 30 (1989), pp. 67–90.

[5] Nelson, D., "Constructible falsity," *The Journal of Symbolic Logic*, vol. 14 (1949), pp. 16–26.

[6] Pagliani, P., "P^0T algebras: Algebraic modelling for E^0 and E^0_+ logics and the algebraic investigation of their connections with other similar constructive logics" (in Italian), Universitá di Milano, Istituto di Cibernetica, Internal Report (1984).

[7] Rasiowa, H., "N-lattices and constructive logic with strong negation," *Fundamenta Mathematicae*, vol. 46 (1958), pp. 61–80.

[8] Rasiowa, H., "Algebraische Charakterisierung der intuitionistischen Logik mit starker Negation," pp. 234–240 in *Proceedings of Constructivity in Mathematics, Amsterdam 1957*, Studies in Logic and the Foundations of Mathematics (1959).

[9] Rasiowa, H., *An Algebraic Approach to Nonclassical Logics*, North Holland, Amsterdam, 1975.

[10] Sendlewski, A., "Some investigations of varieties of N-lattices," *Studia Logica*, vol. 43 (1984), pp. 257–280.

[11] Thomason, R. H., "A semantical study of constructible falsity," *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, vol. 15 (1969), pp. 247–257.

[12] Vakarelov, D., "Notes on N-lattices and constructive logic with strong negation," *Studia Logica*, vol. 36 (1977), pp. 109–125.

[13] Vorob'ev, N. N., "Constructive propositional calculus with strong negation" (in Russian), *Doklady Akademii Nauk*, vol. 85 (1952), pp. 456–468.

[14] Zeman, J. J., "Some calculi with strong negation primitive," *The Journal of Symbolic Logic*, vol. 33 (1968), pp. 97–100.

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